2-1-2009

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Scintillation index of modified Bessel–Gaussian beams propagating in turbulent media

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Received August 4, 2008; revised November 6, 2008; accepted December 8, 2008; posted December 11, 2008 (Doc. ID 99803); published January 29, 2009

The scintillation index is formulated for modified Bessel–Gaussian beams propagating in weakly turbulent media. Numerical calculations applied directly to the derived triple integral show that, for off-axis positions, the modified Bessel–Gaussian beams of higher order zero scintillate less than Gaussian beams at large input beam sizes and low beam orders with the increasing width parameter initially contributing positively to the phenomenon of less scintillation. As the beam order exceeds two, this advantage is diminished. The modified Bessel–Gaussian beam of order zero is a special case, however, exhibiting lowest scintillation at small input beam sizes. When considered against the propagation length, higher-order modified Bessel–Gaussian beams continue to offer less scintillation than those of order zero. At various radial positions, the scintillation index of modified Bessel–Gaussian beams with orders higher than zero attains small values toward the beam edges but rises sharply when approaching the beam axis. The effect of inner and outer scales of turbulence is also studied, and it is found that while increasing the inner scale of turbulence seems to cause increases in scintillation, the influence of the outer scale is hardly noticeable. © 2009 Optical Society of America

OCIS codes: 010.1300, 010.1330, 010.3310

1. INTRODUCTION

Features, production, and behavior on propagation in vacuum of $J$-Bessel–Gaussian beams of all orders [1–11] and $J$-Bessel–Gaussian beams (also known as modified Bessel–Gaussian) [12–14] have been extensively studied. Also, propagation of the second-order statistical properties (e.g., intensity) of $J_0$-Bessel–Gaussian [15], $J_p$-Bessel–Gaussian ($p>0$) [16], and modified Bessel–Gaussian [17] beams in atmospheric turbulence have been recently reported. Intensity fluctuations of beams generated by various sources (including higher-order mode sources) propagating to the receiver plane in turbulent media have also been scrutinized [18–40].

We have recently investigated the evolution of scintillation in weak turbulence for $J_p$-Bessel–Gaussian [41] and $J_p$-Bessel–Gaussian beams, with $p>0$ [42]. It was interesting to find that for all $p\geq 0$, Bessel–Gaussian beams offer favorable levels of scintillation index at large source sizes, the advantage being maintained with increasing beam order. In the current work, we extend our previous work [41,42] and investigate the scintillation index of modified Bessel–Gaussian beams in weak turbulence. Our motivation is to determine the effect of source modes of different types on intensity fluctuations when modified Bessel–Gaussian beams of any order $p\geq 0$ are used in atmospheric optics links.

2. FORMULATION

On the input plane, a modified Bessel–Gaussian beam composed by the product of a modified Bessel function and a Gaussian function has the form [17]

$$E_i(r_i, \theta_i) = \exp(-kw_i^2)\exp(-jp\theta_i)I_p(\Omega r_i).$$

(1)

Here $r_i$ and $\theta_i$ are the set of input plane radial and angular coordinates; $k$ is the wave number given by $k=2\pi/\lambda$, $\lambda$ being the wavelength, via the Gaussian exponential $\exp(-kw_i^2)$; $w_i$ governs the Gaussian input beam size $w_i$ and the focusing parameter $F_i$ through the relation $w = 1/(kw_i^2)^{1/2} + jF_i$; and parameter $p$ defines the order of the modified Bessel function $I_p(\cdot)$ and additionally determines the angular distribution of the field via $\exp(-jp\theta_i)$. The modified Bessel function $I_p(\cdot)$ together with $\exp(-jp\theta_i)$ constitute a solution to the wave equation [43]. Finally $\Omega$ is the width parameter that determines how close or far the zero crossings of the modified Bessel function are with respect to the on-axis position. Being measured in units of $m^{-1}$, $\Omega$ should be distinguished from the beam size $w_i$ as explained and illustrated in [16,17,42].

In the absence of turbulence, at any point within the cross section of the beam located $L$ away from the input plane, the field given in Eq. (1) can be transformed into the output field expression by using the Huygens–Fresnel
integral. Such derived field, denoted as $E_0(r_0, \theta_0, L)$, where $r_0$ and $\theta_0$ are the set of the output plane radial and angular coordinates, is found to be [17]

$$E_0(r_0, \theta_0, L) = \frac{-1}{1+2ijuL} \exp\left[ j(kL-p\theta_0) \right] + \frac{j1/2L - 2wkr^2}{2k(1+2ijuL)} I_p\left( \frac{\Omega r_0}{1+2ijuL} \right). \tag{2}$$

To deduce the scintillation of the output beam under the conditions of weak turbulence, we utilize the function $S(\cdot)$ that was obtained in [44,45] as the result of Rytov’s solution for the log-amplitude correlation function. Thus we incorporate the fluctuations in the complex amplitude of the beam received on the output plane relative to the field propagating in vacuum in the following way [41,42]:

$$S(r_0, \theta_0, \kappa, \varphi, \eta) = \frac{k^2}{2\pi(L-\eta)} E_0(r_0, \theta_0, L) \times \exp\left[ \frac{2(L-\eta)^2 + r_0^2}{2kL} \right] \times \int_{0}^{\pi} r_1 dr_1 \int_{0}^{2\pi} E_0(r_1, \theta_1, \eta)$$

We note that the derivation of Rytov’s solution is provided in [44], and the corresponding function of $S(\cdot)$ in Eq. (3) is denoted as $H(\cdot)$ in Eq A6 of [44]. In Eq. (3), $\kappa$ and $\varphi$ are the modulus and the angular orientation of the spatial frequency and $\eta$ is the distance variable. Inserting for $E_0(r_1, \theta_1, \eta)$ from Eq. (2) into Eq. (3), we can perform the integration in Eq. (3) in a manner analogous to the method used in [16]. That is, the integration over $\theta_1$ can be managed with the combined use of Eqs. 3.937.1 and 3.937.2 of [46]. Then for the resulting expression, we apply Eq. 6.633.2 of [46] and implement the integration over $r_1$. After these developments, the function $S(\cdot)$ becomes

$$S(r_0, \theta_0, \kappa, \varphi, \eta) = (-1)^p jk \frac{k^2(L-\eta)^2 + r_0^2}{2kL} \times \exp\left[ \frac{-j}{2kL} \left( \frac{\Omega r_0}{1+2ijuL} \right)^{0.5p} \right]$$

$$\times \exp\left[ \frac{-j}{2k(1+2ijuL)} \right] + j\varphi \, \theta_0 \right) \right] \times I_p\left( \frac{\Omega r_0}{1+2ijuL} \right) \right] \right)^{-1}. \tag{4}$$

The scintillation index of a beam launched from the input plane after traversing a propagation length of $L$ can then be expressed in terms of the above derived $S(\cdot)$ function as [34–37]

$$\sigma^2(r_0, \theta_0, L) = 4\pi \int_{0}^{\eta} d\eta \int_{0}^{\varphi} d\varphi \int_{0}^{2\pi} \text{Re}\left[ S(r_0, \theta_0, \kappa, \varphi, \eta) \right] \times S(r_0, \theta_0, \kappa, \varphi, \eta) + S(r_0, \theta_0, \kappa, \varphi, \eta) \times S(r_0, \theta_0, \kappa, \varphi, \eta) \times F_n(\kappa) d\varphi, \tag{5}$$

where $F_n(\kappa)$ is the power spectrum function accounting for the fluctuations in the index of refraction of the medium. To include the effects of inner and outer scales of turbulence, we choose $F_n(\kappa)$ to be a modified atmospheric spectrum [27], viz.,

$$F_n(\kappa) = 0.033C_o^2 \left[ 1 + 1.802 \left( \frac{\kappa L_o}{3.3} \right) - 0.254 \left( \frac{\kappa L_o}{3.3} \right)^{7/6} \right] \times \exp\left( \frac{-\kappa^2/10.89}{\kappa^2 + 4\pi^2/L_o^2} \right)^{11/6}. \tag{6}$$

where inner and outer scales of turbulence are symbolized by $L_o$ and $C_o$. $C_o$ is the refractive index structure constant. Upon replacing functions $S(\cdot)$ in Eq. (5) with their form given in Eq. (4), the scintillation index leads to the formula
\[
\sigma^2(r_0, \theta_0, L) = 0.4147C^2 k^2 \int_0^L \int_0^\infty \kappa d\kappa \int_0^{2\pi} \left[ 1 + 1.802 \left( \frac{\kappa L_0}{3.3} \right) - 0.254 \left( \frac{\kappa L_0}{3.3} \right)^{7/6} \right] \\
\times \exp(-\kappa^2/10.89) \text{Re} \left( \exp \left( 2w_c \kappa(L - \eta) \left[ 2k r_0 \cos(\varphi - \theta_0) - \kappa(L - \eta) \right] / k \left( 1 - 4w_c L + 4w_c^2 r_0^2 L^2 \right) \right) \right) \\
\times \int \frac{\Omega r_0}{1 + 2jwL} \left( 1 - \frac{\Omega r_0}{1 - 2jwL} \right)^{-1} \left( 1 + 2jwL \right) \left( \frac{\Omega(L^2 - \eta^2)^2 - 2k r_0 (L - \eta) \cos(\varphi - \theta_0) + k^2 r_0^2 L^2}{k(1 - 2jwL)} \right) \\
\times \left( \frac{\Omega(L^2 - \eta^2)^2 - 2k r_0 (L - \eta) \cos(\varphi - \theta_0) + k^2 r_0^2 L^2}{k(1 - 2jwL)} \right)^{-1/2} \exp(2j\varphi \theta_0) \left( \frac{\Omega r_0}{1 + 2jwL} \right)^{-2} \\
\times \int \frac{\Omega(L^2 - \eta^2)^2 - 2k r_0 (L - \eta) \cos(\varphi - \theta_0) + k^2 r_0^2 L^2}{k(1 - 2jwL)} \left( \frac{\Omega(L^2 - \eta^2)^2 - 2k r_0 (L - \eta) \cos(\varphi - \theta_0) + k^2 r_0^2 L^2}{k(1 - 2jwL)} \right)^{-1/2} \text{d}\varphi,
\]  

(7)

where \( w_r, w_i, \) and \(|w|\) are, respectively, the real and imaginary parts and the absolute value of parameter \( w \). Because of its complexity, Eq. (7) does not allow for an analytic solution. Therefore in Section 3, all the results are obtained from the numeric evaluation of Eq. (7) based on the triple integral routine successfully employed in our previous work [42]. It can be verified that the scintillation index formulation given in Eq. (7) matches perfectly with the formulas offered in our earlier works [31, 34–38, 41, 42], provided that special care is taken to adopt the input beam type and the spectrum beam correctly.

3. RESULTS AND DISCUSSION

Considering the number of input and propagation parameters, it is possible to produce a multitude of graphs from the numerical evaluation of Eq. (7). To cover a reasonable yet effective range of this option in the illustrations below, we will assume collimated beams, use a single operating wavelength, and set the structure constant as \( \lambda = 1.55 \) \( \mu \)m. \( C_2^2 = 10^{-15} \text{ m}^{-2/3} \), additionally keeping the inner and outer scales constants at \( \ell_0 = 1 \) mm and \( L_0 \to \infty \), except for the last two figures, where the variation of the scintillation index is plotted against the changes in these quantities. As shown in [17], single modified Bessel–Gaussian beams have rotational symmetry; thus we may conveniently set \( \theta_0 \) to zero and observe the complete variation of the scintillation index \( \sigma^2(r_0, \theta_0, L) \) over the whole receiver plane just by changing \( r_0 \).

Figure 1 provides the scintillation levels of several beams for an input beam size of \( w_i = 5 \) cm observed at propagation length of \( L = 1.5 \) km at an off-axis point equal to the input beam size, i.e., \( r_0 = 5 \) cm. Figure 1 demonstrates that, at low beam orders, the scintillation of the modified Bessel–Gaussian beams tend to be less than that of the Gaussian beam marked as \( p = 0, \Omega = 0 \) in this figure. However, when the beam order reaches high values, such as \( p = 5 \), this advantage disappears. Furthermore, according to Fig. 1, at low beam orders, rising width parameter (\( \Omega \)) causes a reduction in scintillation, but as we go to higher orders, the opposite effect takes over.

In Fig. 2, the case of the \( I_0 \)-Bessel–Gaussian beam is analyzed and compared with the Gaussian beam at the on-axis point against the variation in the input beam size, where for better viewing logarithmic scales are used for both the vertical and the horizontal axes. So it is seen from Fig. 2 that the \( I_0 \)-Bessel–Gaussian beam exhibits less scintillation only at small input beam sizes, namely, below 1 cm. In this case, the scintillation may be further lowered not only by increasing the width parameter, but also by shifting the useful range slightly toward smaller beam sizes. From [34], we know that at the plane-wave limit, the scintillation index becomes \( \sigma^2 = 1.23C_2^2 k^2 L^{11/6} \). If the numeric values of Fig. 2 are inserted into this expression, then \( \sigma^2 \) turns out to be 0.0199. We observe that this magnitude matches almost exactly the value attained by the curve of \( p = 0, \Omega = 0 \) in Fig. 2 at \( w_i = 10 \) cm, representing an input beam size well toward the plane wave-limit.

Figure 3 illustrates the scintillation index of the \( I_0 \)-Bessel–Gaussian beams for \( p = 1 \) and 2 versus input beam size and again with the Gaussian beam included for comparison purposes. Similar to the inference made for Fig. 1, it is also possible to deduce from Fig. 3 that the modified Bessel–Gaussian beams gradually start to lose their scintillation advantages over Gaussian beams as the beam order is increased, in this particular case for beam
orders above $p=2$. It must be mentioned that, although not displayed here, in the further extension of Fig. 3 to the left, where the smaller beam sizes would lie, the scintillation curves of all higher-order Bessel–Gaussian beams would eventually show steep rises.

Thus from a joint examination of Figs. 1–3 and also from the results of the previous works [41,42], we determine that in terms of offering favorable scintillation characteristics, lowest-order Bessel–Gaussian and lowest-order modified Bessel–Gaussian beams act in reciprocal fashion, whereas higher-order Bessel–Gaussian and higher-order modified Bessel–Gaussian beams act more or less in similar fashion. This means that all Bessel–Gaussian and higher-order modified Bessel–Gaussian beams scintillate less at larger beam sizes, but the favorable region of scintillation for the lowest-order modified Bessel–Gaussian beam is confined to smaller input beam sizes.

Furthermore, comparing Figs. 2 and 3, it is seen that with respect to the fundamental Gaussian beam, lowest- and higher-order modified Bessel–Gaussian beams display opposite behaviors. The physical reason for this may be attributed to the lowest-order modified Bessel–Gaussian beam being a special case. That is, although it has an intensity profile similar to that of the higher-order modified Bessel–Gaussian beams, unlike the higher order

Fig. 1. Scintillation index of modified Bessel–Gaussian beams at various beam orders and width combinations. Note that the units of $\Omega$ are m$^{-1}$.

Fig. 2. Scintillation index of the lowest-order modified Bessel–Gaussian beams versus input beam size at selected width parameters.
counterparts, the lowest-order modified Bessel–Gaussian beam does not possess a singularity at the on-axis position [17].

Figure 4 exhibits the scintillation of beams with orders of $p=1$ and 2 along the propagation axis at $w_i=5$ cm and $r_0=5$ cm plotted at different widths. It is seen in Fig. 4 that the scintillation index of higher-order beams is lower than that of the lowest-order beam (see the curve of $p=0$, $\Omega=1/w_i$). Additionally, from Fig. 4, it is seen that small changes in scintillations occur when the width parameter is below the value of $\Omega=2/w_i$, but that these changes are enhanced when the width parameter reaches the value of $\Omega=3/w_i$.

With the same beams as in Fig. 4, we illustrate in Fig. 5 the variation of the scintillation index against the radial distance, i.e., the distance to the off-axis point from the on-axis position of the same transverse cross section. Figure 5 shows that, toward the outer edges of the beam the scintillation index of the higher-order beams falls rapidly below that of the lowest-order beam, but around the on-axis points, sharp rises are experienced. According to Fig. 5, these falls seem to be directly proportional to rising beam orders, but nearly inversely proportional to rising width sizes. As explained before [42], the rapid rises toward the on-axis points are somewhat artificial and are partially due to the inadequacy of the Rytov theory in not

![Fig. 3. Scintillation index of higher-order modified Bessel–Gaussian beams versus input beam size at selected width parameters.](image)

![Fig. 4. Scintillation index of higher-order modified Bessel–Gaussian beams versus propagation length at selected width parameters.](image)
properly accounting for fluctuations in beams with zero on-axis input intensity.

Finally we investigate the dependence of scintillation index on inner and outer scales of turbulence. To this end, from Figs. 6 and 7, we see that, for the beams examined, the effect of the inner scale increases are felt more than those of the outer scale. Another point of interest is that the relative positioning of beams encountered in Fig. 4 is preserved throughout the inner and outer turbulence scale ranges considered in both Figs. 6 and 7.

4. CONCLUDING REMARKS
Intensity fluctuations at a single point (the scintillation index) are formulated and evaluated for modified Bessel–Gaussian beams propagating in weak atmospheric turbulence. When beams with low orders are considered, the scintillation index of modified Bessel–Gaussian beams is lower than that of the Gaussian beam. However, at relatively high beam orders, the scintillation advantage of modified Bessel–Gaussian beams tends to disappear. Additionally, when the beam orders are low, the intensity...
fluctuations decrease as the width parameter increases, and this trend reverses at higher beam orders. On comparing on-axis scintillation of the \( I_0 \)-Bessel–Gaussian beam to that of the Gaussian beam, it is observed that the \( I_0 \)-Bessel–Gaussian beam has reduced scintillation only at small input beam sizes, where the scintillation attains even lower values when the width parameter is increased.

Combining the results in the current work with our previous results [41,42], when advantageous scintillation characteristics are considered, we can assert that the \( J_0 \)-Bessel–Gaussian and the \( I_0 \)-Bessel–Gaussian beams act in a reciprocal manner, whereas the higher-order Bessel–Gaussian and the higher-order modified Bessel–Gaussian beams show similar behavior with respect to intensity fluctuations in weak turbulence. In other words, all Bessel–Gaussian and higher-order modified Bessel–Gaussian beams have smaller scintillation indices at larger beam sizes, while for the \( I_0 \)-Bessel–Gaussian beam, the intensity fluctuates less only at smaller input beam sizes. For radial positions, as we go away from the origin, the scintillation index of the higher-order modified Bessel–Gaussian beams falls rapidly below that of the lowest-order beam, the tendency in this decrease seeming to be directly proportional to rising beam order, but nearly inversely proportional to rising width sizes.

As a final observation, the effect of the inner and the outer scales of turbulence on the variation of the intensity fluctuations of modified Bessel–Gaussian beams is found to be not substantial.

**ACKNOWLEDGMENTS**

Y. Cai gratefully acknowledges support from the Alexander von Humboldt Foundation. O. Korotkova’s research is funded by the U.S. Air Force Office of Scientific Research (AFOSR) (grant FA 9500810102).

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