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Effect of the Pair-Structure Factor of a Particulate Medium On Scalar Wave Scattering in the First Born Approximation

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Wave scattering from particulate media is always discussed under a number of assumptions with regard to the properties of illumination as well as of individual particles and the collection as a whole [1–4]. Owing to such simplifying assumptions only a few characteristics of the scattered field are usually considered. In particular, in a majority of cases the incident radiation is assumed to be a monochromatic or polychromatic plane wave, either scalar or electromagnetic. Such simplification is not adequate in a number of practical situations, for instance, when either spatial distribution of illumination is not uniform and/or its coherence properties are not perfect.

On the other hand, in the description of random collections of particles it is usually assumed that the scattering potentials of the individual particles are deterministic functions of space and frequency, and their collective properties are determined by the structure factor—a quantity that accounts for the modification of the intensity of a plane wave that is incident at and scattered from the collection along two fixed directions and at a given frequency. However, structure factor cannot provide information about how the correlations in the incident wave are modified on scattering. This very fact limits scattering to the case of the incident plane wave and to the calculation of spectral properties of the scattered waves alone. Moreover, with knowledge of only the scattering potential of a single particle and of the structure factor, it is impossible to determine the coherence properties of the scattered field. Even in the case of a deterministic collection of scatterers the coherence properties of scattered waves may, generally, change [5].

In this Letter we develop the theory of scattering of scalar fields with arbitrary spectral and coherence properties from random collections of particles (deterministic potentials, random locations), which we characterize by a pair-structure factor (cf. [6]). The pair-structure factor is a generalization of the conventional structure factor accounting for the change in correlation properties of light at two given directions of incidence and two given directions of scattering. We stress that the knowledge of the pair-structure factor makes it possible to predict not only spectral but also coherence properties of scattered waves, even in the case when the incident wave has a more complex structure than a plane wave.

Let us consider a collection of $N$ identical particles with scattering potentials $f(r; \omega)$ as deterministic functions of position $r=(x,y,z)$ and frequency $\omega$ but with centers randomly distributed in scattering volume $V$. The scattering potential of a particle is a simple function of the index of refraction $n(r)$, namely,

$$f(r; \omega) = \frac{4\pi^2}{k} \left[ n^2(r) - 1 \right],$$

where $k=2\pi/\lambda$ is the wavenumber and $\lambda$ is the wavelength. The scattering potential $F(r; \omega)$ of the whole collection is then given by the expression [7]

$$F(r, \omega) = \sum_{n=1}^{N} f(r - r_n, \omega),$$

where $r_n$ is the center of the $n$th particle.

For a description of the scalar wave scattering from a random medium a spectral pair-scattering matrix, [8]

$$\mathbb{M}(u_1, u'_1; u_2, u'_2; \omega) = \langle S^*(u_1, u'_1; \omega) S(u_2, u'_2; \omega) \rangle_{rm}$$

may be employed, where the asterisk stands for the complex conjugate and angular brackets with subscript $rm$ denote the statistical average taken over the realization of the medium. In this equation $S(u, u'; \omega)$ is the ordinary scattering matrix that describes the change in the amplitude of a plane wave incident along direction $u'$ and scattered along direction $u$. It was shown [8] that within the accuracy of the first Born approximation, the scattering matrix is related to the scattering potential of the scatterer by the expression
where $\mathbf{K} = \hbar (\mathbf{u} - \mathbf{u}')$ is called the momentum transfer vector and the overbar denotes the (spatial) three-dimensional Fourier transform. On substituting from Eq. (2) into Eq. (4) we find that

$$S(\mathbf{K}; \omega) = \frac{\mathcal{F}(\mathbf{r}; \omega)}{2},$$

(4)

Further, on substituting from Eq. (5) into Eq. (3) we find that

$$M(\mathbf{K}_1, \mathbf{K}_2; \omega) = \frac{\mathcal{F}(\mathbf{K}_1, \omega) \mathcal{F}(\mathbf{K}_2, \omega)}{\mathcal{F}(\mathbf{r}; \omega)},$$

(6)

where

$$Q(\mathbf{K}_1, \mathbf{K}_2; \omega) = \frac{\sum_{n=1}^{N} \sum_{m=1}^{N} e^{-i \mathbf{r}_n \cdot \mathbf{K}_2 - i \mathbf{r}_m \cdot \mathbf{K}_1}}{\mathcal{F}(\mathbf{r}; \omega)},$$

(7)

is the pair-structure factor of the collection (cf. [6]). It provides the measure of the correlation (similarity) between waves along transfer vectors $\mathbf{K}_1$ and $\mathbf{K}_2$. If the momentum transfer vectors coincide, i.e., if $\mathbf{K}_1 = \mathbf{K}_2$, then the pair-structure factor reduces to the ordinary structure factor $S(\mathbf{K})$, viz.,

$$S(\mathbf{K}; \omega) = \frac{\sum_{n=1}^{N} \sum_{m=1}^{N} e^{-i \mathbf{r}_n \cdot \mathbf{K} - i \mathbf{r}_m \cdot \mathbf{K}}}{\mathcal{F}(\mathbf{r}; \omega)},$$

(8)

It is seen from Eq. (6) that in addition to the scattering potential $f(\mathbf{r})$ of an individual particle, the knowledge of the pair-structure factor is sufficient for determining all the second-order statistical properties of fields produced on scattering from a collection of particles while the knowledge of the structure factor $S(\mathbf{K}; \omega)$, together with $f(\mathbf{r})$, is not. Only in a very restricted number of cases, such as the determination of the spectrum at a particular direction of scattering of a plane wave, can we rely solely on the knowledge of $S(\mathbf{K}; \omega)$ and $f(\mathbf{r})$ [3].

Note that by normalizing the pair-structure factor according to the formula

$$q(\mathbf{K}_1, \mathbf{K}_2; \omega) = \frac{Q(\mathbf{K}_1, \mathbf{K}_2; \omega)}{\sqrt{S(\mathbf{K}_1; \omega) S(\mathbf{K}_2; \omega)}},$$

(9)

we obtain a complex-valued quantity whose absolute value varies between 0 and 1 and which may be regarded as a degree of angular correlation of the collection of scatterers. Such a measure is similar to the spectral degree of coherence of optical fields but describes the ability of a random medium to “decorrelate” waves.

To illustrate the importance of the pair-structure factor for scattering of random waves we now assume, without loss of generality, that it can be described by a Schell model, viz.,

$$Q(\mathbf{K}_1, \mathbf{K}_2; \omega) = \sqrt{S(\mathbf{K}_1) S(\mathbf{K}_2)} q(|\mathbf{K}_2 - \mathbf{K}_1|),$$

(10)

i.e., $q$ depends on the distance between the two momentum transfer vectors. Assume also that

$$q(|\mathbf{K}_2 - \mathbf{K}_1|) = \exp \left[ - \frac{\mathbf{K}_2 - \mathbf{K}_1^2}{(\hbar \delta)^2} \right],$$

(11)

where $\delta^2$ is the normalized variance of Gaussian distribution. A typical structure factor of a collection entering Eq. (11) can be found in [9].

We will now recall the general basic equations relating to the statistical properties of random waves scattered from random media. For a more detailed derivation of these expressions see [5, 8]. Let us assume that the incident field $U^{(i)}(\mathbf{r}, \omega)$ is the source plane is described with the help of the cross-spectral density function

$$W^{(i)}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle U^{(i)}(\mathbf{r}_1, \omega) U^{(i)}(\mathbf{r}_2, \omega) \rangle,$$

(12)

or its angular correlation function

$$\mathcal{A}^{(i)}(\mathbf{u}_1, \mathbf{u}_2, \omega) = \frac{\hbar^4}{(2\pi)^4} \int_{-\infty}^{\infty} W^{(i)}(\mathbf{r}_1, \mathbf{r}_2, \omega)$$

$$\times \exp[-i(\mathbf{u}_1 \cdot \mathbf{r}_1 + \mathbf{u}_2 \cdot \mathbf{r}_2)] d^2 \mathbf{r}_1 d^2 \mathbf{r}_2,$$

(13)

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors and the integration is performed over the entire source plane ([10], Section 5.6.3).

It was shown in [5] that the cross-spectral density function of the total (incident + scattered) field in the far zone of the scatterer is given by the expression

$$W^{(i)}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \pm \frac{4\pi^2}{\hbar^2} U^{(i)}(\mathbf{r}_1, \omega)$$

$$\times \mathcal{A}^{(i)}(\mathbf{u}_1, \mathbf{u}_2, \omega) \mathcal{A}^{(i)}(\mathbf{u}_1', \mathbf{u}_2', \omega) d^2 \mathbf{u}_1 d^2 \mathbf{u}_2,$$

(14)

where $r = |\mathbf{r}|$ and both integrations extend only over the homogeneous part of the angular spectrum.

For the spectral density $S^{(i)}(\mathbf{r}; \omega)$, and the spectral degree of coherence $\mu^{(i)}(\mathbf{r}_1, \mathbf{r}_2; \omega)$ of the total far field produced on scattering, we will use the formulas ([10], Section 4.3.2)

$$S^{(i)}(\mathbf{r}; \omega) = W^{(i)}(\mathbf{r}, \mathbf{r}; \omega),$$

(15)

$$\mu^{(i)}(\mathbf{r}_1, \mathbf{r}_2; \omega) = \frac{W^{(i)}(\mathbf{r}_1, \mathbf{r}_2; \omega)}{\sqrt{S^{(i)}(\mathbf{r}_1; \omega) S^{(i)}(\mathbf{r}_2; \omega)}}.$$
Suppose that the scatterers centered at positions \( \mathbf{r}_n = (x_n, y_n, z_n) \) have three-dimensional (soft) Gaussian potentials,

\[
f(\mathbf{r}_n; \omega) = B \exp \left[ -\frac{(x-x_n)^2 + (y-y_n)^2 + (z-z_n)^2}{2\sigma^2} \right],
\]

where variance \( \sigma^2 \) is independent of position but may depend on frequency \( \omega \). The correlation properties of the collection are described by Eqs. (10) and (11).

In Fig. 1 the contour plots of the spectral density of the far field produced on scattering [see Eq. (15)] are shown for two values of the variance of pair-structure factor \( \delta \). On the horizontal scale angle \( \theta \) is the polar angle of the far field and on the vertical scale angle \( \phi \) is the azimuthal angle of the far field. The angles are related to the components of the unit vector \( \mathbf{u} \) by the expressions \( u_x = \cos \theta \cos \phi \), \( u_y = \cos \theta \sin \phi \), and \( u_z = \sin \theta \). Figure 2 illustrates the modulus of the spatial degree of coherence [see Eq. (16)] as a function of the difference \( (\theta_2 - \theta_1) \) between two directions in the far field. It is clear from Figs. 1 and 2 that correlation properties of the scatterers can significantly modify the intensity distribution and coherence properties of the scattered wave.

On finishing we note that the use of the first Born approximation and neglect of the vector nature of waves impose serious limitations on the applicability of the results [11,12]. Indeed, the first Born approximation is valid only for scattering from particles with refractive indices very close to 1. Similarly, the neglect of polarization in scattering may result in a profound loss of information content.

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References