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State of polarization of a stochastic electromagnetic beam in an optical resonator

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On the basis of the unified theory of coherence and polarization, we investigate the behavior of the state of polarization of a stochastic electromagnetic beam in a Gaussian cavity. Formulations both in terms of Stokes parameters and in terms of polarization ellipse are given. We show that the state of polarization stabilizes, except in the case of a lossless cavity, after several passages between the mirrors, exhibiting monotonic or oscillatory behavior depending on the parameters of the resonator. We also find that an initially (spatially) uniformly polarized beam remains nonuniformly polarized even for a large number of passages between the mirrors of the cavity. © 2008 Optical Society of America

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1. INTRODUCTION

The theory of beam propagation in laser resonators was formulated a long time ago for monochromatic scalar fields [1,2]. The transverse modes of the resonator, frequently called the Fox–Li modes, were found to be related to the physical parameters of the cavity and the characteristics of the light source. The generalization of the Fox–Li theory to partially coherent, sometimes called random or stochastic fields, i.e., fields with any state of coherence, was later made [3–5]. The effects of the resonator on the statistical properties of light, such as its spectral density and its state of coherence, were also reported [6,7].

Only recently, with the development of the theory of electromagnetic (vectorial) partially coherent fields (see [8]), it has become possible to predict behavior of polarization properties of light beams on propagation in vacuum and on interaction with various linear media and linear optical systems, which may affect light in a deterministic or random manner (see [9–13]). In [14] the theory of resonator modes was developed for electromagnetic partially coherent fields (see also [15] for an alternative theory), where the transverse electromagnetic modes have been related to the classic Fox–Li modes [1] of the cavity. Very little, however, is known about the behavior of the polarization properties of partially coherent, electromagnetic light beams in resonators. The only publication of this kind (see [16]) addressed the evolution of the degree of polarization of a typical electromagnetic partially coherent beam.

In this paper we will extend the analysis of [16] by calculating the complete state of polarization of a typical electromagnetic partially coherent beam interacting with the laser resonator. We restrict our attention to a broad class of beams, namely, to the electromagnetic Gaussian Schell-model [EGSM] beams [9], which has recently received special attention due to its remarkable tractability on the one hand and due to its potential to exhibit all the phenomena of interest stemming from its electromagnetic as well as its random nature on the other hand. In Section 2 we will review the general theory of interaction of the EGSM beams with optical resonators and, in particular, their cross-spectral matrices, from which all the quantities of interest may be determined in the later sections.

Since, historically, there have been known two alternative descriptions of polarization properties of light, we will also consider them separately. In order to completely describe the polarization properties of a random beam, Stokes introduced (see [17]), as early as in 1852, the set of four independent parameters, since then known as Stokes parameters, which remain extremely popular in experimental optics due to the relative simplicity of their measurement. We note that for deterministic (monochromatic) beams, Stokes theory is also applicable, but in this case only three independent parameters are required to characterize the polarization properties. In Section 3 we will review the general expressions for the Stokes parameters of an electromagnetic partially coherent beam and then the expressions for the particular case of an EGSM beam; finally, we will provide numerical examples of the behavior of the Stokes parameters of the EGSM beams in optical resonators.

The other approach for characterization of the polarization properties of light beams is based on $2 \times 2$ correlation matrices [8] and originates from the fact that the correlation matrix of any partially polarized beam can be locally...
represented as a sum of a completely polarized beam that, as a matter of fact, is indistinguishable from the monochromatic beam, which we will call an equivalent monochromatic field, and a completely unpolarized beam (see [18] for such a decomposition in the space–time domain and [19] in the space–frequency domain). Further, it can be shown that at any point within the beam, the motion of the equivalent monochromatic field is constrained to a closed elliptic curve. The parameters of the ellipse, usually referred to as the polarization ellipse, can be determined from the elements of the correlation matrix. In Section 4 we will review the expressions for the ellipsometric parameters of the electromagnetic partially coherent beams and then will give numerical examples of the behavior of the ellipse of EGSM beams in optical resonators.

2. GENERAL THEORY OF INTERACTION OF AN ELECTROMAGNETIC PARTIALLY COHERENT BEAM WITH A RESONATOR

An optical resonator is composed of two reflecting surfaces (mirrors) with the same or different sizes and curvatures [see Fig. 1(a)]. Originally, the beam of light is sent to one of the mirrors in such a way that it then travels back and forth in the cavity. For simplicity we will assume that both mirrors are spherical, with radius of curvature $R$, mirror spot size $\eta$, and distance between the centers of the mirrors $L$. The interaction of light with such a resonator is equivalent to its propagation through a sequence of thin spherical lenses of focal lengths $f = R/2$ combined with filters with a Gaussian amplitude transmission function [6,7]. Figure 1(b) shows the equivalent “unfolded” version of the resonator. Depending on the value of the stability parameter $g = 1 - L/R$, the resonators are classified as stable ($0 \leq g < 1$) or unstable ($g \geq 1$) [20].

Let the beam entering the cavity be produced by an EGSM source (c.f. [9]). Such a beam can be characterized by the cross-spectral density matrix evaluated at points $(r_1, r_2) = \tilde{r}$ of the form [16]

$$W_{\alpha\beta}(\tilde{r}) = A_{\alpha} A_{\beta} B_{\alpha\beta} \exp \left[ - \frac{ik}{2} \tilde{r}^2 M_{\alpha\beta}^{-1} \tilde{r} \right], \quad (\alpha = x, y; \beta = x, y),$$

(1)

where $k = 2\pi/\lambda$ is the wavenumber, $\lambda$ is the wavelength of light, $A_{\alpha}$ is the square root of the spectral density of electric field component $E_{\alpha}$, $B_{\alpha\beta} = B_{\beta\alpha} \exp(i\phi)$ is the correlation coefficient between $E_{\alpha}$ and $E_{\beta}$ field components, $T$ stands for vector transpose, and $M_{\alpha\beta}^{-1}$ is the $4 \times 4$ matrix of the form

$$M_{\alpha\beta}^{-1} = \begin{pmatrix}
\frac{1}{ik} & \frac{1}{2\sigma_x^2} + \frac{1}{2\sigma_y^2} \\
\frac{i}{k\delta_{x\beta}} & \frac{1}{ik} \\
\frac{i}{k\delta_{y\beta}} & \frac{1}{2\sigma_x^2} + \frac{1}{2\sigma_y^2} \\
\end{pmatrix}, \quad (\alpha = x, y; \beta = x, y),$$

(2)

where $I$ is the $2 \times 2$ identity matrix and parameters $\sigma_{x,y}$ and $\delta_{x,y}$ are the widths of the spectral density and of the spectral degree of coherence of the beam, respectively. In Eqs. (1) and (2) as well as in all the formulas below, the explicit dependence of the cross-spectral density matrix on the oscillation frequency $\omega$ was omitted for simplicity.

We should note, however, that for this class of model approach that after passing $N$ times between the mirrors of the EGSM sources that produce physically realizable beamlike fields. It can be shown with the help of the $ABCD$ matrix approach that after passing $N$ times between the mirrors of the cavity, the cross-spectral density matrix of the beam at points with transverse position vectors $(p_1, p_2) = \tilde{p}$ is given by the expression (see Appendix A) [16]

$$W_{\alpha\beta}(\tilde{p}) = A_{\alpha} A_{\beta} R_{\alpha\beta} \det(\tilde{A} + \tilde{B}M_{\alpha\beta}^{-1})^{-1/2} \exp \left[ - \frac{ik}{2} \tilde{p}^2 M_{\alpha\beta}^{-1} \tilde{p} \right], \quad (\alpha = x, y; \beta = x, y),$$

(3)

where $\det$ stands for the determinant of a matrix and

$$M_{\alpha\beta}^{-1} = (\tilde{C} + \tilde{D}M_{\alpha\beta}^{-1})^{-1},$$

(4)

and $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$ are $4 \times 4$ matrices of the form

$$\tilde{A} = \begin{pmatrix} A & 0I \\ 0I & A^* \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B & 0I \\ 0I & -B^* \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} C & 0I \\ 0I & -C^* \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} D & 0I \\ 0I & D^* \end{pmatrix},$$

(5)

where the asterisk denotes the Hermitian operator. For the resonator of interest, matrices $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$ take the form [23]

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^N,$$

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} I & L \cdot I \\ -\frac{2}{R} - i\frac{\lambda}{\pi\eta^2} I & -I - \frac{2L}{R} - i\frac{\lambda L}{\pi\eta^2} I \end{pmatrix},$$

(6)

where, as before, $R$ is the radius of curvature, $\eta$ is the mirror spot size of the cavity, $L$ is the distance between the centers of the mirrors, and $N$ is the number of passages between the mirrors. The matrix with elements having subscript “1” describes the single pass between the
two mirrors. On substituting from Eqs. (4)–(6) into Eq. (3), one can determine the elements of the cross-spectral density matrix of the EGSM beam from which all second-order statistical properties of the beam can be found.

3. EVOLUTION OF THE STOKES PARAMETERS OF A BEAM IN A RESONATOR

We will now use the analysis of the previous section for calculation of the Stokes parameters of the beam [17] interacting with the optical resonator. Following [24], we express the Stokes parameters of the beam in terms of the elements of the cross-spectral density matrix (3) as

\[
S_0(p) = W_{xx}(p) + W_{yy}(p),
\]

\[
S_1(p) = W_{xx}(p) - W_{yy}(p),
\]

\[
S_2(p) = W_{xy}(p) + W_{yx}(p),
\]

\[
S_3(p) = i[W_{yx}(p) - W_{xy}(p)].
\]

We note that the Stokes parameter \( S_0 \) coincides with the spectral density of the beam.

We will now illustrate the behavior of the Stokes parameters of the beam as a function of the number of passages \( N \) for various values of cavity parameters \( \eta \) and \( g \). For all the figures in this paper, the following parameters of the source of the beam and of the cavity will be chosen (unless different values are specified in the captions): \( A_x = 1.5, \ A_y = 1, \ \phi = \pi/3, \ |B_{yy}| = 0.3, \ \sigma_x = \sigma_y = 1 \text{ cm}, \ \delta_{xx} = 0.225 \text{ mm}, \ \delta_{yy} = 0.25 \text{ mm}, \ L = 35 \text{ cm}, \text{ and } \lambda = 632.8 \text{ nm.} \)

Figure 2 shows the (on-axis) normalized Stokes parameters \( S_1/S_0, S_2/S_0, \text{ and } S_3/S_0 \) calculated from Eq. (7) with increasing values of \( N \) for different values of cavity parameter \( g \), with \( \delta_{xx} = 0.15 \text{ mm} \) and \( \eta = 0.8 \text{ mm} \). From this figure we see that after experiencing the changes for small values of \( N \) (5 < \( N \) < 20), the normalized Stokes parameters saturate at certain values, which may be either larger or smaller than initial values, determined by the cavity parameters and by the initial parameters of the beam. We also note that when the resonator is stable (0 < \( g \) < 10), the changes have oscillatory character while

![Fig. 2](image-url)
they are monotonic for unstable resonators \((g = 1)\), being similar, in this case, to changes when a beam propagates in free space (see Fig. 3 of [24]).

Figure 3 illustrates the evolution of the (on-axis) normalized Stokes parameters with increasing values of \(N\) for different values of the mirror spot size \(\eta\) in a Gaussian plane-parallel cavity \((g = 1)\) with \(\delta_{xx} = 0.15\) mm. It can be readily seen from the figure that with increasing values of mirror spot size \(\eta\), the saturation of all the normalized Stokes parameters occurs less rapidly, requiring a larger number of passes between the mirrors.

From Figs. 2 and 3 a very general conclusion can be drawn: The normalized Stokes parameters of the beam have qualitatively different behavior depending on the values of stability parameter \(g\). This might have the following physical explanation: In stable cavities, the two competing mechanisms of focusing and free-space diffraction act on the beam during each passage between the mirrors. The values of the normalized Stokes parameters change in an oscillatory manner depending on which mechanism prevails for a certain point between the mirrors. On the other hand, for unstable resonators the effect of focusing is negligible and the behavior of the normalized Stokes parameters is practically monotonic, just as in the case of free-space propagation.

4. EVOLUTION OF THE POLARIZATION ELLIPSE OF A BEAM IN A RESONATOR

We will now discuss the behavior of the polarization ellipse of a beam, which is known as an alternative representation of the state of polarization.

It has been recently shown in [19] how the cross-spectral density matrix can be used for finding the (spectral) polarization ellipse associated with the completely polarized portion of the beam as it propagates in free space. We will now use the same approach in order to study the behavior of the ellipse in the resonator.

The cross-spectral density matrix (3) evaluated at coinciding arguments can be uniquely decomposed into polarized and unpolarized matrices:

\[
W(\rho) = W^{(a)}(\rho) + W^{(p)}(\rho),
\]

where

\[
W^{(a)}(\rho) = \begin{pmatrix} A(\rho) & 0 \\ 0 & A(\rho) \end{pmatrix}, \quad W^{(p)}(\rho) = \begin{pmatrix} B(\rho) & D(\rho) \\ D^*(\rho) & C(\rho) \end{pmatrix}
\]

with

\[
A(\rho) = \frac{1}{2} [W_{xx}(\rho) + W_{yy}(\rho) \\
+ \sqrt{(W_{xx}(\rho) - W_{yy}(\rho))^2 + 4|W_{xy}(\rho)|^2}],
\]

\[
B(\rho) = \frac{1}{2} [W_{xx}(\rho) - W_{yy}(\rho) \\
+ \sqrt{(W_{xx}(\rho) - W_{yy}(\rho))^2 + 4|W_{xy}(\rho)|^2}],
\]

\[
C(\rho) = \frac{1}{2} [W_{yy}(\rho) - W_{xx}(\rho) \\
+ \sqrt{(W_{xx}(\rho) - W_{yy}(\rho))^2 + 4|W_{xy}(\rho)|^2}],
\]

\[
D(\rho) = W_{xy}.
\]

Using the fact that matrix \(W^{(p)}\) is a singular matrix at any point, it was demonstrated that its elements may be written as products of “equivalent monochromatic field” components, say, \(\mathcal{E}_x,\) and \(\mathcal{E}_y,\) [19]. The quadratic form associated with such a matrix can be shown to represent the ellipse, known as a spectral polarization ellipse, of the form

\[
W(\rho) = W^{(p)}(\rho),
\]

where

\[
W^{(p)}(\rho) = \begin{pmatrix} A(\rho) & 0 \\ 0 & A(\rho) \end{pmatrix}, \quad W^{(p)}(\rho) = \begin{pmatrix} B(\rho) & D(\rho) \\ D^*(\rho) & C(\rho) \end{pmatrix}
\]

with

\[
A(\rho) = \frac{1}{2} [W_{xx}(\rho) + W_{yy}(\rho) \\
+ \sqrt{(W_{xx}(\rho) - W_{yy}(\rho))^2 + 4|W_{xy}(\rho)|^2}],
\]

\[
B(\rho) = \frac{1}{2} [W_{xx}(\rho) - W_{yy}(\rho) \\
+ \sqrt{(W_{xx}(\rho) - W_{yy}(\rho))^2 + 4|W_{xy}(\rho)|^2}],
\]

\[
C(\rho) = \frac{1}{2} [W_{yy}(\rho) - W_{xx}(\rho) \\
+ \sqrt{(W_{xx}(\rho) - W_{yy}(\rho))^2 + 4|W_{xy}(\rho)|^2}],
\]

\[
D(\rho) = W_{xy}.
\]


where Re and Im stand for real and imaginary parts, respectively, of complex numbers and \( \mathcal{E}_{\phi}^{(n)}(\rho) \) is the \( n \)-th order of Fermat's law. The major and minor semiaxes of the ellipse, \( A_1 \) and \( A_2 \), as well as its degree of ellipticity, \( \varepsilon \), and its orientation angle, \( \theta \), can be related directly to the elements of the cross-spectral density matrix (3) with the help of the expressions

\[
A_{1,2}(\rho) = \frac{1}{\sqrt{2}} \left[ \sqrt{(W_{xx}(\rho) - W_{yy}(\rho))^2 + 4|W_{xy}(\rho)|^2} \right]^{1/2} \pm \sqrt{(W_{xx}(\rho) - W_{yy}(\rho))^2 + 4[Re W_{xy}(\rho)]^2}^{1/2},
\]

(12)

\[
\varepsilon(\rho) = \frac{A_2(\rho)}{A_1(\rho)},
\]

(13)

\[
\theta(\rho) = \frac{1}{2} \arctan \left( \frac{2 \text{Re} W_{xy}(\rho)}{W_{xx}(\rho) - W_{yy}(\rho)} \right).
\]

(14)

In Eq. (12) signs “+” and “−” between the two square roots correspond to \( A_1 \) (major semiaxis) and \( A_2 \) (minor semiaxis), respectively. On substitution from Eq. (3) into Eqs. (11)–(14), we can determine the behavior of the parameters of the polarization ellipse in the cavity.

Figures 4–11 illustrate the behavior of various parameters of the polarization ellipse in the resonator as a function of the number of passages \( N \) between the mirrors for various values of cavity parameters. In particular, Figs. 4 and 5 show the evolution of the orientation ellipse and of the degree of ellipticity for several combinations of stability parameter \( g \) and of the source correlation coefficient \( \delta_w \). One can see from these figures that, independently of the source correlations, while for stable resonators the ellipsometric quantities exhibit an oscillatory regime first and then stabilize for sufficiently large values of \( N \) to some values between oscillation maxima and minima, for unstable resonators they tend monotonically to certain values. As with propagation of the degree of polarization (compare with corresponding figures of [16]) and of the normalized Stokes parameters, the effect of an unstable resonator resembles the effect of the free-space propagation of the polarization ellipse (compare with Figs. 2 and 3 of [19]).

Figures 6 and 7 explore the evolution of the degree of ellipticity and of the orientation angle of the polarization ellipse with increasing \( N \) for various values of mirror spot size \( \eta \) in a Gaussian plane-parallel cavity \( (g=1) \) for the fixed model source (with \( \delta_w=0.15 \text{ mm} \)). As in the case of the other polarization properties, the increasing values of mirror spot size \( \eta \) correspond to slower changes in the ellipsometric parameters.

In Figs. 8 and 9 we illustrate the behavior of the degree of ellipticity and the orientation angle, respectively, with a growing number of passes \( N \), for several fixed values of stability parameter \( g \), in the limiting case of the lossless cavity \( (\eta \rightarrow \infty) \). The parameters of the source were chosen to be the same for all curves (with \( \delta_w=0.15 \text{ mm} \)). On comparing these figures with Figs. 4 and 5, we see that unlike in a stable resonator with losses, in the lossless cavity the parameters of the polarization ellipse never stop oscillating between two values.

As is seen from Figs. 4–9, the dependence of the parameters of the polarization ellipse on the stability parameter \( g \) is similar to that of the normalized Stokes parameters (compare with Figs. 2 and 3), and the physical reason for this is the same as that given in Section 3.

For all the figures above, the evolution of the polariza-

---

**Fig. 4.** Degree of ellipticity (on-axis) versus \( N \) for different values of cavity parameter \( g \) and the source correlation coefficients with \( \eta=0.8 \text{ mm} \): (a) \( \delta_w=0.15 \text{ mm} \), (b) \( \delta_w=0.15 \text{ mm} \), (c) \( g=0.5 \), (d) \( g=1.2 \).
tion properties of the beam were shown only on axis. Figures 10 and 11 illustrate the behavior of the degree of ellipticity and of the orientation angle in a transverse plane for a fixed number of passes \(N = 30\) and several fixed values of mirror spot size \(\eta\). The cavity was chosen to be a plane-parallel cavity \(g = 1\). It can be readily deduced from these figures that the spatial profiles of the polarization properties tend to become Gaussian for large values of \(N\), even though their initial distributions are uniform. Moreover, if in the center of the cavity the state of polarization of the beam is determined by the model source and by the resonator, it turns out to be “more” linearly polarized along the \(x\) direction toward the edge of the beam. In [16] a similar behavior is demonstrated for the degree of polarization of the beam: It tends to zero at the edges of the beam. We note that since such profiles depend on the cavity parameters, by adjusting them it is possible from a given uniformly polarized beam to obtain a beam with a variable, Gaussian-distributed, transverse polarization. In Fig. 12 the intensity distribution is shown for the same beam-cavity scenarios as in Figs. 10 and 11.

To illustrate the spatial distribution of the degree of ellipticity and of the orientation angle of a typical EGSM beam in greater detail, we show them in Fig. 12 as three-dimensional plots versus the \(x\) and \(y\) axes of the beam after \(N = 30\) passes between the mirrors for two selected values of \(\eta\). One sees that, similarly to the situation of Figs. 13 and 14, in the center of the cavity the state of polarization of the beam is determined by the cavity parameters and by the source parameters, but it tends to be linearly polarized along the \(x\) direction for points on the edge of the cavity. In Fig. 15 the three-dimensional plot illustrates the spatial distribution of intensity for the same beam-cavity situation as in Figs. 13 and 14.

![Fig. 5](image1.png)

Fig. 5. Orientation angle \(\theta\) (on-axis) versus \(N\) for different values of cavity parameter \(g\) and the source correlation coefficients with \(\eta = 0.8\) mm; (a) \(\delta_x = 0.15\) mm, (b) \(\delta_x = 0.15\) mm, (c) \(g = 0.5\), (d) \(g = 1.2\).

![Fig. 6](image2.png)

Fig. 6. Degree of ellipticity (on-axis) versus \(N\) for different values of mirror spot size \(\eta\) in a Gaussian plane-parallel cavity \((g = 1)\) with \(\delta_x = 0.15\) mm.

![Fig. 7](image3.png)

Fig. 7. Orientation angle \(\theta\) (on-axis) versus \(N\) for different values of mirror spot size \(\eta\) in a Gaussian plane-parallel cavity \((g = 1)\) with \(\delta_x = 0.15\) mm.
Fig. 8. Degree of ellipticity (on-axis) versus $N$ for different values of $g$ in a lossless cavity ($\eta \rightarrow \infty$) with $\delta_x = 0.15$ mm.

Fig. 9. Orientation angle $\theta$ (on-axis) versus $N$ for different values of $g$ in a lossless cavity ($\eta \rightarrow \infty$) with $\delta_x = 0.15$ mm.

Fig. 10. Degree of ellipticity versus a transverse dimension $x$ for different values of the mirror spot size $\eta$ and the source correlation coefficients in a Gaussian plane-parallel cavity ($g = 1$) with $N=30$. 
5. CONCLUDING REMARKS

We have investigated how various polarization properties of a typical EGSM evolve in optical resonators. In particular, we focused our attention on the Stokes parameters of the beam, normalized by its spectral density, and on two parameters of the polarization ellipse: the degree of ellipticity and the orientation angle. We found that all these quantities have a very similar qualitative dependence on the physical parameters of the cavity and on those of the source of the beam. In particular, there is a striking difference in the behavior of the polarimetric properties of beams that interact with stable and unstable resonators. While for stable resonators during the first several passages the polarimetric parameters may oscillate, for unstable resonators the changes are always strictly monotonic. Moreover, all quantities that we have considered get stabilized for a sufficiently large number \( N \) of passes between the mirrors, except in the case of the lossless cavity.

It is also of interest to note that even for a sufficiently large number of passes \( N \), the transverse polarization properties of the initially uniformly polarized beam are not, in general, uniform. The spatial distributions of all polarimetric properties that we have considered tend asymptotically to Gaussian as \( N \) increases.

The analysis of this paper revealed a number of interesting phenomena relating to the interaction of initially uniformly but arbitrarily polarized stochastic electromagnetic beams with optical resonators and may be used for the spatial modulation of polarization properties of such beams. Moreover, an optical resonator may be viewed as a device capable of generating beams with prescribed polarization properties.

APPENDIX A: DERIVATION OF EQ. (3)

The propagation of the elements of the cross-spectral density matrix through a general astigmatic \( ABCD \) optical system can be studied with the following generalized Collins formula [25]:

\[
W_{a\beta}(\rho) = \frac{k^2}{4\pi^2 (\det(B))^{1/2}} \int \int \int \int W_{a\beta}(\mathbf{r}) \times \exp \left[ -\frac{ik}{2} (\mathbf{r}^T \mathbf{B}^{-1} \mathbf{A} \mathbf{r} - 2\mathbf{r}^T \mathbf{B}^{-1} \rho + \rho^T \mathbf{D} \mathbf{B}^{-1} \rho) \right] d\mathbf{r},
\]

(A1)

where \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \) and \( \mathbf{D} \) are expressed in the form of Eq. (5), and \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \) and \( \mathbf{D} \) are the \( 2 \times 2 \) submatrices of the general astigmatic \( ABCD \) optical system. We note that for free-space propagation (see [25–27]), \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \) and \( \mathbf{D} \) are real quantities, implying that the “*” is not needed any-

![Fig. 11. Orientation angle \( \theta \) versus a transverse dimension \( x \) for different values of the mirror spot size \( \eta \) and the source correlation coefficients in a Gaussian plane-parallel cavity \( (g=1) \) with \( N=30 \).](image1)

![Fig. 12. Spectral density \( S_0 \) of the beam versus a transverse dimension \( x \) for different values of the mirror spot size \( \eta \) and the source correlation coefficients in a Gaussian plane-parallel cavity \( (g=1) \) with \( N=30 \).](image2)
Fig. 13. Spatial distribution of the degree of ellipticity of a typical EGSM beam for $N=30$ and $g=1$ for two different values of the mirror spot size $\eta$.

Fig. 14. Spatial distribution of the orientation angle of a typical EGSM beam for $N=30$ and $g=1$ for two different values of the mirror spot size $\eta$.

Fig. 15. Spatial distribution of spectral density $S_0$ of a typical EGSM beam for $N=30$ and $g=1$ for two different values of the mirror spot size $\eta$. 

$g=1$, $\delta_x=0.2$mm, $\eta=5$mm

$g=1$, $\delta_x=0.2$mm, $\eta=25$mm
where in Eq. (5). However, for a general optical system with loss or gain (e.g., dispersive media, a Gaussian aperture, helical gas lenses, etc.), \( A, B, C, \) and \( D \) take complex values and “*” is then required. Note that \( \tilde{A}, \tilde{B}, \tilde{C}, \) and \( \tilde{D} \) satisfy the following well-known Luneburg relations that describe the symplecticity of a general astigmatic optical system [28]:

\[
(\tilde{B}^{-1}\tilde{A})^T = \tilde{B}^{-1}\tilde{A}, \quad (\tilde{D}\tilde{B}^{-1})^T = \tilde{D}\tilde{B}^{-1},
\]

\[
\tilde{C} - \tilde{D}\tilde{B}^{-1}\tilde{A} = -(\tilde{B}^{-1})^T.
\]

(A2)

Substituting Eq. (1) into Eq. (A1), we obtain (after some operation) the formula

\[
W_{\alpha\beta}(\tilde{p}) = \frac{k^2A_{\alpha\beta}B_{\alpha\beta}}{4\pi^2[\det(\tilde{B})]^{1/2}} \exp \left[ -\frac{ik}{2}\tilde{p}^T\tilde{D}\tilde{B}^{-1}\tilde{p} \right]
\]

\[
+ \frac{ik}{2}\tilde{p}^T\tilde{B}^{-17}(M_{0\alpha\beta} + \tilde{B}^{-1}\tilde{A})^{-1}\tilde{B}^{-17}\tilde{p}
\]

\[
\times \int \int \int \int \exp \left[ -\frac{ik}{2}(M_{0\alpha\beta} + \tilde{B}^{-1}\tilde{A})^{1/2} \tilde{p} \right]
\]

\[
- (M_{0\alpha\beta} + \tilde{B}^{-1}\tilde{A})^{-1/2}\tilde{B}^{-1}\tilde{p}^2 \right] \, d\tilde{r}.
\]

(A3)

Then after applying the integral formula

\[
\int_{-\infty}^{\infty} \exp(-ax^2)dx = \sqrt{\pi/a},
\]

Eq. (A3) reduces (after vector integration) to the expression

\[
W_{\alpha\beta}(\tilde{p}) = \frac{k^2A_{\alpha\beta}B_{\alpha\beta}}{4\pi^2[\det(\tilde{B})]^{1/2}[\det(M_{0\alpha\beta} + \tilde{B}^{-1}\tilde{A})]^{1/2}} \times \exp \left[ -\frac{ik}{2}\tilde{p}^T\tilde{D}\tilde{B}^{-1}\tilde{p} \right]
\]

\[
\times (M_{0\alpha\beta} + \tilde{B}^{-1}\tilde{A})^{-1}\tilde{B}^{-17}.
\]

(A5)

After applying operations

\[
[\det(\tilde{B})]^{-1/2}[\det(M_{0\alpha\beta} + \tilde{B}^{-1}\tilde{A})]^{-1/2} = [\det(\tilde{B} + \tilde{B}M_{0\alpha\beta})]^{-1/2},
\]

(A6)

\[
\tilde{D}\tilde{B}^{-1} - \tilde{B}^{-17}(M_{0\alpha\beta} + \tilde{B}^{-1}\tilde{A})^{-1}\tilde{B}^{-1}
\]

\[
= [\tilde{D}\tilde{B}^{-1}(\tilde{A} + \tilde{B}M_{0\alpha\beta}) - \tilde{B}^{-1}]\tilde{A} + \tilde{B}M_{0\alpha\beta}^{-1}
\]

\[
= (\tilde{C} + \tilde{D}M_{0\alpha\beta})\tilde{A} + \tilde{B}M_{0\alpha\beta}^{-1},
\]

(A7)

and after setting

\[
M_{1\alpha\beta}^{-1} = (\tilde{C} + \tilde{D}M_{0\alpha\beta})(\tilde{A} + \tilde{B}M_{0\alpha\beta})^{-1},
\]

(A8)

Eq. (A5) reduces to Eq. (3) in the text. In the above derivation, we have used the Luneburg relations [Eq. (A2)]. If the EGSM beam travels once between two mirrors [see Fig. 1(b)], the matrices \( A, B, C, \) and \( D \) are given by the second term of Eq. (6) in the text (also see [7,16]).

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