2014-07-01

Hodge Theoretic Aspects of Categorical Spectrum

Yijia Liu
University of Miami, omega8664@gmail.com

UNIVERSITY OF MIAMI

HODGE THEORETIC ASPECTS OF CATEGORICAL SPECTRUM

By

Yijia Liu

A DISSERTATION

Submitted to the Faculty
of the University of Miami
in partial fulfillment of the requirements for
the degree of Doctor of Philosophy

Coral Gables, Florida

August 2014

UNIVERSITY OF MIAMI

A dissertation submitted in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

HODGE THEORETIC ASPECTS OF CATEGORICAL SPECTRUM

Yijia Liu

Approved:

Ludmil Katzarkov, Ph.D. Professor of Mathematics	Drew Armstrong, Ph.D. Associate Professor of Mathematics
Shulim Kaliman, Ph.D. Professor of Mathematics	M. Brian Blake, Ph.D. Dean of the Graduate School
Rafael Nepomechie, Ph.D. Professor of Physics	

LIU, YIJIA (Ph.D., Mathematics)
Hodge Theoretic Aspects of Categorical Spectrum (August 2014)

Abstract of a dissertation at the University of Miami.

Dissertation supervised by Professor Ludmil Katzarkov.
No. of pages of text. (77)

Historically, the notion of generation time was first introduced by Rouquier [Rou08], in his approach to Representation Theory. Then Orlov [Orl09b] generalized this idea to the notion of Orlov Spectrum, by considering generation times over all possible generators. Later in the work of Ballard, Favero and Katzarkov [BFK12], it appears that this invariant carries deep geometric information and it is connected to Birational Geometry. In this thesis we will consider in the case of singularity category of type A and this category with restricted generations. After realizing them combinatorially we use the combinatorics to calculate the spectra of them. This is the baby procedure of creating noncommutative base loci. We will introduce the notion of categorical Okounkov body to analyze this categorical base loci and demonstrate on the A_n example that jump numbers for categorical multiplier ideal sheaves are the Orlov spectra. The connection established above shall lead to an analogy with the classical Hodge theory.

Acknowledgments

I would like to express my special appreciation and thanks to my advisor Professor Dr. Ludmil Katzarkov: You have been a tremendous mentor for me. Your advice on both research as well as on my career have been invaluable.

I would also like to thank my committee members, Dr. Shulim Kaliman, Dr. Drew Armstrong, Dr. Rafael Nepomechie for serving as my committee members and their encouragement, insightful comments along the way.

I am incredibly grateful to Dr. Nikolai Saveliev and many other staff in Math department who also have dedicated so much time over the years and have always been so encouraging.

I would also like to thank my family and friends for their consistent support, especially my wife, Linlin Gao.

Table of Contents

List of Figures

Chapter 1

Introduction

Orlov Spectrum is an invariant of a triangulated category[1]. Historically, the notion of generation time was introduced by Rouquier [Rou08], in his approach to Representation Theory. Then Orlov [Orl09b] generalized this idea to the notion of Orlov Spectrum, by considering generation times over all possible generators. He realized that in the case of categories of geometric origin, such as the derived category of coherent sheaves, this invariant carries deep geometric information. Based on calculations of Rouquier and himself, Orlov proposed the conjecture that for any smooth algebraic variety X, the Krull dimension of X and the Rouquier dimension (i.e. the least number in Orlov Spectrum) of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ are equal. Many examples were verified in above mentioned papers, such as smooth affine varieties, projective spaces and smooth curves.

Later in the work of Ballard, Favero and Katzarkov [BFK12], connections of the spectra to Birational Geometry are made. They expected that finding gaps (i.e. missing numbers) in spectra is a new approach to deal with questions of rationality, by stating the conjecture that any gap of spectrum of a variety of dimension n has length no more than $n-2$.

[1] See Appendix A for definition

In this thesis we start with an example - spectra of the singularity category of type A [2] in chapter 2. Then in chapter 3 we experiment on this example by imposing obstructing polygons in the corresponding combinatorial picture of this A_n category and calculate the spectra after such restrictions on the generators. In this way in chapter 4 we try to develop categorical linear systems and base loci based on above examples. We start by conjecturing that such generation-restricted A_n category is connected to some localization category. The obstructing polygons, which correspond to localization functors, become a noncommutative linear system. The elaborated combinations of polygons become a noncommutative base locus. To study above structures, we introduce the notions of categorical Okounkov body and multiplier ideal sheaf. Classical Okounkov body is studied by Lazarsfeld and Mustaţă [LM09] motivated by earlier idea of Okounkov [Oko97, Oko96]. It measures how Picard groups in a flag of subvarieties of X fit together interacting with a big divisor D on X. The categorical Okounkov body is a way of measuring asymptotic interaction of two endofunctors by a flag of subcategories. Following approaches of pioneering works by Seidel [Sei14], Ein, Lazarsfeld, Mustaţă, Nakamaye, Popa [ELM$^+$06], Budur [Bud12], we see categorical restricted Okounkov body as a way of characterizing the base loci of a category. At the end we indicate a categorical analogue of the multiplier ideal sheaf with filtration which we conjecture is related to the Orlov spectrum of the category.

1.1 Orlov Spectrum

We first recall some definitions and expositions from [Orl09b] and [BFK12].

Let $\mathcal{T}$ be a triangulated category.

[2] See Appendix B

For a full subcategory $\mathcal{I}$ of $\mathcal{T}$, we denote by $\langle \mathcal{I} \rangle$ the full subcategory of $\mathcal{T}$ whose objects are isomorphic to summands of finite direct sums of shifts of objects in $\mathcal{I}$. Note also $\langle \mathcal{I} \rangle$ is the smallest full subcategory containing $\mathcal{I}$ which is closed under isomorphisms, shifts, summands and finite direct sums.

Given two full subcategories, $\mathcal{I}_1$ and $\mathcal{I}_2$, we denote by $\mathcal{I}_1 * \mathcal{I}_2$ the full subcategory of objects, X, which fits in a distinguished triangle,

$$I_1 \to X \to I_2 \to I_1[1], \tag{1.1.1}$$

with $I_i \in \mathcal{I}_1$. Further set

$$\mathcal{I}_1 \diamond \mathcal{I}_2 := \langle \mathcal{I}_1 * \mathcal{I}_2 \rangle. \tag{1.1.2}$$

By setting

$$\langle \mathcal{I} \rangle_0 := \langle \mathcal{I} \rangle, \tag{1.1.3}$$

we are able to inductively define

$$\langle \mathcal{I} \rangle_n := \langle \mathcal{I} \rangle_{n-1} \diamond \langle \mathcal{I} \rangle. \tag{1.1.4}$$

Similarly we define

$$\langle \mathcal{I} \rangle_\infty := \bigcup_{n \geq 0} \langle \mathcal{I} \rangle_n. \tag{1.1.5}$$

The operations, $*$ and $\diamond$, were introduced in [BvdB03] where their associativity is proven, due to the octahedral axiom A.14. It follows from the associativity that

$$\langle \mathcal{I} \rangle_n \diamond \langle \mathcal{I} \rangle_m = \langle \mathcal{I} \rangle_{n+m+1}. \tag{1.1.6}$$

Remark 1.1.1. For an object, $X \in \mathcal{T}$, we identify X with the full subcategory consisting of X notationally and in this way we can define similarly $\langle X \rangle_n$.

Remark 1.1.2. $\langle \mathcal{I} \rangle_{n+1}$ is the full subcategory of objects $X \in \mathcal{T}$ such that there is a distinguished triangle

$$X_1 \to X \to X_2 \to X_1[1] \tag{1.1.7}$$

with $X_1 \in \langle \mathcal{I} \rangle_i$ and $X_2 \in \langle \mathcal{I} \rangle_j$, where $i + j = n$, closed under summands of finite direct sums of shifts, and $\langle \mathcal{I} \rangle_0 := \langle \mathcal{I} \rangle$. Under this convention, the index equals the number of distinguished triangles allowed to use to obtain new objects from objects in $\mathcal{I}$.

Definition 1.1.3. Let X be an object of a triangulated category, $\mathcal{T}$. If there is an n with $\langle X \rangle_n = \mathcal{T}$, we set

$$g_{\mathcal{T}}(X) := \min \{n \geq 0 \mid \langle X \rangle_n = \mathcal{T}\}. \tag{1.1.8}$$

Otherwise, we set $g_{\mathcal{T}}(X) := \infty$. We call $g_{\mathcal{T}}(X)$ the **generation time** of X. When $\mathcal{T}$ is clear from context, we omit it and simply write $g(X)$. If $\langle X \rangle_\infty$ equals $\mathcal{T}$, we say that X is a **generator**. If $g(X)$ is finite, we say that X is a **strong generator**.

Definition 1.1.4. The **Orlov spectrum** of $\mathcal{T}$, denoted $\operatorname{OSpec} \mathcal{T}$, is the set

$$\operatorname{OSpec} \mathcal{T} := \{g(G) \mid G \in \mathcal{T},\ g(G) < \infty\} \subset \mathbb{Z}_{\geq 0}. \tag{1.1.9}$$

The **Rouquier dimension** of $\mathcal{T}$, denoted $\operatorname{rdim} \mathcal{T}$, is the infimum of $\operatorname{OSpec} \mathcal{T}$. The **ultimate dimension** of $\mathcal{T}$, denoted $\operatorname{udim} \mathcal{T}$, is the supremum of $\operatorname{OSpec} \mathcal{T}$. Conventionally, both of above notions are defined as ∞ when $\operatorname{OSpec} \mathcal{T}$ is empty.

It is also convenient to recall the following definitions that appeared in [ABIM10] and [BFK12].

Definition 1.1.5. Let G be an object of a triangulated category, $\mathcal{T}$. If there is an n with $X \in \langle G \rangle_n$, we set

$$\mathrm{Lvl}^G_{\mathcal{T}}(X) := \min \{n \geq 0 \mid X \in \langle G \rangle_n\}. \tag{1.1.10}$$

Otherwise, we set $\mathrm{Lvl}^G_{\mathcal{T}}(X) = \infty$. This number is called the **level of X with respect to G**, or simply the level of X when G is implicit.

The above two definitions indicate that:

Remark 1.1.6. For a generator G of a triangulated category $\mathcal{T}$,

$$g(G) = \max_{X \in \mathcal{T}} \mathrm{Lvl}^G(X) \tag{1.1.11}$$

Definition 1.1.7. Let I be a subset of $\mathbb{Z}$. We say that I has a **gap** of length s if, for some $a \in \mathbb{Z}$, $[a, a+s+1] \cap I = \{a, a+s+1\}$. We say that a triangulated category, $\mathcal{T}$, has a **gap** of length s if $\mathrm{OSpec}\, \mathcal{T}$ has a gap of length s.

Definition 1.1.8. Let $\mathcal{T}$ be a triangulated category, $f : X \to Y$ be a morphism, and $\mathcal{I}$ be a full subcategory. We say that $f : X \to Y$ is $\mathcal{I}$ **ghost** if, for all $I \in \mathcal{I}$, the induced map, $\mathrm{Hom}_{\mathcal{T}}(I, X) \to \mathrm{Hom}_{\mathcal{T}}(I, Y)$, is zero. We say that f is $\mathcal{I}$ **co-ghost** if, for all $I \in \mathcal{I}$, the induced map, $\mathrm{Hom}_{\mathcal{T}}(Y, I) \to \mathrm{Hom}_{\mathcal{T}}(X, I)$, is zero. If G is an object of $\mathcal{T}$, we will say that f is G **ghost** if f is $\langle G \rangle_0$ ghost and f is G **co-ghost** if f is $\langle G \rangle_0$ co-ghost.

The following lemma appears in [BFK12], as a corollary of earlier work by Kelly [Kel65], Rouquier [Rou08], Krause, Kussin [KK06] and Oppermann [O^+09], It is a useful tool towards calculating generation time, in particular the lower bound of generation time.

Lemma 1.1.9 (Ghost/Co-ghost Lemma and Converse)**.** Let $\mathcal{T}$ be a k-linear Ext-finite triangulated category and let G and X_0 be objects in $\mathcal{T}$. The following are equivalent:

1. $X_0 \in \langle G \rangle_n$ and $X_0 \notin \langle G \rangle_{n-1}$;

2. there exists a sequence,

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n, \tag{1.1.12}$$

   of maps in $\mathcal{T}$ such that all the f_i are ghost for G and $f_n \circ \cdots \circ f_1 \neq 0$. Furthermore there is no such sequence for $n+1$.

3. there exists a sequence,

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0, \tag{1.1.13}$$

   of maps in $\mathcal{T}$ such that all the f_i are co-ghost for G and $f_1 \circ \cdots \circ f_n \neq 0$. Furthermore there is no such sequence for $n+1$.

Proof. [BFK12], page 373. □

Chapter 2

Orlov Spectrum of Type A_n

The triangulated category we are considering in this thesis is the singularity category associated with isolated hypersurface A_n singularity [3]. It is denoted by $\mathrm{D}_{\mathrm{sg}}^{\mathrm{gr}}(A_n)$ in the graded case and $\mathrm{D}_{\mathrm{sg}}(A_n)$ in the ungraded case.

Convention. Throughout this thesis k will always denote an algebraically closed field of characteristic zero and we always mean by T or [1] the translation functor. Also, the Auslander-Reiten translation will be denoted by τ. Here we also abuse the notation and identify A_n with the algebra $k[x]/x^{n+1}$.

2.1 Graded Case

In the graded case, by [Orl09a] and [Tak05] the triangulated category of singularities $\mathrm{D}_{\mathrm{sg}}^{\mathrm{gr}}(A_n)$ has a full exceptional sequence and is equivalent to the bounded derived category of finite dimensional representations of the Dynkin quiver of type A_n, denoted $\mathrm{D}^{\mathrm{b}}(\operatorname{rep} A_n)$. So in this way, we can give a combinatorial description of the triangulated category [4].

[3] See Appendix B

[4] I greatly appreciate Johan Steen for explaining to me this construction.

2.1.1 Quiver Representation

First let us recall basics of the theory of quiver representations. Let Q be a quiver. Thus, Q is an oriented graph given by a set of vertices Q_0, a set of arrows Q_1 and two maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ taking an arrow to its source vertex respectively its target vertex. We assume that Q is finite (both Q_0 and Q_1 are finite) and acyclic (there are no oriented cycles in Q).

Definition 2.1.1.

- A **representation** V of Q is a collection $\{V_i \mid i \in Q_0\}$ of k-vector spaces together with a collection $\{V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)} \mid \alpha \in Q_1\}$ of k-linear maps.

- A **morphism** from a representation V to a representation W (both of the same quiver Q), denoted $\varphi : V \to W$, is a collection $\{\varphi_i : \varphi_i(V_i) \to \varphi_i(W_i) \mid i \in Q_0\}$ of k-linear maps such that for any arrow $\alpha : i \to j$ we have

$$W_\alpha \varphi_i = \varphi_j V_\alpha. \tag{2.1.1}$$

  This is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} V_i \arrow[r, "\varphi_i"] \arrow[d, "V_\alpha"'] & W_i \arrow[d, "W_\alpha"] \\ V_j \arrow[r, "\varphi_j"] & W_j \end{tikzcd} \tag{2.1.2}$$

- Note that the **isomorphisms** of representations are exactly the invertible morphisms, that is, those morphisms $\varphi : V \to W$ for which there exists a morphism $\psi : W \to V$ such that $\psi\varphi = 1_V$ and $\varphi\psi = 1_W$.

Then for a quiver Q, the category of representations of Q, $\mathrm{Rep}_k(Q)$ (or $\mathrm{Rep}(Q)$) is defined as the abelian category whose objects are representations of Q and whose morphisms are defined as above. Similarly, the category of finite dimensional representations of Q is denoted by $\mathrm{rep}_k(Q)$ (or $\mathrm{rep}(Q)$).

Example 2.1.2 (**Zero representation**)**.** The representation of any quiver Q which assigns to each vertex the zero space (and consequently to each arrow the zero map).

Definition 2.1.3.

- A representation W is a **subrepresentation** of a representation V (both of the same quiver Q) if the inclusions $\{W_i \hookrightarrow V_i \mid i \in Q_0\}$ define a morphism of representations.
- A representation S is an **irreducible** (or a **simple**) representation if S has exactly two subrepresentations, namely the zero representation and S itself.

Example 2.1.4 (**Simple representations**)**.** For any quiver Q, fix $i \in Q_0$, we can define a representation by assigning each $j \in Q_0$

$$S(i)_j = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2.1.3}$$

and each $\alpha \in Q_1$

$$S(i)_\alpha = 0. \tag{2.1.4}$$

Obviously, they are all simple representations of Q. Indeed, every simple object is in such form and therefore there exists a one-to-one correspondence between simples and vertices of Q.

Definition 2.1.5.

- For two representations V and W of a quiver Q we can define a new representation, called the **direct sum** of V and W, denoted $V \oplus W$, by setting $(V \oplus W)(i) = V(i) \oplus W(i)$ for each vertex i and $(V \oplus W)(\alpha) = V(\alpha) \oplus W(\alpha)$ for each arrow α of Q.

- A representation of Q is called (an) **indecomposable** if it is not isomorphic to the direct sum of two non-zero representations.

It is well-known that the category $\mathrm{Rep}(Q)$ is equivalent to $\mathrm{Mod}(kQ)$, the category of left modules over the **path algebra** kQ: the vector space generated by paths in the quiver Q, with multiplication defined by extending the composition of paths bilinearly, setting $vw = 0$ whenever $s(v) \neq t(w)$. So in this sense, we don't distinguish between indecomposable(simple) representations and indecomposable(simple) modules. We could also define projective(surjective) representations which are associated with projective(surjective) modules. Similarly, this equivalence restricts to the equivalence of $\mathrm{rep}(Q)$ and $\mathrm{mod}(kQ)$, the category of finitely generated left modules over kQ.

Remark 2.1.6. The path algebra kQ of a finite quiver Q without oriented cycles is **hereditary**, which means that all submodules of projective modules are again projective. Therefore, the abelian category $\mathrm{Mod}(kQ)(\mathrm{mod}(kQ))$ is a **hereditary category**: for any two objects X, Y,

$$\mathrm{Ext}^n(X, Y) = 0 \tag{2.1.5}$$

for any $n \geqslant 2$.

Recall that **bounded derived category** of an abelian category $\mathcal{A}$, denoted by $\mathrm{D^b}(\mathcal{A})$, is obtained from the category of bounded complexes in $\mathcal{A}$ by formally inverting all quasi-isomorphisms, and it is a triangulated category with a natural translation functor $\mathcal{T}$ the shift functor $[\,\cdot\,] : \mathrm{D^b}(\mathcal{A}) \to \mathrm{D^b}(\mathcal{A})$: for any complex $C^\bullet$

and $n \in \mathbb{Z}$, $C^\bullet[n]$ is the complex having degree m component

$$(C^\bullet[n])^m = C^{m+n} \tag{2.1.6}$$

for any $m \in \mathbb{Z}$ and the differentials $C^m[n] \to C^{m+1}[n]$ of which are $(-1)^n$ times the differentials $C^{m+n} \to C^{m+n+1}$ in $C^\bullet$.

Remark 2.1.7. The distinguished triangle in $\mathrm{D^b}(\mathcal{A})$ is the triangle that is isomorphic in $\mathrm{D^b}(\mathcal{A})$ to the triangle

$$X \to Y \to Cone(f) \to X[1] \tag{2.1.7}$$

for some map of chain complexes $f : X \to Y$ and the mapping cone of f, denoted $Cone(f)$. In particular, for a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, the triangle $X \to Y \to Z \to X[1]$ is distinguished in $\mathrm{D^b}(\mathcal{A})$.

So now we can accordingly denote the bounded derived category of $\mathrm{mod}(kQ)$ by $\mathrm{D^b}(\mathrm{mod}\, kQ)$, or $\mathrm{D^b}(kQ)$ for short. Note it is naturally equivalent to $\mathrm{D^b}(\mathrm{rep}\, A_n)$. The following useful lemma was proven by Happel [Hap88]:

Lemma 2.1.8. Let $X^\bullet$ be an indecomposable object in $\mathrm{D^b}(kQ)$. Then $X^\bullet$ is isomorphic to a stalk complex with indecomposable stalk, namely

$$\cdots \to 0 \to M \to 0 \to \cdots , \tag{2.1.8}$$

for an indecomposable module M in $\mathrm{D^b}(kQ)$.

Proof. [Hap88], page 49. □

Therefore, we could notationally identify indecomposable module X with stalk complex concentrated in degree 0 and accordingly define $X[n]$. Furthermore, to understand the structure of $\mathrm{D^b}(kQ)$, we have the following theorem in [Len07]:

Theorem 2.1.9. *For hereditary category $\mathcal{A}$, the bounded derived category $\mathrm{D^b}(\mathcal{A})$ is naturally equivalent to the repetitive category $\bigvee_{n\in\mathbb{Z}} \mathcal{A}[n]$, where each $\mathcal{A}[n]$ is a copy of $\mathcal{A}$, with objects written $X[n]$ for X in $\mathcal{A}$, and morphisms given by*

$$Hom_{\mathrm{D^b}(\mathcal{A})}(X[n], Y[m]) = Ext_{\mathcal{A}}^{m-n}(X, Y). \tag{2.1.9}$$

Here, $\bigvee_{n\in\mathbb{Z}} \mathcal{A}[n]$ stands for the additive closure of the union of all $\mathcal{A}[i]$, with only trivial morphism from $\mathcal{A}[i]$ to $\mathcal{A}[j]$ for $i > j$.

Proof. [Len07], page 9. □

Remark 2.1.10. Since $\mathcal{A}$ is hereditary, by equation 2.1.5 the only nontrivial morphisms in $\mathrm{D^b}(\mathcal{A})$ are $\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(X[n], Y[n])$ and $\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(X[n], Y[n+1])$.

2.1.2 Representation of A_n Quiver

Let Q be a quiver such that the underlying graph is a simply-laced Dynkin diagram of type A_n. We abuse the notation and call it A_n as well. Since all such quivers are derived Morita equivalent - see e.g. [Bon89], we simply consider the quiver with the orientation:

$$\overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \cdots \longleftarrow \overset{n}{\bullet}. \tag{2.1.10}$$

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be the Auslander-Reiten quiver of $\mathrm{mod}(kA_n)$, in which Γ_0 of vertices is identified with the collection of the isomorphism classes of indecomposable finite dimensional left kA_n-modules and Γ_1 of arrows whose direction and number is given by the dimension of the space of irreducible morphisms between these isomorphism classes.

For each integers $0 \leq i < j \leq n$, we write $X_{i,j}$ the indecomposable kA_n-module whose corresponding representation is given by

$$(\overset{1}{0} \leftarrow \cdots \leftarrow \overset{i}{0} \leftarrow \overset{i+1}{k} \leftarrow \cdots \leftarrow \overset{j}{k} \leftarrow \overset{j+1}{0} \leftarrow \cdots \leftarrow \overset{n}{0}). \tag{2.1.11}$$

Note that

$$\dim X_{i,j} = j - i. \tag{2.1.12}$$

Then it is known that $\Gamma_0 = \{X_{i,j} \mid 0 \leq i < j \leq n\}$ and Γ is of the following form:

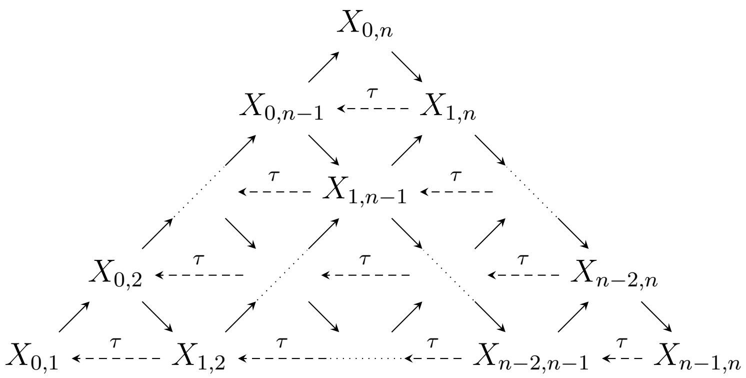
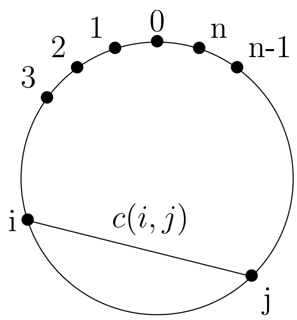


Figure 2.1: AR-quiver of A_n

Remark 2.1.11. In above figure, $P_i = X_{0,i}$ are the projective modules; $I_i = X_{i-1,n}$ are the injective modules; $S_i = X_{i-1,i}$ are the simple modules. Notice that $X_{i,j} = P_j/P_i$, $i < j$. In addition, we have the Auslander-Reiten translation τ from non-projective objects to non-injective objects, such that there exists an almost split sequence $0 \to \tau M \to N \to M \to 0$.

Now we consider the circle with $n+1$ points labeled $0, 1, 2, \ldots, n$ counter clockwise on it. We write $c(i,j)(= c(j,i))$ the chord between the points i and j. We denote by C_{n+1} the set of chords on the circle.



Recall that a partition of a set is a pairwise disjoint set of non-empty subsets. For a finite set, say $\{0, 1, ..., n\}$ without loss of generality. A non-crossing partition of it is a partition in which no two such subsets "cross" each other, i.e., for a and b belong to one and x and y to another, they cannot be arranged in cyclic order $axby$. By Ingalls and Thomas [IT09], there is a lattice isomorphism between non-crossing partitions of the vertices $\{0, 1, ..., n\}$ on the circle and wide subcategories of $\mathrm{mod}(kA_n)$, i.e. full abelian subcategories closed under extensions where the inclusion functor is exact. In particular, there is a bijection between vertices of the Auslander Reiten quiver $\Gamma_0 = \{X_{i,j} | 0 \leq i < j \leq n\}$ - indecomposables $X_{i,j}$ and the set of chords $C_{n+1} = \{c(i,j) | 0 \leq i < j \leq n\}$.

$$\begin{aligned} \Phi : \Gamma_0 &\to C_{n+1} \\ X_{i,j} &\mapsto c(i,j). \end{aligned} \tag{2.1.13}$$

So from now on we will use $X_{i,j}$ to represent the chord $c(i,j)$.

Now let us classify the distinguished triangles in $\mathrm{D^b}(kA_n)$. By Lemma 2.1.8, the indecomposables in $\mathrm{D^b}(kA_n)$ are simply $\{X_{i,j}[n] \mid n \in \mathbb{Z}\}$, after identification to the stalk complex. Moreover we could extend the Auslander-Reiten Quiver in Figure 2.1 to the Auslander-Reiten Quiver of $\mathrm{D^b}(kA_n)$.

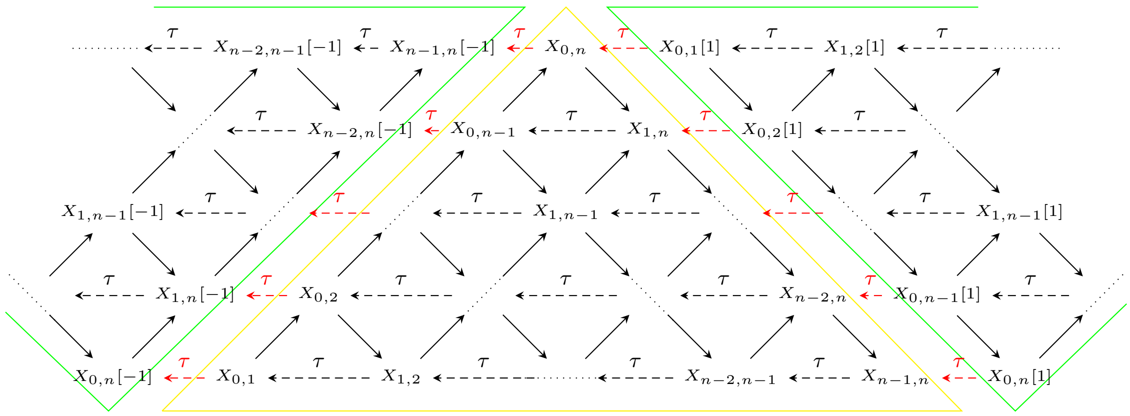


Figure 2.2: Derived AR-quiver of $\mathrm{D}^{\mathrm{b}}(kA_n)$

Let $f : X \to Y$ be a morphism in $\mathrm{D}^{\mathrm{b}}(kA_n)$. Then by Remark 2.1.7 f could be extended to a distinguished triangle:

$$
\begin{tikzcd}
& T(\mathrm{Ker}\, f) \oplus \mathrm{coKer}\, f \arrow[dl, "h"', "{[1]}"] & \\
X \arrow[rr, "f"] & & Y \arrow[ul, "g"']
\end{tikzcd}
\tag{2.1.14}
$$

Also note the only possibilities of nontrivial morphisms are given by Theorem 2.1.9, it suffices to consider the morphism $f : X_{i,j} \to X_{k,l}$ and $f : X_{i,j} \to X_{k,l}[1]$. Let us calculate the cones for both cases.

- For $i \leqslant k < j \leqslant l$, there exists a unique non-trivial morphism of representations $f : X_{i,j} \to X_{k,l}$ given by:

$$
f_p = \begin{cases} \mathrm{id} & \text{if } k < p \leqslant j \\ 0 & \text{otherwise} \end{cases} \tag{2.1.15}
$$

Therefore it give rise to a nontrivial extension/distinguished triangle:

$$\begin{tikzcd} & X_{i,k}[1] \oplus X_{j,l} \arrow[dl, "h"', "{[1]}"] & \\ X_{i,j} \arrow[rr, "f"] & & X_{k,l} \arrow[ul, "g"'] \end{tikzcd} \tag{2.1.16}$$

For $k \leqslant i < l \leqslant j$, we could only have the trivial one:

$$\begin{tikzcd} & X_{i,j}[1] \oplus X_{k,l} \arrow[dl, "h"', "{[1]}"] & \\ X_{i,j} \arrow[rr, "f"] & & X_{k,l} \arrow[ul, "g"'] \end{tikzcd} \tag{2.1.17}$$

But we could use the map $f' : X_{k,l} \to X_{i,j}$ to obtain the unique nontrivial extension/distinguished triangle:

$$\begin{tikzcd} & X_{k,i}[1] \oplus X_{l,j} \arrow[dl, "h'"', "{[1]}"] & \\ X_{k,l} \arrow[rr, "f'"] & & X_{i,j} \arrow[ul, "g'"'] \end{tikzcd} \tag{2.1.18}$$

- $f : X_{i,j} \to X_{k,l}[1]$ is the trivial map of complexes, which gives rise to only trivial extension/distinguished triangle:

$$
\begin{tikzcd}
& X_{i,j}[1]\oplus X_{k,l}[1] \arrow[dl, "h"', "{[1]}"] & \\
X_{i,j} \arrow[rr, "f"] & & X_{k,l}[1] \arrow[ul, "g"']
\end{tikzcd}
\tag{2.1.19}
$$

In addition, by Remark 2.1.7, a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(kA_n)$ could also come from an exact sequence in $\mathrm{mod}(kA_n)$. Without loss of generality, we consider only the exact sequence as a nontrivial extension of $X_{i,j}$ and $X_{k,l}$ which happens, as we are going to see, only when $i \leqslant k < j \leqslant l$ or $i \leqslant k = j \leqslant l$.

- Assume $i \leqslant k < j \leqslant l$. What would fit in the extension

$$0 \to X_{i,j} \to \mathbf{?} \to X_{j,l} \to 0 \tag{2.1.20}$$

non-trivially? This can be readily seen in the Auslander-Reiten Quiver (Figure 2.1). Note arrows are irreducible morphisms, while going upward means injective and downward surjective. So starting from $X_{i,j}$, we can go upstairs to $X_{i,l}$ and go downstairs to $X_{j,l}$. The right candidate to compensate the extension is $X_{k,j}$. They are actually the vertices of a parallelogram:

$$
\begin{tikzcd}
& X_{i,l} \arrow[dr, dotted] & \\
X_{i,j} \arrow[ur, dotted] \arrow[dr, dotted] & & X_{j,l} \\
& X_{k,j} \arrow[ur, dotted] &
\end{tikzcd}
$$

Therefore we could obtain the extension in $\mathrm{mod}(kA_n)$:

$$0 \to X_{i,j} \to X_{i,l} \oplus X_{k,j} \to X_{k,l} \to 0 \tag{2.1.21}$$

and pass to the distinguished triangle in $\mathrm{D}^{\mathrm{b}}(kA_n)$:

$$\begin{tikzcd} & X_{k,l} \arrow[dl, "{[1]}"'] & \\ X_{i,j} \arrow[rr] & & X_{i,l} \oplus X_{k,j} \arrow[ul] \end{tikzcd} \tag{2.1.22}$$

- For $i \leqslant j = k \leqslant l$, the dimension of $X_{i,l}$ equals the sum of the dimensions of $X_{j,l}$ and $X_{k,l}$. We need nothing to compensate hence obtain the extension

$$0 \to X_{i,j} \to X_{i,l} \to X_{j,l} = X_{k,l} \to 0 \tag{2.1.23}$$

and pass to the distinguished triangle:

$$\begin{tikzcd} & X_{j,l} = X_{k,l} \arrow[dl, "{[1]}"'] & \\ X_{i,j} \arrow[rr] & & X_{i,l} \arrow[ul] \end{tikzcd} \tag{2.1.24}$$

- However, if $i < j < k < l$, the dimension of $X_{i,l}$ is already larger than the sum of the dimensions of $X_{j,l}$ and $X_{k,l}$, therefore no way to build a nontrivial extension.

Now that we classified the triangles in $\mathrm{D}^{\mathrm{b}}(kA_n)$, let us neglect the shifts and look at the corresponding picture in the circle. Since $c(i,j) = c(j,i)$, we identify $X_{i,j}$ with $X_{j,i}$ as the nontrivial one of them. Then the distinguished triangles 2.1.16, 2.1.18 and 2.1.22 can be interpreted in the following figure:

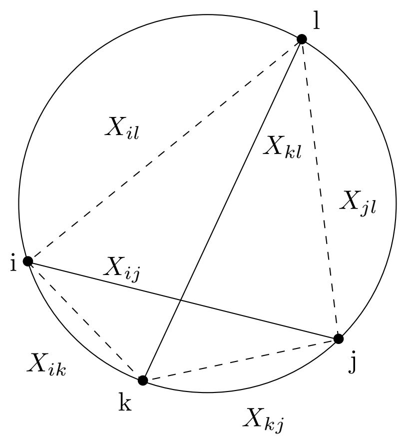


Figure 2.3: Extension

The indecomposables become chords in the circle. The condition $i \leqslant k < j \leqslant l$ or $k \leqslant i < l \leqslant j$ becomes the condition that the chords $X_{i,j}$ and $X_{k,l}$ intersects. Then our classifications above imply that the chords of the sides of the parallelogram can be given by the extension on the chords of its diagonals. In other words $X_{i,l}$, $X_{k,j}$, $X_{i,k}$ and $X_{j,l}$ are generated in one step starting from $G = X_{i,j} \oplus X_{k,l}$. Formally for any one of these sides, say X, $\mathrm{Lvl}^G(X) = 1$. Using this fact, an upper bound for each generator can be given by counting the lowest steps to reach the whole category, namely all diagonals and sides in this picture.

Indeed, the generation pattern is classified into the following three cases:

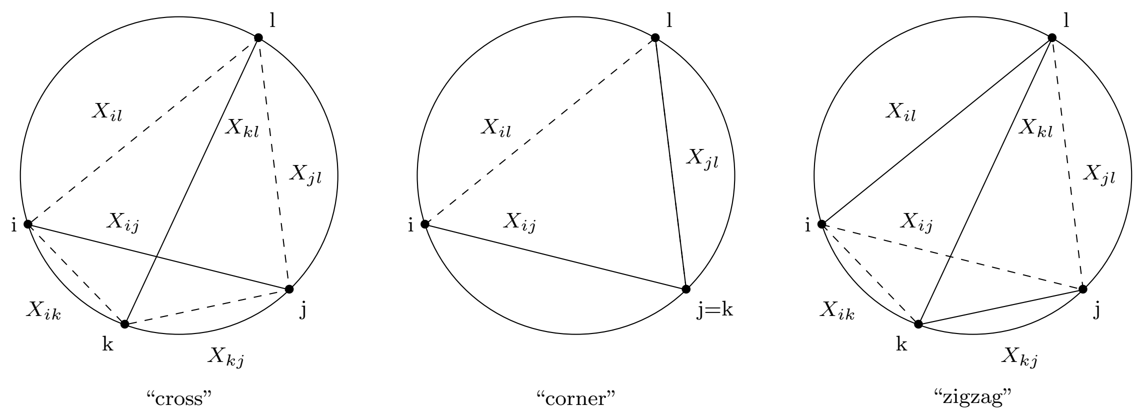


Figure 2.4: Generation Patterns

The "cross" case is what we discussed on previous page using distinguished triangles 2.1.16, 2.1.18 or 2.1.22; The "corner" case is the special case when $i \leqslant j = k \leqslant l$, using distinguished triangle 2.1.24; The "zigzag" case is the rotation of the distinguished triangle 2.1.16, 2.1.16 or 2.1.22, which we knew is also distinguished, from the axioms of triangulated category ((TR2) in Definition A.3).

Based on these generation patterns, we have the following lemma:

Lemma 2.1.12. An object G of $\mathrm{D}^{\mathrm{b}}(kA_n)$ is a (strong) generator if and only if its corresponding diagram is path-connected and passes through all vertices.

Proof. Decompose G into a sum of indecomposables and obtain the chordal diagram by drawing the correspond chords in the circle. Assume that the diagram has two disjoint path-connected components, by the generation pattern, only chords within each component could be generated and there is no way to generate the chords connecting these two components (no nontrivial cone and extension when two chords are disjoint). Finally since there are only finitely many indecomposables up to shifts, a generator is automatically strong. □

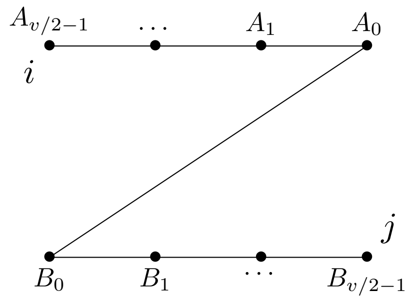
This lemma implies:

Remark 2.1.13. For any generator G of $\mathrm{D}^{\mathrm{b}}(kA_n)$ and any two vertices i and j in its chordal diagram, there is path of v vertices connecting i and j.

Therefore we could have the following estimate on the level of X_{ij} which is called the counting formula:

Lemma 2.1.14 (Counting formula).

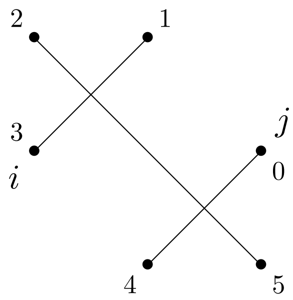
1. If the path connecting i and j is in the following shape, say "Big Z",



then

$$\mathrm{Lvl}^G(X_{ij}) \leqslant v/2 - 1 \tag{2.1.25}$$

2. If the path connecting i and j is in the following shape, say "Double Cross",



then

$$\mathrm{Lvl}^G(X_{ij}) \leqslant 2 \tag{2.1.26}$$

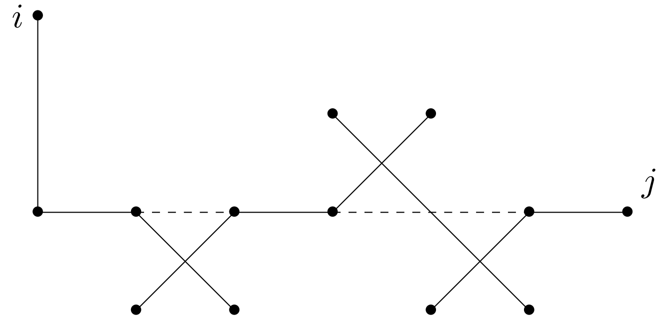
3. For any general path connecting i and j, we have

$$\mathrm{Lvl}^G(X_{ij}) \leqslant v - 2 - \#\{\text{crosses}\} \tag{2.1.27}$$

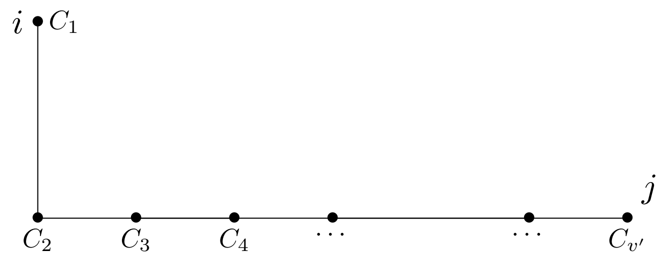
Proof. (1) Let us apply "zigzag" rule to generate new objects. First use one distinguished triangle to generate A_1B_1 by looking at zigzag A_1-A_0-B_0-B_1. Then A_2-A_1-B_1-B_2 becomes a new zigzag and we can use another one to generate A_2B_2. Continue until we obtain the chord $A_{v/2-1}B_{v/2-1}$, i.e. X_{ij}. In total we used $(v/2-1)$ triangles, therefore $\mathrm{Lvl}^G(X_{ij}) \leqslant v/2 - 1$.

(2)Apply "cross" rule on the cross 0-2-4-5 to generate the chord 2-4. Then 1-2-3-4 becomes a new cross which will generate the chord 0-3. In total we used two triangles, therefore $\mathrm{Lvl}^G(X_{ij}) \leqslant 2$.

(3)Any path connecting i and j looks like a chain with c_1 single crosses and c_2 double crosses as follows



First, we can use "cross" rule and "double cross" rule to generate all dashed objects in the path. In total we need to use $c_1 + 2c_2$ triangles. Now the path becomes a chain of $v' = v - 2c_1 - 4c_2$ vertices:



Secondly, for such chain, we use "corner" rule to generate new objects. Using one distinguished triangle, corner C_1-C_2-C_3 produces chord C_1C_3 and a new corner C_2-C_3-C_4. Then using another triangle we will produce chord C_1C_4. Continue until we obtain the chord $C_1C_{v'}$, i.e. X_{ij}. In total we used $v' - 2$ triangles.

Therefore, we can use $(c_1+2c_2)+(v'-2) = v-2-(c_1+2c_2) = v-2-\#\{\text{crosses}\}$ triangles to generate X_{ij}, which implies $\mathrm{Lvl}^G(X_{ij}) \leqslant v - 2 - \#\{\text{crosses}\}$. $\square$

Corollary 2.1.15. *The generation time of any generator G of $\mathrm{D^b}(kA_n)$ is bounded from above by $n-1$.*

Proof. Any path contains at most $n+1$ vertices in the chordal diagram of $\mathrm{D^b}(kA_n)$, so by Lemma 2.1.14 (3) any object has level no more than $n+1-2=n-1$. Therefore $g(G)\leqslant n-1$ as desired. $\square$

Based on the combinatorial interpretation, we are ready to give a different way to calculate the Orlov Spectrum of $\mathrm{D^b}(kA_n)$ from that in [BFK12].

Theorem 2.1.16.

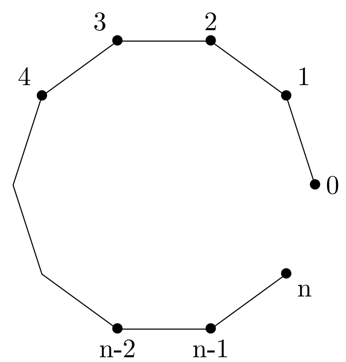
$$\operatorname{OSpec}\mathrm{D^b}(kA_n)=\{0,\ 1,\ ...,\ n-1\} \tag{2.1.28}$$

In particular,

$$g(\bigoplus_{i=1}^{n} S_i)=n-1. \tag{2.1.29}$$

Proof. By Corollary 2.1.15, $\operatorname{OSpec}\mathrm{D^b}(kA_n)\subseteq[0,\ n-1]$.

- For $G=\bigoplus_{i=1}^{n} S_i$, then the corresponding chordal diagram is a path:



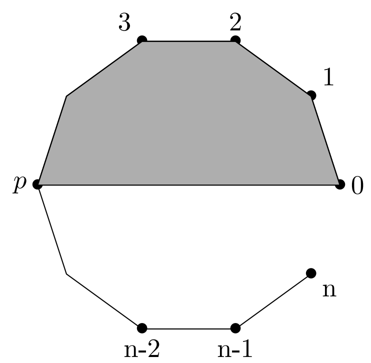
We knew $g(G)\leqslant n-1$ by Corollary 2.1.15. We will use the Ghost Lemma (Lemma 1.1.9) to show the lower bound of $g(G)$ is also $n-1$. Indeed, recall that $I_i = X_{i-1,n}$ are the injectives. Then

$$I_1 \to I_2 \to \cdots \to I_n \tag{2.1.30}$$

with natural morphisms as projections is the desired $\bigoplus_{i=1}^{n} S_i$-ghost sequence, since each $f_k : I_k \to I_{k+1}$ is $\bigoplus_{i=1}^{n} S_i$-ghost, and their composition is the nontrivial projection $I_1 \to I_n$. Therefore, we conclude that $g(G) = g(\bigoplus_{i=1}^{n} S_i) = n-1$ and $n-1 \in \operatorname{OSpec} \mathrm{D^b}(kA_n)$.

- Now for $2 \leqslant p \leqslant n-1$, consider the generator G_p associated with the diagram below, i.e. the direct sum of all simples and all diagonals and sides of the shadowed area. To be precise,

$$G_p = \bigoplus_{i=1}^{n} S_i \oplus \bigoplus_{0 \leqslant i < j \leqslant p} X_{ij} \tag{2.1.31}$$



Note we could now use a path of $(n+1)-(p-1) = n-p+2$ vertices to connect any two vertices. Thus $g(G_p) \leqslant n-p$. And the lower bound is also $n-p$ by the G_p-ghost sequence:

$$I_p \to I_{p+1} \to \cdots \to I_n. \tag{2.1.32}$$

So $g(G_p) = n - p$ and we conclude $\{1, 2, \ldots, n-2\} \subseteq \mathrm{OSpec}\,\mathrm{D}^{\mathrm{b}}(kA_n)$

- Last but not least, $g(\bigoplus_{i,j} X_{i,j}) = 0$ (because it is a finite direct sum), which completes the proof.

□

Indeed, since we classified all generation patterns, this also can be calculated by considering all possible paths in the diagram. We introduce the notion of the generation time of a path.

Definition 2.1.17. For any path P connecting i and j in chordal diagram corresponding to a generator G , the least number of distinguished triangle needed to generate $X_{i,j}$ is said to be the **generation time of the path** P , denoted $g_G(P)$ or $g(P)$ when G is implicit.

Remark 2.1.18. The counting formula indicates that for a path with v vertices:

- If $P =$ "Big Z", then $g(P) = v/2 - 1$.
- If $P =$ "Double Cross", then $g(P) = 2$.

In general, to obtain chord $X_{i,j}$ means to reduce the vertices of the path to 2 vertices. We observe that "Big Z" (including "zigzag" case), "cross" (including "Double Cross" case) rules reduce 2 vertices for each distinguished triangle used while "corner" rule only 1. So we apply the former rules on P until we have a chain then apply the corner rule. We record the number of distinguished triangles we used along the way and this number is the generation time of P. Then level of X_{ij} is nothing else but the least number of the generation times of all possible paths connecting i and j.

Therefore we have the following second counting formula:

Theorem 2.1.19 (2^{nd} **Counting formula**)**.**

1. *Assume that P is a path connecting i and j in chordal diagram corresponding to a generator G. P consists of v vertices, C crosses and D big Z's, where each Z_d contains $2p_d$ vertices. Then*

$$g(P) = v - 2 - \sum_{1 \leqslant d \leqslant D} (p_d - 1) - C \tag{2.1.33}$$

2. *For a generator G,*

$$\mathrm{Lvl}^G(X_{ij}) = \min_P g(P), \tag{2.1.34}$$

where P runs through all possible paths connecting i and j in chordal diagram corresponding to G.

3. *For a generator G,*

$$g(G) = \max_{i<j} \min_{P_{ij}} g(P_{ij}), \tag{2.1.35}$$

Proof. (1) and (2) follow immediately from the discussion above and (3) follows from Remark 1.1.6. □

2.2 Ungraded Case

For the ungraded case, $\mathrm{D_{sg}}(A_n)$, namely the stable derived category of the ring $A_n = k[u]/u^{n+1}$, was fully analyzed in [BFK12] and they proved that

Theorem 2.2.1 ([BFK12])**.**

$$\mathrm{OSpec}\, \mathrm{D_{sg}}(A_n) = \{\lceil \frac{\lfloor (n+1)/2 \rfloor}{s} \rceil - 1 \ : \ s \in \mathbb{N}\} \tag{2.2.36}$$

where $\lfloor\alpha\rfloor$ *is the greatest integer less than* α*,* $\lceil\alpha\rceil$ *is the least integer greater than* α*.*

Proof. [BFK12], page 402. $\square$

Remark 2.2.2. Combinatorially, notice in the Figure 2.2, the Auslander-Reiten functor τ acting on the indecomposables (omitting the shifts) is simply the clockwise rotation by one position. Since $\mathrm{D}_{\mathrm{sg}}(A_n) \cong \mathrm{D}^{\mathrm{b}}(kA_n)/\tau$ (see [Ami07]), the Orlov Spectrum of $\mathrm{D}_{\mathrm{sg}}(A_n)$ are simply the collection of the generation times given by the rotation invariant diagrams in previous section.

What is interesting for $\mathrm{D}_{\mathrm{sg}}(A_n)$ is that now the gap appears.

Example 2.2.3 (A_8, 1 gap)**.**

$$\operatorname{OSpec}\mathrm{D}_{\mathrm{sg}}(A_8) = \{0, 1, \text{\textvisiblespace}, 3\} \tag{2.2.37}$$

Example 2.2.4 (A_{18}, 2 gaps)**.**

$$\operatorname{OSpec}\mathrm{D}_{\mathrm{sg}}(A_{18}) = \{0, 1, 2, \text{\textvisiblespace}, 4, \text{\textvisiblespace}, 8\} \tag{2.2.38}$$

And as n goes to infinity, we could obtain as many gaps as we want. This phenomenon will reoccur in the generation-restricted A_n case in the next Chapter.

2.3 Estimate on Semi-Orthogonal Decompositions

In this section we discuss the estimate on the generation time if we could describe a triangulated category as a span of two components. Recall some definitions and lemmas from [Bon89] and [BK89].

Let $\mathcal{T}$ be a k-linear triangulated category with finite-dimensional morphism spaces. A functor $S : \mathcal{T} \to \mathcal{T}$ is called a Serre functor, if S is an auto-equivalence of $\mathcal{T}$ and there are isomorphisms

$$\mathrm{Hom}_{\mathcal{T}}(X, SY) \cong \mathrm{Hom}_{\mathcal{T}}(Y, X), \tag{2.3.39}$$

for $X, Y \in \mathcal{T}$.

Recall a full triangulated subcategory $\mathcal{I}$ of $\mathcal{T}$ is called **right (left) admissible** if the inclusion functor has a right (left) adjoint. The **right (left) orthogonal** $\mathcal{I}^{\perp}$ (${}^{\perp}\mathcal{I}$) of an admissible subcategory is the full category formed by objects B so that $\mathrm{Hom}_{\mathcal{T}}(A, B) = 0$ ($\mathrm{Hom}_{\mathcal{T}}(B, A) = 0$), for any $A \in \mathcal{I}$. We say $\mathcal{I}$ is **admissible** if it is both right and left admissible.

Definition 2.3.1. A **semi-orthogonal decomposition** of a triangulated category, $\mathcal{T}$, is a sequence of full triangulated subcategories, $\mathcal{A}_1, \ldots, \mathcal{A}_m$, in $\mathcal{T}$ such that $\mathcal{A}_i \subset \mathcal{A}_j^{\perp}$ for $i < j$ and, for every object $T \in \mathcal{T}$, there exists a diagram:

$$\begin{tikzcd}
0 \arrow[rr] & & T_{m-1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & T_2 \arrow[rr] & & T_1 \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
& A_m \arrow[ul] & & & & A_2 \arrow[ul] & & A_1 \arrow[ul] &
\end{tikzcd}$$

where all triangles are distinguished and $A_k \in \mathcal{A}_k$. We shall denote a semi-orthogonal decomposition by $\langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$.

A case of particular importance is if each $\mathcal{A}_i$ is equivalent to $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, k)$ as a triangulated category. Let A_i denote the object in $\mathcal{T}$ corresponding to k in $\mathcal{A}_i$. In this case, we call $A_1, \ldots, A_m$ an **exceptional collection**. If, in addition, $\mathrm{Hom}_{\mathcal{T}}(A_i, A_j[l]) = 0$ for $l \neq 0$, we say that the exceptional collection, $A_1, \ldots, A_n$, is **strong**.

The proofs of the following lemmas can be found in [BK89]:

Lemma 2.3.2. Let $\mathcal{A}$ be a full triangulated subcategory of a triangulated category $\mathcal{T}$ with Serre functor. Then the following are equivalent:

1. $\mathcal{A}$ is left admissible

2. $\mathcal{A}$ is right admissible

3. $\mathcal{A}$ is admissible

Lemma 2.3.3. If $\langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$ is a semi-orthogonal decomposition of a triangulated category $\mathcal{T}$ with Serre functor, then $\mathcal{A}_i$ is admissible for all i. Furthermore, if $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semi-orthogonal decomposition, then $\mathcal{B} = {}^{\perp}\mathcal{A}$.

The following property is a useful tool for us to get the estimate on generation time:

Proposition 2.3.4. *Let $\mathcal{I}$ be an admissible subcategory of $\mathcal{T}$, then $\mathcal{I}$ and $\mathcal{I}^{\perp}$ generate $\mathcal{T}$ in one step.*

Proof. Any $C \in \mathcal{T}$ can be included in a distinguished triangle $A \to C \to B \to A[1]$, where $A \in \mathcal{I}$ and $B \in \mathcal{I}^{\perp}$. Indeed, we could let $A = i^!(C)$, where $i^!$ is a right adjoint of the inclusion functor i. Then the identity morphism $A \to A$ defines a morphism $A \to C$ that can be extended to a distinguished triangle $A \to C \to B \to A[1]$. For any $A' \in \mathcal{I}$, we have an exact sequence:

$$\begin{tikzcd}
\mathrm{Hom}(A', A) \arrow[r, equal] & \mathrm{Hom}(A', i^!(C)) \arrow[r] & \mathrm{Hom}(A', C) \arrow[dll, rounded corners] \\
\mathrm{Hom}(A', B) \arrow[r] & \mathrm{Hom}(A', A[1]) \arrow[r] & \mathrm{Hom}(A', C[1])
\end{tikzcd}$$

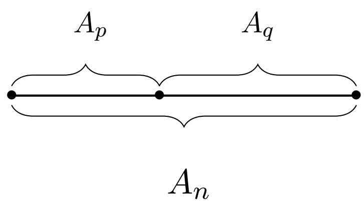
where the first and last morphisms are isomorphisms. This shows $\mathrm{Hom}(A', B) = 0$ for all $A' \in \mathcal{I}$, hence $B \in \mathcal{I}^{\perp}$. $\square$

In particular, for our case $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(kA_n)$, we have the following straightforward theorem:

Theorem 2.3.5. *A_n has semi-orthogonal decompositions*

$$A_n = \langle A_p, A_q \rangle, \quad \textit{where } p + q = n. \tag{2.3.40}$$

If we let X_p and X_q be the strong generators of A_p and A_q, with generation time $g(X_p)$ and $g(X_q)$ respectively. Then $X_p \oplus X_q$ is a generator for A_n, whose generation time is bounded above by $g(X_p) + g(X_q) + 1$.



Proof. We have a full exceptional sequence $\langle X_{0,1}, X_{0,2}, \ldots, X_{0,n} \rangle$.
Therefore $\langle X_{0,1}, X_{0,2}, \ldots, X_{0,p} \rangle$ is exceptional for all p, which implies that A_p is an admissible subcategory of A_n. Similarly, A_q is admissible as well. And it is obvious that $A_p \subset A_q^{\perp}$. The second part of this theorem is immediate, as you generate them separately and then apply Property 2.3.4. $\square$

Chapter 3

Spectra of Generation-Restricted A_n

In this chapter, we put some restrictions on the choice of generators of A_n category and consider the set of generation time of all qualified generators.

3.1 One-Polygon Case

Let I be a nonempty subset of $\{0, 1, ..., n\}$ with cardinality $|I| = k$. We mark points in I on the circle picture described in previous section. The convex hull of these k marked point forms a closed k-polygon inside the disk. And we only consider the generator without an indecomposable summand that has an intersection with the interior of this polygon. In other words, the indecomposable summand could either be one side of the polygon, intersect the polygon only at vertices or be totally disjoint from it. We define the spectrum of this I-restricted A_n category by the collection of generation time of all qualified generators, denoted $\operatorname{OSpec}_I A_n$.

Marking 1 point won't impose any obstruction. When $|I| = 2, 3$, we have the following observations.

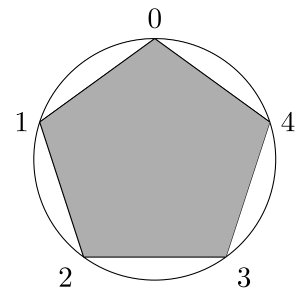
Remark 3.1.1. Segments, namely marking 2 points, kill only 0 in the spectrum if it is a diagonal of the $(n+1)$-gon. Otherwise, the spectrum remains unchanged.

Remark 3.1.2. Triangles, namely marking 3 points, kill only 0 in the spectrum for $n \geqslant 3$. More precisely,

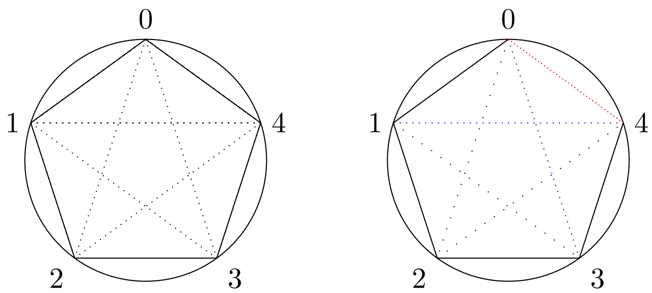
$$\operatorname{OSpec}_I A_n = \begin{cases} \{0,\ 1\} & \text{if } n = 2 \\ \{1,\ 2,\ ...,\ n-1\} & \text{if } n \geqslant 3 \end{cases} \tag{3.1.1}$$

However, if we impose a n-polygon with $n \geqslant 4$, gap could appear.

Example 3.1.3 (A_4 with pentagon)**.**



The figures below give us the only two choice of generators up to rotations, since we require the diagram to be path-connected and pass through all vertices.



The left one is the direct sum of all sides of pentagon, whose generation time is 1 since any of the rest indecomposables lies in a distinguished triangle with two given sides. And the right is the direct sum of all simples, whose generation time is 3 by Theorem 2.1.16. Therefore, in this case, the spectrum is $\{1, 3\}$, with a gap at 2.

Similarly, for A_n we have the following straightforward lemma.

Lemma 3.1.4. If we mark all $n+1$ vertices in A_n chordal diagram, then gap of maximal length appears. The spectrum in this case is $\{\lfloor \frac{n-1}{2} \rfloor,\ n-1\}$.

Proof. After marking all vertices on the diagram allowed indecomposables correspond to the sides of the outer $(n+1)$-gon. Up to rotations, which doesn't change the generation time, the only two choices of generators are the direct sum of simples, namely $G_1 = \bigoplus_{i=1}^{n} S_i$ and the direct sum of all indecomposables corresponding to the $n+1$ sides, namely $G_2 = I_1 \oplus \bigoplus_{i=1}^{n} S_i$.

For G_1, we knew by Theorem 2.1.16 $g(G_1) = n - 1$.

For G_2, the longest path connecting 2 vertices is the chain starting from 0 to the farthest vertex from it, which is the vertex $\lfloor \frac{n+1}{2} \rfloor$. So by counting formula, $g(G_2)$ is bounded above by $z = \lfloor \frac{n+1}{2} \rfloor + 1 - 2 = \lfloor \frac{n-1}{2} \rfloor$. It is also a lower bound since we have a G_2-ghost sequence (since $2z + 1 \leqslant n$):

$$X_{0,z+1} \to X_{1,z+2} \to \cdots \to X_{z,2z+1}. \tag{3.1.2}$$

Hence $g(G_2) = \lfloor \frac{n-1}{2} \rfloor$. □

One may ask: will the spectrum depend on the position of the imposed polygon? The answer is yes, which can be seen from the following example.

Example 3.1.5 (A_9 with octagon)**.**

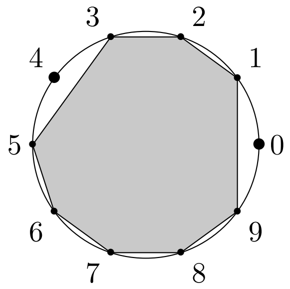
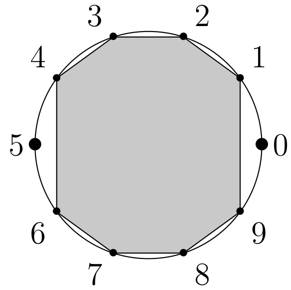
$I = \{1, 2, 3, 5, 6, 7, 8, 9\}$	$I' = \{1, 2, 3, 4, 6, 7, 8, 9\}$
	
$\operatorname{OSpec}_I A_9 = \{3, 4, \text{␣}, 6, 7, 8\}$	$\operatorname{OSpec}_{I'} A_9 = \{4, \text{␣}, 6, 7, 8\}$

Figure 3.1: A_9 with octagon

To distinguish above two cases, we introduced the notion of mirror points.

Definition 3.1.6. For a subset I of $S = \{0, 1, ..., n\}$, we say $\{x, y\}$ is a pair of **mirror points** with respect to I, if

$$|\{i \in I| \ x < i < y\}| = |\{i \in I| \ i < x \text{ or } i > y\}| \tag{3.1.3}$$

for $x, y \in S - I$ so that $x < y$.

Remark 3.1.7. Mirror points with respect to I ($|I| = k$), is a pair of points lying at antipodal parts of the areas divided by the circle and the k-polygon. They can only exist in the even polygon case with $k < n$ (so at least 2 points outside I), since if k is odd, one side in equation 3.1.3 will be odd with another side even.

In fact the position of the polygon only affects the Rouquier dimension by at most 1. In general, we will prove the following formula for k-gon in A_n:

Theorem 3.1.8 (Orlov spectrum restricted by 1 polygon)**.**

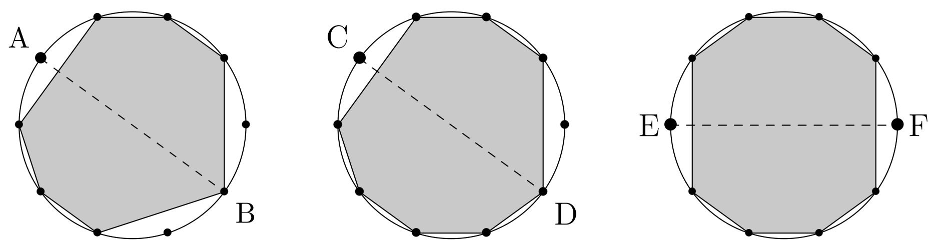
$$
\mathrm{OSpec}_I\, A_n = \begin{cases} \{\lfloor \frac{n-1}{2} \rfloor\} \cup \{n-1\} & \textit{if} \quad k = n+1 \quad (3.1.4a) \\ \{\lfloor \frac{k-1}{2} \rfloor,\ ...,\ \lfloor \frac{n-1}{2} \rfloor\} \cup \{k-2,\ ...,\ n-1\} & \\ \qquad \textit{if} \quad k \leqslant n \textit{ with no mirror points} & (3.1.4b) \\ \{\lfloor \frac{k-1}{2} \rfloor + 1,\ ...,\ \lfloor \frac{n-1}{2} \rfloor\} \cup \{k-2,\ ...,\ n-1\} & \\ \qquad \textit{if} \quad k < n \textit{ with a pair of mirror points} & (3.1.4c) \end{cases}
$$

where $k = |I|$.

To show this formula, let us first prove the following two lemmas.

Lemma 3.1.9. For $|I| = k$ and $k \leqslant n$, denote by G_I the sum of all allowed objects, then

$$
g(G_I) = \begin{cases} \lfloor \frac{k-1}{2} \rfloor & \text{if } I \text{ has no mirror points} \\ \lfloor \frac{k-1}{2} \rfloor + 1 & \text{if } I \text{ has mirror points} \end{cases} \tag{3.1.5}
$$



Proof. We consider the following cases:

- Assume I is odd. Pick any point vertex not in I (possible since $k \leqslant n$) and denote it by A. Then pick the vertex B in I so that $|\{i \in I|\ \min(A,B) < i < \max(A,B)\}| = |\{i \in I|\ i < \min(A,B) \text{ or } i > \max(A,B)\}|$, which means chord AB divide the k-gon into two parts and these two parts have the same number of vertices in I.

Note that any two vertices can be connected by a chain of no more than $\frac{k-1}{2}+2$ vertices. Therefore for any chord c, $\mathrm{Lvl}^{G^I}(c) \leqslant \frac{k-1}{2}$ by first counting formula. Besides $\mathrm{Lvl}^{G^I}(AB) = \frac{k-1}{2}$ by second counting formula. Hence $g(G_I) = \frac{k-1}{2} = \lfloor\frac{k-1}{2}\rfloor$.

- Assume k is even and I has no mirror points. Pick any point vertex not in I (possible since $k \leqslant n$) and denote it by C. Then pick the vertex D in I so that $|\{i \in I|\ \min(C,D) < i < \max(C,D)\}| = |\{i \in I|\ i < \min(C,D) \text{ or } i > \max(C,D)\}| + 1$.

  Note that any two vertices can be connected by a chain of no more than $\frac{k}{2}+1$ vertices. Therefore for any chord c, $\mathrm{Lvl}^{G^I}(c) \leqslant \frac{k}{2}-1$ by first counting formula. Besides $\mathrm{Lvl}^{G^I}(CD) = \frac{k}{2}-1$ by second counting formula. Hence $g(G_I) = \frac{k}{2}-1 = \lfloor\frac{k-1}{2}\rfloor$.

- Assume I has a pair of mirror points. Denote these two points by E and F. Then now any two vertices can be connected by a chain of no more than $\frac{k}{2}+2$ vertices. Therefore for any chord c, $\mathrm{Lvl}^{G^I}(c) \leqslant \frac{k}{2}$ by first counting formula. Besides $\mathrm{Lvl}^{G^I}(EF) = \frac{k}{2}$ by second counting formula. Hence $g(G_I) = \frac{k}{2} = \lfloor\frac{k-1}{2}\rfloor + 1$.

$\square$

Corollary 3.1.10. *For $|I| = k$ and $k \leqslant n$,*

$$\{\lfloor\frac{k-1}{2}\rfloor + 1,\ ...,\ \lfloor\frac{n-1}{2}\rfloor\} \subseteq \mathrm{OSpec}_I\, A_n \tag{3.1.6}$$

and

$$\lfloor \frac{k-1}{2} \rfloor \in \operatorname{OSpec}_I A_n \textit{ if } I \textit{ has no mirror points.} \tag{3.1.7}$$

Proof. Every k-gon is contained in a j-gon J, where $j \geqslant k$ is odd, thus no mirror points. Let G_J = sum of all allowed objects, then $g(G_J) = \frac{j-1}{2}$. All desired generation time can be realized this way. □

Lemma 3.1.11. If I contains consecutive numbers (here we identify 0 with $n+1$) i.e. the k-gon has one side on the boundary of the outer $(n+1)$-gon. Let $G'_I = \oplus$ all allowed objects but this chord, then

$$g(G'_I) = k - 2 \tag{3.1.8}$$

Proof. Observe any two vertices on the circle can be connected by a path of at most k points. So by counting formula, $g(G'_I) \leqslant k-2$. But the level of the missing chord is $k-2$ by second counting lemma. Therefore the generation time is $k-2$, as desired. □

Corollary 3.1.12.

$$\{k-1, \ ..., \ n-1\} \subset \operatorname{OSpec}_I A_n$$

Proof. Every k-gon is contained in a $k+1$-gon which has one side on the boundary. Hence it is contained in a j-gon J with $j \geqslant k+1$, which has one side on the boundary. Let $G'_J = \oplus$ all allowed objects but this chord, then $g(G'_J) = j-2$ as shown in above lemma. As j can be chosen to be any integer from $k+1$ to $n+1$, we obtain all desired entries. □

Remark 3.1.13.

- If $k \leqslant \lfloor \frac{n-1}{2} \rfloor + 2$, then $\lfloor \frac{n-1}{2} \rfloor \geqslant k-2$. So $\{\lfloor \frac{k-1}{2} \rfloor, \ ..., \ \lfloor \frac{n-1}{2} \rfloor\} \cup \{k-1, \ ..., \ n-1\} = \{\lfloor \frac{k-1}{2} \rfloor, \ ..., \ n-1\}$ and $\{\lfloor \frac{k-1}{2} \rfloor + 1, \ ..., \ \lfloor \frac{n-1}{2} \rfloor\} \cup \{k-1, \ ..., \ n-1\} =$

$\{\lfloor\frac{k-1}{2}\rfloor+1,\ \ldots,\ n-1\}$. We have already all possible generation times and no gap in $\operatorname{OSpec}_I A_n$.

- If $k \geqslant \lfloor\frac{n-1}{2}\rfloor+3$, then $n+1-k<k$, so there are less than k vertices outside of the k-gon, namely at least one side of the k-gon is on the boundary of the $(n+1)$-gon. Then Lemma 3.1.11 implies that $k-2 \in \operatorname{OSpec}_I A_n$.

The remark shows that

Corollary 3.1.14. *If $k \leqslant \lfloor\frac{n-1}{2}\rfloor+3$, then*

$$\operatorname{OSpec}_I A_n = \begin{cases} \{\lfloor\frac{k-1}{2}\rfloor,\ \ldots,\ n-1\} & \textit{if } k\textit{-gon has no mirror points} \\ \{\lfloor\frac{k-1}{2}\rfloor+1,\ \ldots,\ n-1\} & \textit{if } k\textit{-gon has mirror points} \end{cases} \tag{3.1.9}$$

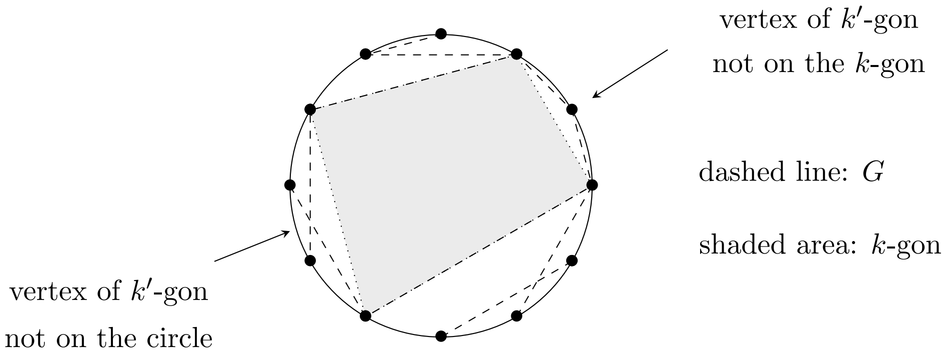
This fits in the Theorem 3.1.8, now let us prove the theorem.

Proof of Theorem 3.1.8.

When $k=n+1$, the statement in Theorem 3.1.8 is immediate from Lemma 3.1.4.

When $k \leqslant n$, according to Corollary 3.1.14, we only consider the case when $k > \lfloor\frac{n-1}{2}\rfloor+3$. Note that we have all desired generation times in the spectrum by Corollary 3.1.10, Corollary 3.1.12 and Remark 3.1.13. It suffices to show that no generator has generation time in $(\lfloor\frac{n-1}{2}\rfloor, k-2)$.

For a generator G, assume $g(G) \in (\lfloor\frac{n-1}{2}\rfloor, k-2)$. Then its corresponding chordal diagram must bound a closed k'-polygon (of some vertex possibly the intersection of a cross) which contains our k-gon (hence $k' \geqslant k$). Otherwise G contains all but one edges (not two by path-connectedness) of a k'-gon ($k' \geqslant k$) and the missing edge is an edge of the outer $n+1$-gon, then G has at least generation time $k-2$ by Lemma 3.1.11. Schematically, the diagram associated with G is shown below, where G is a collection of chords connecting all vertices which has no interior intersection with the k-gon - shaded area (might contain some edges of the k-gon).



Now we claim that $g(G) \leqslant \lfloor \frac{n-1}{2} \rfloor$, i.e. the level of any chord $\leqslant \lfloor \frac{n-1}{2} \rfloor$ or equivalently for any two vertices there is a path connecting them having generation time $\leqslant \lfloor \frac{n-1}{2} \rfloor$, which contradicts our assumption that $g(G) \in (\lfloor \frac{n-1}{2} \rfloor, k-2)$. We'll use first and second counting formulas to make an estimate.

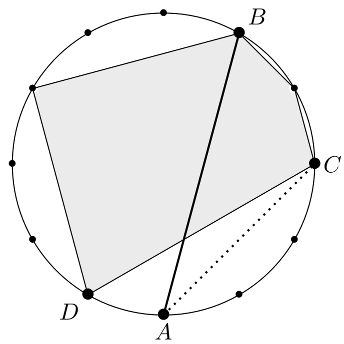
Notice this k'-gon in the diagram may have some vertices not on the circle (like shown above). We treat them separately. Let i be the number of the vertices of k'-gon which is on the circle and i_1 the number of the vertices of k'-gon which is not on the circle, i.e. the intersections of crosses.

Since k'-gon contains k-gon, we have $i \geqslant k > \lfloor \frac{n-1}{2} \rfloor + 3$ and outside this i-gon, we have $(n+1) - i$ points. If one area outside this i-gon has j points, then $j \leqslant (n+1) - i$ implies that $(n+1) - j \geqslant i > \lfloor \frac{n-1}{2} \rfloor + 3$, therefore $j < \lfloor \frac{n-1}{2} \rfloor$. Notice the j points in this area along with one edge of the i-gon construct a $(j+2)$-gon, we can regard it as the polygon inside the A_{j+1} circle. Hence to generate the chords inside this $(j+2)$-gon, according to Lemma 2.1.14 (3) on the A_{j+1} case, we need at most j triangles, which is less than $\lfloor \frac{n-1}{2} \rfloor$.

Lastly, it remains to consider a chord that intersects the interior of the i-gon. Assume the chord has endpoints A and B. We claim that $\mathrm{Lvl}(AB) \leqslant \lfloor \frac{n-1}{2} \rfloor$ and then the theorem follows. Let us consider the following cases.

- Assume both A and B are on the i-gon:

  First we use i_1 triangles to complete the sides of the i-gon. Then by Lemma 3.1.4 we need at most $\lfloor \frac{i-2}{2} \rfloor$ triangles to generate the diagonals inside i-gon, including the chord AB. Note each cross that contributes a vertex in i_1 contains 2 points on the circle outside i-gon. Therefore $2i_1 + i \leqslant n+1$ which implies $i_1 \leqslant \frac{n+1-i}{2}$. Therefore, $\mathrm{Lvl}(AB) \leqslant i_1 + \lfloor \frac{i-2}{2} \rfloor \leqslant \frac{n+1-i}{2} + \frac{i-2}{2} = \frac{n-1}{2}$, which implies that $\mathrm{Lvl}(AB) \leqslant \lfloor \frac{n-1}{2} \rfloor$ since $\mathrm{Lvl}(AB) \in \mathbb{Z}$.



There is a path connecting A, C in diagram of G

shaded area: i-gon

- Assume B is on the i-gon but A is in an area of j points outside i-gon without a cross from i_1 (So $j \leqslant n + 1 - i - 2i_1$):

  Denote the closest vertices of i-gon to A by C and D. Notice to be a generator, G is path-connected. So from A, there is a path of $(m + 1)$ $(m \leqslant j)$ points and m_1 = number of crosses on the path, to one of C and D (say C).

  Suppose clockwise D and B are l edges of i-gon away, and as a result counterclockwise C and B are $(i - 1 - l)$ edges of i-gon away. First, use i_1 triangles to complete the sides of the i-gon and m_1 triangles to make the above path a chain of $t = (m + 1 - 2m_1)$ vertices. Let us compare two different paths to generate AB.

Counterclockwise, we have a chain P_1 from A to C then to B of $t+(i-1-l)$ vertices. Therefore, by Lemma 2.1.19 (1), $g(P_1) = t + (i-1-l) - 2 = i + m - 2m_1 - l$.

Clockwise, we use the the path P_2 from A to C then to D and B clockwise. The "big Z" in $B - D - C - A$ will reduce our steps dramatically. After $t-1$ steps, we could generate a chord from A to $(t-1)th$ position above D. If we haven't reach B yet, then use $l - (t-1) = (l - t + 1))$ extra steps. In total, $\max\{m - 2m_1,\ l\}$ steps. So $g(P_2) = \max\{m - 2m_1,\ l\}$.

We summarize:

$$\mathrm{Lvl}(AB) \leqslant i_1 + m_1 + \min\{g(P_1),\ g(P_2)\} \tag{3.1.10}$$

where

$$\begin{cases} g(P_1) = i + m - 2m_1 - l \\ g(P_2) = \max\{m - 2m_1,\ l\} \end{cases} \tag{3.1.11}$$

subject to conditions:

$$\begin{cases} i \geqslant k > \lfloor \frac{n-1}{2} \rfloor + 3 \\ m \leqslant j \leqslant n + 1 - i - 2i_1 \\ 0 \leqslant l \leqslant i \end{cases} \tag{3.1.12}$$

Therefore,

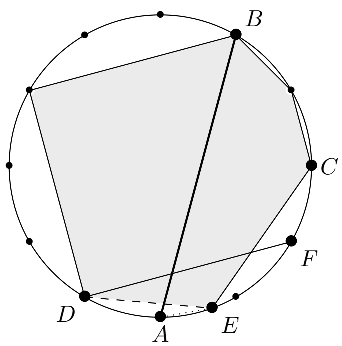
- if $0 \leqslant l \leqslant m - 2m_1$:

$$\begin{aligned}
\mathrm{Lvl}(AB) &\leqslant i_1 + m_1 + \min\{i + m - 2m_1 - 2 - l,\ m - 2m_1\} \\
&\leqslant i_1 + \min\{(m - m_1) + (i - 2 - l),\ (m - m_1)\} \\
&\leqslant i_1 + (m - m_1) \leqslant i_1 + m \\
&\leqslant i_1 + n + 1 - i - 2i_1 \leqslant n + 1 - i \\
&\leqslant n - \lfloor \frac{n-1}{2} \rfloor - 3 < \lfloor \frac{n-1}{2} \rfloor
\end{aligned} \tag{3.1.13}$$

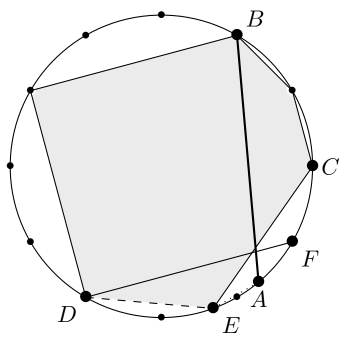
– if $l > m - 2m_1$:

$$\begin{aligned}
\mathrm{Lvl}(AB) &\leqslant i_1 + m_1 + \min\{i + m - 2m_1 - 2 - l,\ l\} \\
&\leqslant i_1 + m_1 + \frac{i + m - 2m_1 - 2}{2} \\
&= \frac{i + 2i_1 + m}{2} - 1 \\
&\leqslant \frac{n+1}{2} - 1 = \frac{n-1}{2}
\end{aligned} \tag{3.1.14}$$

Therefore, $\mathrm{Lvl}(AB) \leqslant \lfloor \frac{n-1}{2} \rfloor$ since $\mathrm{Lvl}(AB) \in \mathbb{Z}$.



or



shaded area: $(i+1)$-gon

shaded area: $(i+1)$-gon

- Assume B is on the i-gon but A is in an area of j points outside i-gon with a cross from i_1 (so $j \leqslant n + 1 - i - 2(i_1 - 1)$):

  Denote the vertices of such cross by C, D, E and F. If there is a chain from

A to C or D, then it is in the same case as above since now $m \leqslant j-2$. Without loss of generality, we can assume from A there is a path of $(m+1)$ $(m \leqslant j-2)$ points and $m_1 =$ number of crosses on the path, to E. First, use $(i_1 - 1)$ triangles to complete the sides of the i-gon except CD, 1 extra triangle to build DE and m_1 triangles to make the above path a chain of $t = (m + 1 - 2m_1)$ vertices. Now we have an $(i+1)$-gon as shown in above figure.

Suppose clockwise D and B are l edges away, and as a result counterclockwise E and B are $(i-l)$ edges away. Similarly we have two different paths P_1 and P_2 to generate AB with $g(P_1) = (i + m - 2m_1 - 1 - l)$ and $g(P_2) = \max\{m - 2m_1,\ l\}$.

We summarize:

$$\mathrm{Lvl}(AB) \leqslant i_1 + m_1 + min\{g(P_1),\ g(P_2)\} \tag{3.1.15}$$

where

$$\begin{cases} g(P_1) = i + m - 2m_1 - 1 - l \\ g(P_2) = \max\{m - 2m_1,\ l\} \end{cases} \tag{3.1.16}$$

subject to conditions:

$$\begin{cases} i \geqslant k > \lfloor \frac{n-1}{2} \rfloor + 3 \\ m \leqslant j - 2 \leqslant n + 1 - i - 2i_1 \\ 0 \leqslant l \leqslant i \end{cases} \tag{3.1.17}$$

Therefore,

– if $0 \leqslant l \leqslant m - 2m_1$:

$$\begin{aligned}\mathrm{Lvl}(AB) &\leqslant i_1 + m_1 + min\{i + m - 2m_1 - 1 - l,\ m - 2m_1\} \\ &\leqslant i_1 + min\{(m - m_1) + (i - 1 - l),\ (m - m_1)\} \\ &\leqslant i_1 + (m - m_1) \leqslant i_1 + m \\ &\leqslant i_1 + n + 1 - i - 2i_1 \leqslant n + 1 - i \leqslant \lfloor \frac{n-1}{2} \rfloor\end{aligned} \tag{3.1.18}$$

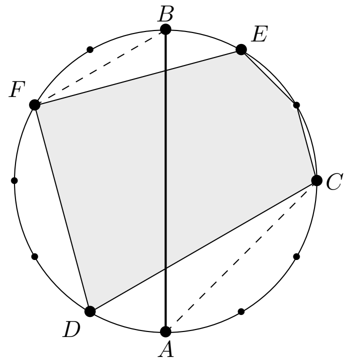
– if $l > m - 2m_1$ and $l \leqslant \frac{i+m-2m_1-2}{2}$:

$$\begin{aligned}\mathrm{Lvl}(AB) &\leqslant i_1 + m_1 + min\{i + m - 2m_1 - 1 - l,\ l\} \\ &\leqslant i_1 + m_1 + \frac{i + m - 2m_1 - 2}{2} \\ &= \frac{i + 2i_1 + m - 2}{2} \\ &\leqslant \frac{n-1}{2}\end{aligned} \tag{3.1.19}$$

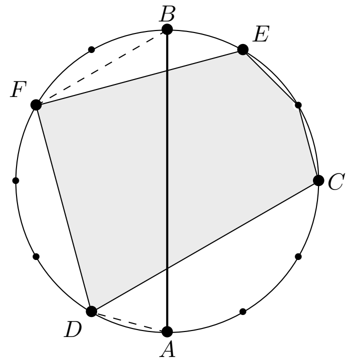
– if $l > m - 2m_1$ and $l \geqslant \frac{i+m-2m_1}{2}$:

$$\begin{aligned}\mathrm{Lvl}(AB) &\leqslant i_1 + m_1 + min\{i + m - 2m_1 - 1 - l,\ l\} \\ &\leqslant i_1 + m_1 + \frac{i + m - 2m_1 - 2}{2} \\ &= \frac{i + 2i_1 + m - 2}{2} \\ &\leqslant \frac{n-1}{2}\end{aligned} \tag{3.1.20}$$

Therefore, $\mathrm{Lvl}(AB) \leqslant \lfloor \frac{n-1}{2} \rfloor$ since $\mathrm{Lvl}(AB) \in \mathbb{Z}$.



or



- Assume A, B are in two disjoint areas of j and j' points respectively outside i-gon and the picture is as shown above (the case when any of such area contains a cross of i_1 is similar).

  Denote the closest vertices of i-gon to A by C and D and the closest vertices of i-gon to B by E and F. Then from A, there is a path of $(m+1)$ $(m \leqslant j)$ points and m_1 = number of crosses on the path, to one of C and D (say C) and from B, there is a path of $(r+1)$ $(r \leqslant j')$ points and r_1 = number of crosses on the path, to one of E and F (say F). For simplicity, we can assume $m \leqslant r$ and ignore those crosses since they can only reduce the level of AB as we have seen in the previous two cases. Namely the path connecting A, C is a chain of $(m+1)$ vertices and the path connecting B, F a chain of $(r+1)$ vertices.

  First, use i_1 triangles to complete the sides of the i-gon. Suppose clockwise D and F are l edges of i-gon away, and as a result counterclockwise C and E are $(i-2-l)$ edges of i-gon away.

  Counterclockwise, we have a chain P_1 from A to C, then to F on the i-gon, then going clockwise to B of $m+1+r+1+(i-2-l)$ vertices. Notice we have a "Big Z" $B-F-E-A$, by Lemma 2.1.19 (1), $g(P_1) = \max\{r,\ m+i-2-l\} = \max\{r,\ m+i-2-l\}$.

Clockwise, we use the path P_2 from A to C then to D and B clockwise. Notice we have a "Big Z" $B-D-C-A$, by Lemma 2.1.19 (1), $g(P_2) = \max\{m,\ l+r\} = \max\{m,\ l+r\} = l+r$.

We summarize:

$$\mathrm{Lvl}(AB) \leqslant i_1 + \min\{g(P_1),\ g(P_2)\} \tag{3.1.21}$$

where

$$\begin{cases} g(P_1) = \max\{r,\ m+i-2-l\} \\ g(P_2) = l+r \end{cases} \tag{3.1.22}$$

subject to conditions:

$$\begin{cases} i \geqslant k > \lfloor \frac{n-1}{2} \rfloor + 3 \\ m+r \leqslant j+j' \leqslant n+1-i-2i_1 \\ 0 \leqslant l \leqslant i-1 \\ m \leqslant r \end{cases} \tag{3.1.23}$$

Therefore,

- if $i-2-(r-m) \leqslant l \leqslant i-1$:

$$\begin{aligned} \mathrm{Lvl}(AB) &\leqslant i_1 + \min\{r,\ l+r\} \\ &\leqslant i_1 + r \\ &\leqslant n+1-i \\ &\leqslant n - \lfloor \frac{n-1}{2} \rfloor - 3 < \lfloor \frac{n-1}{2} \rfloor \end{aligned} \tag{3.1.24}$$

– if $\frac{i-(r-m)}{2} \leqslant l \leqslant i-3-(r-m)$:

$$\begin{aligned}
\mathrm{Lvl}(AB) &\leqslant i_1 + \min\{m+i-2-l,\ l+r\} \\
&\leqslant i_1 + m + i - 2 - l \\
&\leqslant \frac{i+2i_1+m+r}{2} - 2 \\
&< \frac{n-1}{2}
\end{aligned} \tag{3.1.25}$$

– if $0 \leqslant l \leqslant \frac{i-(r-m)}{2} - 1$:

$$\begin{aligned}
\mathrm{Lvl}(AB) &\leqslant i_1 + \min\{m+i-2-l,\ l+r\} \\
&\leqslant i_1 + l + r \\
&\leqslant \frac{i+2i_1+m+r}{2} - 1 \\
&\leqslant \frac{n-1}{2}
\end{aligned} \tag{3.1.26}$$

Therefore, $\mathrm{Lvl}(AB) \leqslant \lfloor\frac{n-1}{2}\rfloor$ since $\mathrm{Lvl}(AB) \in \mathbb{Z}$.

By the same argument, $\mathrm{Lvl}(AB) \leqslant \lfloor\frac{n-1}{2}\rfloor$ for the second picture as well.

$\square$

An immediate result of Theorem 3.1.8 is:

Remark 3.1.15. With one degeneration, the spectrum contains at most one gap.

Therefore, to look for more gaps we need to impose more polygons.

3.2 Two-Polygon Case

Now denote by I and J, the set of vertices index (subsets of $\{0, 1, ..., n\}$) of two polygons. Similarly we could define the set of generation times over all qualified generators by $\mathrm{OSpec}_{I,J}$. Observe the following lemma:

Lemma 3.2.1. If the interiors of I and J intersect, then $\mathrm{OSpec}_{I,J} = \mathrm{OSpec}_{I \cup J}$.

Proof. Imposing two polygons or just one polygon which is the convexification of these polygons, we will have the same condition of choice of indecomposables thus same set of qualified generators. □

The spectrum in above case could be given by Theorem 3.1.8. Therefore we only need to consider the case while two polygons either disjoint, sharing only a vertex or sharing only an edge. Without loss of generality, assume that $|I| = i \geqslant j = |J|$.

If we impose two polygons, we could have more gaps, for instance:

Example 3.2.2 (A_{25}: 2 gaps)**.** Impose a 16-gon and a 12-gon i.e. mark $I = \{0, 1, ..., 15\}$ and $J = \{0, 15, 16, ..., 25\}$. Then the 26-gon is divided into a 16-gon and a 12-gon sharing a common edge - the chord $X_{0,15}$.

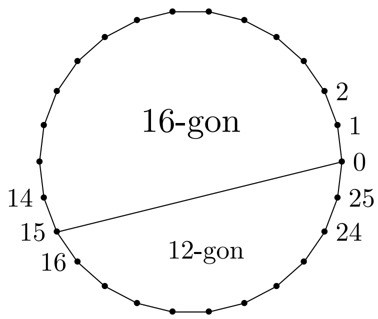


Figure 3.2: A_{25}: 2 gaps

Note by Theorem 3.1.8, if we impose only I. Then

$$\operatorname{OSpec}_I A_{25} = \{7, 8, 9, 10, 11, 12\} \cup \{14, 15, \ldots, 24\}, \tag{3.2.27}$$

which has a gap at 13. Now imposing J, the Rouquier dimension is lifted to 8, namely G = sum of all allowed indecomposables has generation time 8. For any other generator, the diagram need to be path connected and pass through all vertices. If we take any indecomposable away from G, we have 3 choices. Taking $X_{0,15}$, the rest has generation time 12 by Lemma 3.1.4; taking any side of I other than $X_{0,15}$, the rest has generation time at least 14 by Lemma 3.1.11; taking any side of J other than $X_{0,15}$, the rest has generation time at least 10 by Lemma 3.1.11. Therefore, 9 is a gap. Indeed one can calculate that

$$\operatorname{OSpec}_{I,J} A_{25} = \{8\} \cup \{10, 11, 12\} \cup \{14, 15, \ldots, 24\}. \tag{3.2.28}$$

Therefore we have two gaps at 9 and 13.

Furthermore, it is quite likely that putting more elaborated combinations of polygons will lead to

Conjecture 3.2.3. *In a similar way one can obtain any number of gaps.*

In the next chapter we will try to generalize calculations done here to indicate a general theory. The obstructing polygons correspond to localization functors - become a base locus of a noncommutative linear system. We will define a categorical multiplier ideal sheaf and based on the examples done in chapter 2 and chapter 3 we outline a conjectural connection with the Orlov spectra of a category.

Chapter 4

Hodge Theoretic Analogy

The calculations we have done in previous chapters indicate a generalized theory of linear systems and base loci in categorical sense. The obstructing polygons correspond to localization functors and in this way such generation-restricted A_n categories correspond to localization categories:

$$\begin{aligned} I_1\text{-restricted } A_n \quad &\longleftrightarrow \quad A_n/_{\Gamma_1}; \\ (I_1, I_2)\text{-restricted } A_n \quad &\longleftrightarrow \quad A_n/_{\Gamma_1/_{\Gamma_2}}; \\ &\cdots \end{aligned}$$

The structures of such categories can be realized by certain convex bodies, studied by Lazarsfeld and Mustaţă [LM09] motivated by earlier idea of Okounkov [Oko97, Oko96]. Many asymptotic invariants of the linear system are encoded in this so called "Okounkov body".

4.1 Categorical Okounkov Bodies

Let us recall first the construction of classical Okounkov bodies in [LM09]. The construction is defined on an admissible flag of a d-dimensional projective variety X, namely a flag of irreducible subvarieties:

$$X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_d = pt, \tag{4.1.1}$$

where each Y_i is non-singular at Y_d with $\operatorname{codim}_X Y_i = i$. Then for any big divisor D on X, the flag determines a valuation map ν, which maps any nonzero section s on D to a d-tuple of non-negative integers $(\nu_1, \nu_2, \ldots, \nu_d)$. Start by setting $\nu_1 = ord_{Y_1}(s)$. Then we set $\nu_2 = ord_{Y_2}(s_1)$, where s_1, on Y_1, is a restriction of a section non-vanishing on Y_1 determined by s. So on and so forth, we can obtain all the remaining ν_i's. Then the **Okounkov body** of D, denoted $\Delta(D)$, is the closed convex hull of $\bigcup\limits_{m \geqslant 1} \frac{1}{m}\nu(mD)$.

Now let translate above notions into their analogues in categorical language. For instance, in our example of A_n category. The flag of subvarieties becomes the flag of subcategories

$$A_n \supset A_n/_{\Gamma_1} \supset A_n/_{\Gamma_1/_{\Gamma_2}} \supset \cdots \tag{4.1.2}$$

with localization functors:

$$\varphi: \quad A_n \xrightarrow{\varphi_1} A_n/_{\Gamma_1} \xrightarrow{\varphi_2} A_n/_{\Gamma_1/_{\Gamma_2}} \longrightarrow \cdots . \tag{4.1.3}$$

The sections become natural transformations between these localization functors and restriction functors, and valuations ν_i are the maximal numbers of liftings of the natural transformations, which measure how far along one can lift these natural transformations. For example, in the figure below, we have $\nu_1 = 2, \nu_2 = 3,$

$\nu_3 = \ldots.$

$$\begin{tikzcd}
\cdots \arrow[r, "\varphi_1"] & A_n/_{\Gamma_1} \arrow[r, "\varphi_1"] \arrow[drr, "\nu_1"'] & A_n/_{\Gamma_1} \arrow[r, "\varphi_1"] \arrow[dr, dotted] & \mathrm{Id}(A_n/_{\Gamma_1}) \arrow[d] \\
 & & & A_n/_{\Gamma_1}
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, "\varphi_2"] & A_n/_{\Gamma_1/_{\Gamma_2}} \arrow[r, "\varphi_2"] \arrow[drrr, "\nu_2"'] & A_n/_{\Gamma_1/_{\Gamma_2}} \arrow[r, "\varphi_2"] \arrow[drr, dotted] & A_n/_{\Gamma_1/_{\Gamma_2}} \arrow[r, "\varphi_2"] \arrow[dr, dotted] & \mathrm{Id}(A_n/_{\Gamma_1/_{\Gamma_2}}) \arrow[d] \\
 & & & & A_n/_{\Gamma_1/_{\Gamma_2}}
\end{tikzcd}$$

$\ldots\ldots$

Figure 4.1: $\nu_1 = 2$, $\nu_2 = 3$, $\ldots$

Now we denote by φ^m the localization functors:

$$\varphi^m : \quad A_n \overset{\varphi_1^m}{\longrightarrow} A_n/_{\Gamma_1} \overset{\varphi_2^m}{\longrightarrow} A_n/_{\Gamma_1/_{\Gamma_2}} \longrightarrow \cdots . \tag{4.1.4}$$

Then

$$\nu(\varphi^m) = (\nu_1, \nu_2, \ldots, \nu_l), \tag{4.1.5}$$

where l is the number of localizations and ν_i is the maximal numbers of liftings of the natural transformation between φ_i^m and restriction functor, similar to what is shown in above figure. Now we define:

Definition 4.1.1 (Categorical Okounkov Body)**.** The **categorical Okounkov body**, denoted $\delta(\varphi)$, is defined to be the closed convex hull of $\bigcup\limits_{m\geqslant 1} \frac{1}{m}\nu(\varphi^m)$.

Remark 4.1.2. Classical Okounkov body $\Delta(D)$ measures how Picard groups in the flag of subvarieties fit together interacting with D - see e.g. [LM09]. Therefore, similarly $\delta(D)$ measures how restriction functor and localization functor interact asymptotically in respect of the flag of subcategories.

In general, the definition of categorical Okounkov body can be extended to a flag of subcategories $R_1,\ R_2,\ \ldots,\ R_l$ of a category with spherical and restriction functors and natural transformations between them. Examples of such flags of categories mainly come from derived categories of flags of subvarieties. For example:

Example 4.1.3. The cube of categories below is given by quadrics Q_1, Q_2 and Q_3 in $\mathbb{P}^3$ and their intersections. Derived categories of Q_1, $Q_1 \cap Q_2$ and $Q_1 \cap Q_2 \cap Q_3$ define a flag of categories R_1, R_2, R_3, as shown below.

$$
\begin{tikzcd}
& \mathrm{D^b}(X) \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & \mathrm{D^b}(Q_2) \arrow[dl] \arrow[dd] \\
\mathrm{D^b}(Q_1) \arrow[rr] \arrow[dd] & & \mathrm{D^b}(Q_1 \cap Q_2) \arrow[dd] & \\
& \mathrm{D^b}(Q_3) \arrow[rr, dotted] \arrow[dl, dotted] & & \mathrm{D^b}(Q_2 \cap Q_3)) \arrow[dl] \\
\mathrm{D^b}(Q_1 \cap Q_3) \arrow[rr] & & \mathrm{D^b}(Q_1 \cap Q_2 \cap Q_3) &
\end{tikzcd}
$$

For a flag of subcategories $R_1,\ R_2,\ \ldots,\ R_l$ of a category we denote by S the restriction functor and t the spherical functor of a twist by a divisor. Then each ν_i is the maximal number of liftings of the natural transformation. For example, in the figure below, we have $\nu_1 = 2, \nu_2 = 3,\ \nu_3 = \ldots.$ Then we can define the categorical Okounkov body associated with this flag following Definition 4.1.1.

$$
\begin{tikzcd}
\cdots \arrow[r, "S"] & \{-3R_1\} \arrow[r, "S"] & \{-2R_1\} \arrow[r, "S"] \arrow[drr, "\nu_1 = k_1 = 2"'] & \{-R_1\} \arrow[r, "S"] \arrow[dr, dotted] & Id \arrow[d, "t"] \\
 & & & & \{-D\}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots \arrow[r, "S"] & \{-3R_2\} \arrow[r, "S"] \arrow[drrr, "\nu_2 = k_2 = 3"'] & \{-2R_2\} \arrow[r, "S"] \arrow[drr, dotted] & \{-R_2\} \arrow[r, "S"] \arrow[dr, dotted] & Id_{R_1} \arrow[d, "t"] \\
 & & & & \{-k_1R_1 - D\}
\end{tikzcd}
$$

$$\cdots\cdots$$

Remark 4.1.4. Definition 4.1.1 is a categorification of the usual definition of Okounkov body. Namely the classical Okounkov body of a flag of subvarieties and the categorical Okounkov body of a flag of derived subcategories corresponding to these subvarieties are the same. Classically k_i is the multiplicity with which D passes through R_i.

Now if we consider the categorical Okounkov bodies associated with the localization categories of A_n, the calculations in Example 3.2.2 in previous chapter indicate the following conjecture:

Conjecture 4.1.5. *Orlov spectrum of $A_n/_{\Gamma_1/\Gamma_2}$ has a second gap if and only if the associated Okounkov body $\delta(\varphi)$ is non-polyhedral.*

We are going to introduce the notion of multiplier ideal sheaf and propose a more general statement in the coming section. As we are going to see, the categorical Okounkov body will play an important role in classifying base loci of the category.

4.2 Categorical Linear Systems and Base Loci

In this section we are going to define the notions of categorical linear system and categorical base locus. These two notions are closely connected to the categorical invariants we've been studying in this thesis - Orlov spectra and the gaps.

Assume that $\mathcal{C}$ is a category with k functors t_1, t_2, ..., t_k and the identity functor Id.

Definition 4.2.1. All natural transformations $\mathfrak{n}_i : \mathrm{Id} \to t_i$ form a noncommutative linear system of divisors $Cone(\mathfrak{n}_i)$. We call this linear system a **categorical linear system**.

Definition 4.2.2. The orthogonal complement of all t_i in $\mathcal{C}$ is called a **categorical base locus** of the linear system generalized by (t_1, t_2, ..., t_k).

Example 4.2.3. For a Landau-Ginzburg model (see e.g. [AKO08])

$$w : Y \to \mathbb{P}^1, \tag{4.2.6}$$

where Y is a fibration over $\mathbb{P}^1$ with compact fibers, we consider the Fukaya-Seidel category FS associated with w. Let F be a functor, Id the identity functor and $\mathfrak{n}$ a natural transformation $\mathfrak{n} : \mathrm{Id} \to F$. Then all natural transformations $\mathfrak{n} : \mathrm{Id} \to F$ form a noncommutative linear system. The common subset of objects (common subcategory) of two or more natural transformations $\mathfrak{n}_i : \mathrm{Id} \to F$ is a base locus.

In our A_n example the obstructing polygons (marking vertices) correspond to localization functors. So all natural transformations from identity functor to localization functor become a noncommutative linear system. The elaborated combinations of polygons become a noncommutative base locus. So as an analogy, for the categorical base loci for Fukaya-Seidel categories we can think of natural transformations of rotation functors and identity functor as paths around the fiber at infinity. Intersections of these paths are the categorical base loci i.e. marked points (localized thimbles) on the fiber at infinity.

Now we are ready to define categorical multiplier ideal sheaf following the procedure we used in defining categorical Okounkov body. We recall first the classical definition. Let X be a smooth variety with an effective Cartier divisor D on it. Then for a log resolution $\mu : Y \to X$ of D, the multiplier ideal sheaf, denoted $K_\alpha(D)$ or $J(X, \alpha D)$ is defined to be

$$K_\alpha(D) = J(X, \alpha D) = \mu_*(\mathcal{O}_Y(K_{Y/X} - \lfloor \alpha \mu^* D \rfloor)). \tag{4.2.7}$$

Remark 4.2.4. $\mu_*(\mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X$.

We categorify this definition:

Definition 4.2.5. Let $\mathcal{C}$ be a category with functor F. Then a sequence of **categorical multiplier ideal sheaves** is a sequence of sheaves of categories

$$J(\lambda_1, \ldots, \lambda_k) : \; J(\mathcal{C}, \lambda_k F) \subset \cdots \subset J(\mathcal{C}, \lambda_1 F) \tag{4.2.8}$$

defined by a sequence of functors $\lambda_i F$, where each λ_i, called the **jump number**, is the number of possible liftings of the point-like object p.

We denote the image of the functor $\lambda_i F$ by K_{p,λ_i}. The analogy of categorical multiplier ideal sheaf with the classical one is demonstrated in the table below. Classically n_{x,λ_i} is the dimension of the eigenspace with monodromy $e^{i.\lambda_i}$ for the Mixed Hodge Structure associated with the function f defining D. K_{x,λ_i} measure singularities of the pair (X, D). Categorically n_{p,λ_i} is the dimension of the eigenspace with monodromy $e^{i.\lambda_i}$ for the noncommutative Mixed Hodge Structure

([KKP]) associated with the category K_{p,λ_i} and the functor F. The category K_{p,λ_i} measures the pair (LG model, categorical base loci).

Table 4.1: Categorical Multiplier Ideal Sheaf

<table>
<tr><th>CLASSICAL</th><th>CATEGORICAL</th></tr>
<tr><td>$J(X, \lambda_k D) \subset \cdots \subset J(X, \lambda_1 D) \subset \mathcal{O}_X$</td><td>$J(C, \lambda_k F) \subset \cdots \subset J(C, \lambda_1 F)$</td></tr>
<tr><td>

$$\begin{tikzcd} \cdots & 2m_x \arrow[l] & 1m_x \arrow[l] & \mathrm{Id} \arrow[l] \\ & & & \mathcal{O}_x(+D_{\lambda_i}) \arrow[ull] \arrow[ul] \arrow[u, "\otimes - D_{\lambda_i}"'] \end{tikzcd}$$

$K_{x,\lambda_i} = K_{\lambda_i}(D), x$ point in D</td><td>

$$\begin{tikzcd} \cdots & 2m_p \arrow[l] & 1m_p \arrow[l] & \mathrm{Id} \arrow[l] \\ & & & F_{\lambda_i} \arrow[ull] \arrow[ul] \arrow[u] \end{tikzcd}$$

K_{p,λ_i}, p point-like object</td></tr>
<tr><td>$\dim K_{x,\lambda_i} = n_{x,\lambda_i}$</td><td>$\dim K_{p,\lambda_i} = n_{p,\lambda_i}$</td></tr>
</table>

We can generalize this idea to define categorical multiplier ideal sheaf for several functors (classically several divisors). By restricting to flags of subcategories we can obtain the Okounkov body associated with it.

In the case when the subcategories $J(C, \lambda_k F) \subset \cdots \subset J(C, \lambda_1 F)$ are triangulated. By taking triangles we move up within these subcategories and in this way we can build the Orlov spectrum of $\mathcal{C}$. This point of view is verified by our A_n example. We have a sheaf of generators (a sheaf of localized categories) for which the jump number determines how many sides do we take from the whole polygon in order to form the forbidden part. The calculations in previous chapter are recorded in the table below.

Table 4.2

Spectra	Sheaves and Jump numbers	
$\{\lfloor\frac{n-1}{2}\rfloor\}\cup\{n-1\}$	$\lambda_n=\frac{n}{n+1}$	generators in Lemma 3.1.4
$\{(\lfloor\frac{k-1}{2}\rfloor),\ ...,\ \lfloor\frac{n-1}{2}\rfloor\}$ $\cup\{k-2,\ ...,\ n-1\}$	$\lambda_k=\frac{k}{n+1}$	generators in Corollary 3.1.10 3.1.12
$\{0,\ 1,\ \dots,\ n-1\}$	$\lambda_1=\frac{1}{n+1}$	generators in Theorem 2.1.16

Theorem 3.1.8 now can be interpreted as the following theorem.

Theorem 4.2.6. *The multiplier ideal sheaf for the category A_n and the localization functor - restricting a k-gon has jump numbers $\frac{k}{n+1}$. The sequence of multiplier ideal sheaves $J(\lambda_1,\dots,\lambda_k)$ determines the Orlov spectrum of generation-restricted A_n.*

Proof. It is a direct consequence of the definition of $J(\lambda_1,\dots,\lambda_k)$, Theorem 3.1.8 and Table 4.2. □

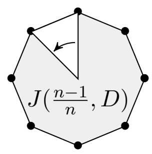
In this case the categorical multiplier ideal sheaf is a sequence of localizations $J(C,\lambda_k F)\subset\cdots\subset J(C,\lambda_1 F)$. Marking a polygon corresponds to localizing by subcategory. The localization by the biggest polygon produces the first nontrivial category $J(C,\lambda_k F)$ and by the smallest $J(C,\lambda_1 F)$. In the table below we represent the multiplier ideal sheaf in this case as rotation by angles of λ_j of the localization functor F.

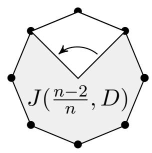
Table 4.3

<table>
<tr><th>The Multiplier Ideal Sheaf</th><th>Functor Localization by A_n</th></tr>
<tr><td>

$$\begin{tikzcd} \cdots \arrow[r, "t"] & (-2p) \arrow[r, "t"] \arrow[drr] & (-p) \arrow[r, "t"] \arrow[dr, dotted] & \mathrm{Id} \arrow[d, "\lambda_j s"] \\ & & & -\lambda_j F \end{tikzcd}$$

</td><td>

A_n  $j = n - 1$

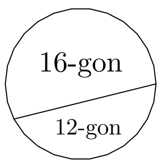
 $j = n - 2$

</td></tr>
</table>

Now we try to connect the non-polyhedralness of restricted categorical Okounkov bodies with the appearance of gaps in the Orlov spectra of the category. Example 3.2.2 suggests the table below:

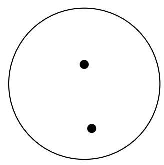
Table 4.4: Analogies

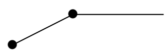
<table>
<tr><td>



Example 3.2.2
2 gaps

</td><td>

 Multiplicities
Structure on
Cat. Base Loci



restricted Okounkov body is non-polyhedral

</td></tr>
</table>

In general we conjecture:

Conjecture 4.2.7. *Let $\mathcal{C}$ be a triangulated category (e.g. Fukaya-Seidel category) and F is a functor on it.*

1. *The categorical base locus is the subcategory defined by the vanishing of restricted categorical Okounkov bodies for F and a flag of categories R_1, $\ldots$, R_l.*

2. *The complexity of the categorical base locus of the functor F is measured by the jump numbers of the categorical multiplier ideal sheaf. These jump numbers determine the filtration of the mixed noncommutative Hodge structure associated with categorical base locus of the functor F.*

3. *The categorical multiplier ideal sheaf of the functor F determines the gaps of Orlov spectrum of the category $\mathcal{C}$.*

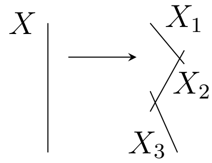
If above connection of the Orlov spectra with categorical multiplier ideal sheaves is established many of the standard properties of the Mixed Hodge structures will hold - functoriality, strictness, hyperplane sections. We record these generic expectations in the table below. (This is an example of three-dimensional Fano manifold X.)

Similar as in A_n example, here degenerations (adding markings) corresponding to the canonical multiplier ideal sheaf using the following formula.

$$(\operatorname{OSpec} X_1 + \operatorname{OSpec} X_2 + \operatorname{OSpec} X_3)_m - R \tag{4.2.9}$$

Here $\operatorname{OSpec} X_i$'s are Orlov spectra of the big polytope and the boundary of the small marked polytope, and R is the number coming from repeating simplexes. $\operatorname{OSpec} X_1 + \operatorname{OSpec} X_2 + \operatorname{OSpec} X_3$ denotes the Minkowski sum and $(\operatorname{OSpec} X_1 + \operatorname{OSpec} X_2 + \operatorname{OSpec} X_3)_m$ denotes Minkowski sum with the contributions from the monodromy around the fiber at infinity, which determines the gap. $< E_1, \ldots, E_n, \mathcal{A} >$ is a semi-orthogonal decomposition of X with phantom $\mathcal{A}$ - the subcategory with trivial K-theory.

Table 4.5: Generic Hodge Theoretic Properties of Orlov Spectra

Functoriality		$\operatorname{OSpec} X \supset$ $(\operatorname{OSpec} X_1 + \operatorname{OSpec} X_2 + \operatorname{OSpec} X_3)_m - R$ Minkowski sum depending on monodromy; repetition
Hyperplane Section	$X,\, X_H = X \cap H$	$\mathrm{GAP}(\mathrm{D^b}(X_H))$ $\geqslant \mathrm{GAP}(\mathrm{D^b}(X))$
Strictness	$< E_1, \ldots, E_n, \mathcal{A} >$ generic phantom	$\operatorname{OSpec}(E_1, \ldots, E_n, \mathcal{A})$ $= \operatorname{OSpec}(E_1, \ldots, E_n)$

If above mentioned expectations are established they can become a powerful computational tool. Conjecturally the gap in the Orlov spectra generically becomes a number totally computable on the B side of Homological Mirror Symmetry (HMS) and it is equal to the dimension of some Grassmannians in the Mixed Hodge Structure of the degeneration of X.

This is rather speculative due to lack of fully developed examples. I expect to compute more examples in order to provide more credibility to this approach in future work.

Appendices

Appendix A

Triangulated Category

A triangulated category is an additive category equipped with a "translation functor" and a collection of "distinguished triangles". The distinguished triangles in a triangulated category play a similar role to that of the short exact sequences in abelian categories. A central class of examples is the derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$, which was first introduced by Jean-Louis Verdier[Ver96] in his Ph.D. thesis, under the supervision of Grothendieck. Motivated by some special "triangles" structure on a derived category, he wrote down axioms for the basic properties of these triangles and defined the notion of a triangulated category.

For precise definition, we have the following notions of translation functor and triangles:

Definition A.1. A **translation functor** on an additive category $\mathcal{D}$ is an additive automorphism

$$T : \mathcal{D} \to \mathcal{D}.$$

Denote $X[n] = T^n(X)$ and $f[n] = T^n(f)$ for the morphism $f : X \to Y$.

Definition A.2.

- A **triangle** (X, Y, Z, u, v, w) is a six tuple of objects X, Y, and Z of $\mathcal{D}$ and morphisms $u : X \to Y, v : Y \to Z$ and $w : Z \to X[1]$, which can be written in the form:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \tag{A.1}$$

or

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \cdot \tag{A.2}$$

for short.

- A **morphism of triangles** is a commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd} \tag{A.3}$$

Definition A.3. A **triangulated category** $\mathcal{T}$ is an abelian category $\mathcal{D}$ equipped with a translation functor and a collection of triangles, called **distinguished triangles**, satisfying the following axioms:

TR 0: Any triangle isomorphic to a distinguished triangle is distinguished.

TR 1: For any object X, the trivial triangle

$$X \xrightarrow{\mathrm{id}} X \to 0 \to \cdot \tag{A.4}$$

is distinguished.

TR 2: Any morphism $u : X \to Y$ can be completed to a distinguished triangle

$$X \xrightarrow{u} Y \to Z \to \cdot \tag{A.5}$$

for some object Z, which is called a **mapping cone** of u.

TR 3: Rotations of distinguished triangles are distinguished. i.e.

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \tag{A.6}$$

is distinguished if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1] \tag{A.7}$$

and

$$Z[-1] \xrightarrow{-w[-1]} X \xrightarrow{u} Y \xrightarrow{v} Z \tag{A.8}$$

are distinguished.

TR 4: Any map between two morphisms can be extended to a morphism of triangles between their mapping cones. i.e. there exists some map h (not necessarily unique) which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd} \tag{A.9}$$

TR 5: For any morphisms $u : X \to Y$ and $v : Y \to Z$ and their composition $vu : X \to Z$. The three distinguished triangles formed by these three morphisms can be made into the vertices of a distinguished triangle so that everything commutes.

Formally, given distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{k} \cdot \tag{A.10}$$

$$Y \xrightarrow{v} Z \xrightarrow{l} X' \xrightarrow{i} \cdot \tag{A.11}$$

$$X \xrightarrow{vu} Z \xrightarrow{m} Y' \xrightarrow{n} \cdot \tag{A.12}$$

there exists a distinguished triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{h} \cdot \tag{A.13}$$

such that every face of the following octahedron commutes:

$$\begin{tikzcd}
 & & Y' \arrow[dr, dashed, "g"] & & \\
 & & & X' \arrow[dlll, dotted, "j{[1]} \circ i", "h"'] \arrow[dddl, dotted, "i"] & \\
Z' \arrow[uurr, dashed, "f"] \arrow[dr, "k"] & & & & Z \arrow[uull, "m"'] \arrow[ul, dotted, "l"'] \\
 & X \arrow[uuur, dash, "n"] \arrow[urrr, "v \circ u"] \arrow[dr, "u"] & & & \\
 & & Y \arrow[uull, "j"] \arrow[uurr, "v"'] & &
\end{tikzcd} \tag{A.14}$$

Figure A.1: Octahedral Axiom

Therefore, this axiom is also called the **Octahedral Axiom**.

Example A.4. Let $\mathcal{A}$ be an abelian category. The derived categories $D(\mathcal{A})$ are triangulated.

For $n \in \mathbb{Z}$ and a complex X, the complex $X[n]$ is defined by

$$X[n]^i = X^{n+i}, \tag{A.15}$$

with differential

$$d_{X[n]} = (-1)^n d_X. \tag{A.16}$$

A distinguished triangle in $D(\mathcal{A})$ is a triangle isomorphic to the triangle

$$X \to Y \to Cone(f) \to X[1] \tag{A.17}$$

for some map of complexes $f : X \to Y$. In particular, any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ can be completed to a distinguished triangle $X \to Y \to Z \to X[1]$ in $D(\mathcal{A})$.

Appendix B

Singularity Category

The singularity category originates from the local theory of singularities as a matrix factorization introduced in the paper of Eisenbud [Eis80], where he described MCM modules over local rings using this construction. Later in [Orl04], Orlov developed the notion of "triangulated category of singularities". He associated with a commutative graded Noetherian ring R the graded singularity category $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(R)$, which is defined as the Verdier quotient of bounded derived category of graded modules $\mathrm{D}^{\mathrm{b}}(\operatorname{gr} R)$ by its full triangulated category of perfect complexes. It is a graded analogue of the singularity category $\mathrm{D}_{\mathrm{sg}}(S)$, which reflects many properties of the singularities of $\operatorname{Spec} R$.

Let us recall the formal definitions, by looking at the ungraded case first.

Definition B.1.

If S is a commutative Noetherian k-algebra.

- The **singularity category** of S, denoted $\mathrm{D}_{\mathrm{sg}}(S)$, is the Verdier quotient of the bounded derived category of finitely generated modules $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} S)$ by the subcategory consisting of all bounded complexes of finitely generated projective modules $\mathrm{D}^{\mathrm{b}}(\operatorname{proj} S)$.

Assume further that S is local with maximal ideal $\mathfrak{m}_S$, then

- $(S, \mathfrak{m}_S)$ is an **isolated singularity** if for any prime ideal $\mathfrak{p} \neq \mathfrak{m}_S$, $S_{\mathfrak{p}}$ is a regular ring.
- $(S, \mathfrak{m}_S)$ is a **hypersurface singularity** if S is isomorphic to $R/(w)$, where $(R, \mathfrak{m}_R)$ is a regular local Noetherian k-algebra and $w \in \mathfrak{m}_R$.

In the case of isolated hypersurface singularity, to study the category $\mathrm{D}_{\mathrm{sg}}(S)$, it is useful to consider the following two constructions. Recall that a S-module M is called a **maximal Cohen-Macaulay** (MCM for short) module, if depth M $=\dim S$.

Remark B.2. M is an MCM module if and only if $\mathrm{Ext}^i_R(M, S) = 0$, for $i \neq 0$.

The first construction is the category $\underline{\mathrm{MCM}}(S)$ consisting of the same objects as $\mathrm{MCM}(S)$, the full subcategory of $\mathrm{mod}\, S$ of MCM modules and

$$\mathrm{Hom}_{\underline{\mathrm{MCM}}(S)}(M, N) = \mathrm{Hom}_S(M, N)/\sim \tag{B.1}$$

where $f \sim g$ if there exists maps $p : M \to P$ and $q : P \to N$ with $f - g = qp$ and P projective, i.e. $f - g$ factors through a projective module.

The second construction is the homotopy category of matrix factorization of w, $\mathrm{HMF}(w)$. The objects are **matrix factorizations**, which are sequences of R-modules,

$$P_0 \overset{A}{\to} P_1 \overset{B}{\to} P_0, \tag{B.2}$$

where P_i are finitely generated projective R-modules, $AB = w\,\mathrm{id}_{P_1}$, and $BA = w\,\mathrm{id}_{P_0}$. Write P for a matrix factorization (P_0, P_1, A, B) and A_P B_P for the maps in the matrix factorization. A morphism between two matrix factorizations, P and Q, consists of R-module maps $f_0 : P_0 \to Q_0$ and $f_1 : P_1 \to Q_1$ making the following diagram commute:

$$\begin{tikzcd} P_0 \arrow[r, "A_P"] \arrow[d, "f_0"] & P_1 \arrow[r, "B_P"] \arrow[d, "f_1"] & P_0 \arrow[d, "f_0"] \\ Q_0 \arrow[r, "A_Q"] & Q_1 \arrow[r, "B_Q"] & Q_0 \end{tikzcd} \tag{B.3}$$

A homotopy between two morphisms $f, g : P \to Q$ is a pair of maps $h_0 : P_0 \to Q_1$ and $h_1 : P_1 \to Q_0$ so that $f_0 - g_0 = B_Q h_0 + h_1 A_P$ and $f_1 - g_1 = A_Q h_1 + h_0 B_P$. Morphisms in $\mathrm{HMF}(w)$ are homotopy classes of morphisms between P and Q.

Both of above categories are naturally triangulated. We have the following result from [Buc87] or [Orl04]:

Theorem B.3. *For an isolated hypersurface singularity S, the following categories are equivalent as triangulated categories:*

$$\mathrm{D}_{\mathrm{sg}}(S) \simeq \underline{\mathrm{MCM}}(S) \simeq \mathrm{HMF}(w). \tag{B.4}$$

In particular, the triangulated structure of $\underline{\mathrm{MCM}}(S)$ is described in [Buc87] as follows. Such structure on a additive category is determined by its distinguished triangles. To describe these, let $f : M \to N$ be any S-linear map of MCM S-modules. Choose an embedding $i : M \to Q$ of M into a finitely generated projective S-module such that its cokernel is still MCM. Then define a mapping-cone $Cone(f)$ of f as the push-out of f and i, so that there is a commutative diagram of short exact sequences of S-modules:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "f"] & Q \arrow[r, "p"] \arrow[d] & \mathrm{coKer}(i) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r, "i'"] & Cone(f) \arrow[r, "p'"] & \mathrm{coKer}(i) \arrow[r] & 0 \end{tikzcd} \tag{B.5}$$

Here $\text{coKer}(i)$ represents $T(M)$, the translate of M, and $\Omega_S(\text{coKer}(i))$ is represented by M. We call

$$M \xrightarrow{f} N \xrightarrow{i'} Cone(f) \xrightarrow{-p'} TM = \text{coKer}(i) \tag{B.6}$$

a **typical triangle**, and its image in $\underline{\text{MCM}}(S)$ yields exactly all distinguished triangles in $\underline{\text{MCM}}(S)$.

In this thesis, we consider only simple hypersurface singularities, in which case we have only finitely many indecomposable objects, due to the following theorem of characterization of singularities:

Theorem B.4. *[Knö87] [BGS87] An isolated hypersurface singularity is simple if and only if it has finite CM-representation type. i.e. there are only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules.*

Recall classification of simple hypersurface singularities up to isomorphisms is done by Arnold [Arn81]:

Theorem B.5. *Let $S = k[[x_0, ..., x_d]]$ where k is an algebraically closed field of characteristic zero. Then $R = S/(f + x_2^2 + ... + x_d^2)$ is a* ***simple hypersurface singularity*** *if and only if f is equal to one of the following polynomials after a suitable change of variables:*

$$A_n: \quad x_0^{n+1} + x_1^2 \quad (n \geq 1) \tag{B.7}$$

$$D_n: \quad x_0^{n-1} + x_0 x_1^2 \quad (n \geq 4) \tag{B.8}$$

$$E_6: \quad x_0^4 + x_1^3 \tag{B.9}$$

$$E_7: \quad x_0^3 x_1 + x_1^3 \tag{B.10}$$

$$E_8: \quad x_0^5 + x_1^3 \tag{B.11}$$

Note that, by Knörrer periodicity [Knö87], any two categories in the following forms are equivalent:

$$\underline{\mathrm{MCM}}(S/(f)) \cong \underline{\mathrm{MCM}}(S[[y,z]]/(f+y^2+z^2)) \tag{B.12}$$

So the question reduces to only dimension 1 and 2. Indeed, they actually would give us the same spectra by the following fact:

Remark B.6. Let R be a complete regular ring. For any $f \in R$, we have

$$\mathrm{D_{sg}}(R) = \mathrm{D_{sg}}(R/(f)) \cong \mathrm{D_{sg}}(R[[y]]/(f+y^2)) \tag{B.13}$$

Now let's take the gradings into consideration. Assume that $A = \bigoplus_{n\geq 0} A_n$ is a graded Noetherian k-algebra with $A_0 = k$. Write $\operatorname{gr} A$, for the abelian category of finitely generated graded A-modules with morphisms the degree zero A-module homomorphisms. In $\mathrm{D^b}(\operatorname{gr} A)$, we have the full thick (extension closed) triangulated subcategory $\operatorname{perf} A$, consisting of all bounded complexes of finite rank free A-modules.

Definition B.7. The **graded singularity category** of A, denoted $\mathrm{D^{gr}_{sg}}(A)$, is the Verdier quotient of $\mathrm{D^b}(\operatorname{gr} A)$ by $\operatorname{perf} A$.

Similarly we have the description of graded matrix factorization, see [Orl09a]. The definition is a repetition of that of the category of matrix factorizations while taking care of the grading. Let $A = k[x_0,\ldots,x_n]/(f)$ with f homogeneous of degree d. A **graded matrix factorization** P is a sequence of graded free A-modules:

$$P_0 \xrightarrow{p_0} P_1 \xrightarrow{p_1} P_0(d), \tag{B.14}$$

where p_0 and p_1 are morphisms in $\operatorname{gr} A$ so that $p_0p_1 = p_1p_0 = f$. Morphisms from

P to Q are pairs of maps, $f_0 : P_0 \to Q_1$ and $f_1 : P_1 \to Q_1$, so that squares in the diagram

$$\begin{tikzcd} P_0 \arrow[r, "p_0"] \arrow[d, "f_0"] & P_1 \arrow[r, "p_1"] \arrow[d, "f_1"] & P_0(d) \arrow[d, "f_0(d)"] \\ Q_0 \arrow[r, "q_0"] & Q_1 \arrow[r, "q_1"] & Q_0(d) \end{tikzcd}$$

commute. A homotopy between $f : P \to Q$ and $g : P \to Q$ is a pair of maps $h_0 : P_0 \to P_1(-d)$ and $h_1 : P_1 \to Q_0$ so that $f_0 - g_0 = q_1h_0 + h_1p_0$ and $f_1 - g_1 = q_0h_1 + h_0p_1$. We also have a shift [1] which takes P to matrix factorization

$$P_1 \xrightarrow{p_1} P_0(d) \xrightarrow{p_0(d)} P_1(d).$$

Let $\mathrm{HMF}^{\mathrm{gr}}(f)$ denote the homotopy category of the category of graded matrix factorizations. In [Orl09a], Orlov proves the following:

Theorem B.8 (Orlov Theorem)**.** *There is an equivalence of triangulated categories:*

$$\mathrm{HMF}^{\mathrm{gr}}(f) \simeq \mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A). \tag{B.15}$$

In the case of simple hyper surface singularities, we have the following theorem [KST07].

Theorem B.9. *Let f be a polynomial of type ADE in B.5, and let Q be a Dynkin quiver of the corresponding type with a fixed orientation. Then we have the equivalence of the triangulated categories:*

$$\mathrm{HMF}^{\mathrm{gr}}(f) \simeq \mathrm{D}^{\mathrm{b}}(kQ) \tag{B.16}$$

Therefore, instead of $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$, in practical sense we can consider its equivalent categories:

i) $\mathrm{D^b}(kQ)$, the bounded derived category of finitely generated module over path algebra of the corresponding Dynkin quiver Q.

ii) $\mathrm{D^b}(\mathrm{rep}\, Q)$, the bounded derived category of finite dimensional representations of the Dynkin quiver Q.

Bibliography

[ABIM10] Luchezar L. Avramov, Ragnar-Olaf Buchweitz, Srikanth B. Iyengar, and Claudia Miller. Homology of perfect complexes. *Adv. Math.*, 223(5):1731–1781, 2010.

[AKO08] Denis Auroux, Ludmil Katzarkov, and Dmitri Orlov. Mirror symmetry for weighted projective planes and their noncommutative deformations. *Ann. of Math. (2)*, 167(3):867–943, 2008.

[Ami07] Claire Amiot. On the structure of triangulated categories with finitely many indecomposables. *Bull. Soc. Math. France*, 135(3):435–474, 2007.

[Arn81] V. I. Arnol′d. *Singularity theory*, volume 53 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1981. Selected papers, Translated from the Russian, With an introduction by C. T. C. Wall.

[BFK12] Matthew Ballard, David Favero, and Ludmil Katzarkov. Orlov spectra: bounds and gaps. *Invent. Math.*, 189(2):359–430, 2012.

[BGS87] R.-O. Buchweitz, G.-M. Greuel, and F.-O. Schreyer. Cohen-Macaulay modules on hypersurface singularities. II. *Invent. Math.*, 88(1):165–182, 1987.

[BK89] A. I. Bondal and M. M. Kapranov. Representable functors, Serre functors, and reconstructions. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(6):1183–1205, 1337, 1989.

[Bon89] A. I. Bondal. Representations of associative algebras and coherent sheaves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):25–44, 1989.

[Buc87] Ragnar-Olaf Buchweitz. Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings. *Unpublished manuscript*, page 155, 1987.

[Bud12] Nero Budur. Bernstein-Sato ideals and local systems. *arXiv preprint arXiv:1209.3725*, 2012.

[BvdB03] A. Bondal and M. van den Bergh. Generators and representability of functors in commutative and noncommutative geometry. *Mosc. Math. J.*, 3(1):1–36, 258, 2003.

[Eis80] David Eisenbud. Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc.*, 260(1):35–64, 1980.

[ELM+06] Lawrence Ein, Robert Lazarsfeld, Mircea Mustata, Michael Nakamaye, and Mihnea Popa. Asymptotic invariants of base loci. In *ANNALES-INSTITUT FOURIER*, volume 56, page 1701. Association des Annales de l'Institute Fourier; 1999, 2006.

[Hap88] Dieter Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1988.

[IT09] Colin Ingalls and Hugh Thomas. Noncrossing partitions and representations of quivers. *Compos. Math.*, 145(6):1533–1562, 2009.

[Kel65] GM Kelly. Chain maps inducing zero homology maps. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 61, pages 847–854. Cambridge Univ Press, 1965.

[KK06] Henning Krause and Dirk Kussin. Rouquier's theorem on representation dimension. *Contemporary Mathematics*, 406:95, 2006.

[KKP] L. Katzarkov, M. Kontsevich, and T. Pantev. Tian Todorov Theorems and LG models in prep.

[Knö87] Horst Knörrer. Cohen-Macaulay modules on hypersurface singularities. I. *Invent. Math.*, 88(1):153–164, 1987.

[KST07] Hiroshige Kajiura, Kyoji Saito, and Atsushi Takahashi. Matrix factorization and representations of quivers. II. Type *ADE* case. *Adv. Math.*, 211(1):327–362, 2007.

[Len07] Helmut Lenzing. Hereditary categories. *London Mathematical Society Lecture Note Series*, 332:105, 2007.

[LM09] Robert Lazarsfeld and Mircea Mustaţă. Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):783–835, 2009.

[O+09] Steffen Oppermann et al. Lower bounds for Auslander's representation dimension. *Duke Mathematical Journal*, 148(2):211–249, 2009.

[Oko96] Andrei Okounkov. Brunn-Minkowski inequality for multiplicities. *Invent. Math.*, 125(3):405–411, 1996.

[Oko97] A Yu Okounkov. Note on the Hilbert polynomial of a spherical variety. *Functional Analysis and Its Applications*, 31(2):138–140, 1997.

[Orl04] D. O. Orlov. Triangulated categories of singularities and D-branes in Landau-Ginzburg models. *Tr. Mat. Inst. Steklova*, 246(Algebr. Geom. Metody, Svyazi i Prilozh.):240–262, 2004.

[Orl09a] Dmitri Orlov. Derived categories of coherent sheaves and triangulated categories of singularities. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, volume 270 of *Progr. Math.*, pages 503–531. Birkhäuser Boston, Inc., Boston, MA, 2009.

[Orl09b] Dmitri Orlov. Remarks on generators and dimensions of triangulated categories. *Mosc. Math. J.*, 9(1):153–159, back matter, 2009.

[Rou08] Raphaël Rouquier. Dimensions of triangulated categories. *J. K-Theory*, 1(2):193–256, 2008.

[Sei14] P. Seidel. Fukaya A_∞-structures associated to Lefschetz fibrations.II. *ArXiv e-prints*, April 2014.

[Tak05] Atsushi Takahashi. Matrix factorizations and representations of quivers I. *arXiv preprint math/0506347*, 2005.

[Ver96] Jean-Louis Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, (239):xii+253 pp. (1997), 1996. With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.