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Prayat Poudel
University of Miami, p.prayat@gmail.com

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UNIVERSITY OF MIAMI

LINK HOMOLOGY AND EQUIVARIANT GAUGE THEORY

By

Prayat Poudel

A DISSERTATION

Submitted to the Faculty
of the University of Miami
in partial fulfillment of the requirements for
the degree of Doctor of Philosophy

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LINK HOMOLOGY AND EQUIVARIANT GAUGE THEORY

Prayat Poudel

Approved:

Nikolai Saveliev, Ph.D.
Professor of Mathematics

Kenneth Baker, Ph.D.
Professor of Mathematics

Alexander Dvorsky, Ph.D.
Professor of Mathematics

M. Brian Blake, Ph.D.
Dean of the Graduate School

Orlando Alvarez, Ph.D.
Professor of Physics

POUDEL, PRAYAT (Ph.D., Mathematics)
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The singular instanton Floer homology was defined by Kronheimer and Mrowka in connection with their proof that the Khovanov homology is an unknot detector. We study this theory for knots and two-component links using equivariant gauge theory on their double branched covers. We show that the special generator in the singular instanton Floer homology of a knot is graded by the knot signature mod 4, thereby providing a purely topological way of fixing the absolute grading in the theory. Our approach also results in explicit computations of the generators of the singular instanton Floer chain complex for several classes of knots with simple double branched covers, such as two-bridge knots, torus knots, and Montesinos knots, as well as for several families of two-components links.

The instanton Floer homology of admissible bundles on 3-manifolds was defined by Floer in the late 1980s. Taubes proved that, for integral homology spheres, its Euler characteristic equals twice the Casson invariant. We extend this result to all closed oriented 3-manifolds with positive first Betti number by establishing a similar relationship between the Lescop invariant of the manifold and its instanton Floer homology. Our formula matches the one conjectured in the physics literature.

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Contents

Chapter 1

Introduction

The Floer homology $I_*(\Sigma, \mathcal{L})$ for two-component links $\mathcal{L} \subset \Sigma$ in homology 3-spheres was defined by Kronheimer and Mrowka [22] using singular $SO(3)$ instantons. An important special case of this theory is the singular instanton knot Floer homology $I^\natural(k)$ for knots $k \subset S^3$ obtained by applying $I_*(S^3, \mathcal{L})$ to the link $\mathcal{L}$, which is a connected sum of k with the Hopf link H. Kronheimer and Mrowka [22] used $I^\natural(k)$ and its close cousin $I^\sharp(k)$ to prove that Khovanov homology is an unknot-detector.

The definition of groups $I_*(\Sigma, \mathcal{L})$ uses singular gauge theory, which makes them difficult to compute. We propose a new approach to these computations which uses equivariant gauge theory in place of the singular one. Given a two-component link $\mathcal{L}$ in an integral homology sphere Σ, we pass to the double branched cover $M \to \Sigma$ with branch set $\mathcal{L}$ and observe that the singular connections on Σ used in the definition of $I_*(\Sigma, \mathcal{L})$ pull back to equivariant smooth connections on M. The generators of the Floer chain complex $IC_*(\Sigma, \mathcal{L})$, whose homology is $I_*(\Sigma, \mathcal{L})$, are then derived from the equivariant representations $\pi_1(M) \to SO(3)$, and their mod 4 Floer indices can be computed using equivariant rather than singular index theory.

We use this approach to determine the index of the special generator in the Floer chain complex $IC^{\natural}(k)$ of a knot $k \subset S^3$, see Section 2.4. This fixes the absolute grading on $I^{\natural}(k)$ and confirms the conjecture of Hedden, Herald and Kirk [17].

Theorem. *For any knot $k \subset S^3$, the index of the special generator in the Floer chain complex $IC^{\natural}(k)$ equals* $\operatorname{sign} k$ *mod 4.*

We also achieve significant simplifications in computing the generators of the Floer chain complexes $IC^{\natural}(k)$ and $IC_*(\Sigma, \mathcal{L})$ for knots and links with simple double branched covers, such as torus knots, two-bridge knots, and general Montesinos knots and links, whose double branched covers are Seifert fibered manifolds. Explicit calculations for these knots and links are possible because the gauge theory on Seifert fibered manifolds is sufficiently well developed, see Fintushel and Stern [11] and, in the equivariant setting, Collin–Saveliev [8] and Saveliev [37]. Some of these results concerning two-bridge and torus knots were obtained earlier by Hedden, Herald, and Kirk [17] using pillowcase techniques, which are completely different from our equivariant methods.

Chapter 2 begins with a sketch of the definition of $I_*(\Sigma, \mathcal{L})$ mainly following Kronheimer and Mrowka [22] but using the language of projective representations developed in [34]. We obtain a purely algebraic description of the generators in $IC_*(\Sigma, \mathcal{L})$ as well as of the natural $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ action on them, which is crucial to the rest of the chapter.

Equivariant gauge theory is developed in Section 2.2. The section begins with a computation of $\mathbb{Z}/2$ cohomology rings of double branched covers $M \to \Sigma$ of two-component links, followed by a computation of the characteristic classes of $SO(3)$ bundles on M pulled back from orbifold bundles on Σ. The results are used to establish a bijective correspondence between equivariant $SO(3)$ representations of $\pi_1 M$ and orbifold $SO(3)$ representations of $\pi_1\Sigma$.

The next three sections are dedicated to the singular knot Floer homology $I^\natural(k)$ for knots $k \subset S^3$. Section 2.3 describes generators in the chain complex $IC^\natural(k)$ in terms of equivariant representations $\pi_1(Y) \to SO(3)$ on the double branched cover $Y \to S^3$ with branch set the knot K. These representations fall into three different categories: trivial, reducible non-trivial, and irreducible. Each equivariant irreducible representation $\pi_1(Y) \to SO(3)$ gives rise to four generators in $IC^\natural(k)$, while each non-trivial reducible representations $\pi_1(Y) \to SO(3)$ contributes two generators.

The trivial representation $\theta : \pi_1(Y) \to SO(3)$ gives rise to a special generator $\alpha \in IC^\natural(k)$ which was used in [22] to fix an absolute grading on $I^\natural(k)$. We pass to the double branched cover and use Taubes [42] index theory on manifolds with periodic ends to show that the Floer grading of α equals $\operatorname{sign}(k) \bmod 4$.

Section 2.6 contains calculations of $IC_*(\Sigma, \mathcal{L})$ for several two-component links $\mathcal{L}$ not of the form $k \# H$. In the special case of the pretzel link $\mathcal{L} = P(2, -3, -6)$ in the 3-sphere, we provide an independent verification of our answer by computing the Floer homology of Harper–Saveliev [16] of $\mathcal{L}$: the latter theory is isomorphic to $I_*(\Sigma, \mathcal{L})$ but does not use singular connections in its definition.

Finally, Section 2.7 contains proofs of some topological results, which were postponed earlier for the sake of exposition.

Chapter 3 studies a different version of instanton Floer homology in connection with a combinatorial invariant of 3-manifolds called the Lescop invariant. The latter is a rational valued invariant $\lambda_L(M)$ defined by Lescop [27] for an arbitrary closed oriented 3-manifold M as a generalization of the Casson invariant [2]. The Casson invariant, while only defined for integral homology 3-spheres, has a very useful gauge theoretic interpretation, due to Taubes [41], as half the Euler characteristic of the instanton Floer homology [12]. We provide a similar interpretation of the Lescop invariant for *all* 3-manifolds with positive first Betti number using a version of instanton Floer homology $I_*(M,P)$ defined by Floer [14] for admissible $SO(3)$ bundles $P \to M$. In fact, our formula matches the one conjectured in the physics literature, where the Lescop invariant arises as a partition function of the Donaldson-Witten theory of a 4-manifold of the form $S^1 \times M$; see Mariño–Moore [28].

Theorem. *Let M be a closed oriented connected 3-manifold with $b_1(M) \geq 1$, and let $\lambda_L(M)$ be its Lescop invariant. Then there exists an admissible bundle P over M such that*

$$\lambda_L(M) = -\frac{1}{2}\,\chi(I_*\,(M,P)) - \frac{1}{12}\,|\operatorname{Tor}(H_1(M))|, \quad \text{if } b_1(M) = 1, \quad \text{and}$$

$$\lambda_L(M) = \frac{1}{2}\,(-1)^{b_1(M)} \cdot \chi(I_*\,(M,P)), \quad \text{if } b_1(M) \geq 2,$$

where $\chi(I_*\,(M,P))$ stands for the Euler characteristic of the instanton Floer homology of the pair (M,P); see Section 3.1.

In addition, we show that $\chi(I_*\,(M,P))$ is independent of the choice of admissible bundle P and hence the above formulas hold for *any* admissible bundle P. Still lacking is a gauge theoretic interpretation of the Lescop invariant for rational homology 3-

spheres with non-trivial torsion because there is no satisfactory definition of instanton Floer homology for such manifolds.

Our proof of the above theorem proceeds by induction on the first Betti number $b_1(M)$ of the manifold and uses the Floer exact triangle [14]. In case M has no torsion in homology, we start the induction at $b_1(M) = 0$ and use Taubes' theorem [41]. In the presence of torsion, due to the aforementioned problem with defining instanton Floer homology for rational homology spheres, we start at $b_1(M) = 1$ and use an extension of Taubes' theorem due to Masataka [29].

This chapter also contains applications of Theorem 1 to the singular instanton knot homology of Kronheimer and Mrowka [23] and to the instanton homology of two component links of Harper and Saveliev [16], together with an example explaining the factor $|\operatorname{Tor}(H_1(M))|$ in the Lescop invariant from a gauge-theoretic viewpoint.

Chapter 2

Link Homology and Equivariant Gauge Theory

2.1 Link homology

In this section, we sketch the definition of the singular instanton homology $I_*(\Sigma, \mathcal{L})$ of a two-component link $\mathcal{L} \subset \Sigma$ in an integral homology sphere using the language of projective representations. Complete details of the construction can be found in Kronheimer and Mrowka [22].

2.1.1 The Chern–Simons functional

Given a two-component link $\mathcal{L}$ in an integral homology sphere Σ, the second homology of its exterior $X = \Sigma - \operatorname{int} N(\mathcal{L})$ is isomorphic to a copy of $\mathbb{Z}$ spanned by either one of the boundary tori of X. Let $P \to X$ be the unique $SO(3)$ bundle with a non-trivial second Stiefel–Whitney class $w_2(P) \in H^2(X; \mathbb{Z}/2) = \mathbb{Z}/2$. The flat connections in this bundle serve as the starting point for building $I_*(\Sigma, \mathcal{L})$. Since $w_2(P)$ evaluates non-trivially on the boundary tori, these connections are necessarily irreducible and have order two holonomy along the meridians of the link components. Therefore, they give rise to orbifold flat connections in an orbifold $SO(3)$ bundle on Σ, which we again call P. The homology sphere Σ itself is viewed as an orbifold with the cone angle π along the singular set $\mathcal{L}$ and with a compatible orbifold Riemannian metric.

Kronheimer and Mrowka [22] interpret the gauge equivalence classes of the orbifold flat connections in P as the critical points of an orbifold Chern–Simons functional

$$\mathbf{cs} : \mathcal{B}\,(\Sigma, \mathcal{L}) \to \mathbb{R}/\mathbb{Z}, \tag{2.1}$$

and define $I_*(\Sigma, \mathcal{L})$ as its Morse homology. An important feature of this construction is the use of the determinant-one gauge group $\mathcal{G}$ in the definition of the configuration space,

$$\mathcal{B}\,(\Sigma, \mathcal{L}) = \mathcal{A}\,(\Sigma, \mathcal{L})/\mathcal{G}_S,$$

where $\mathcal{A}\,(\Sigma, \mathcal{L})$ is an affine space of connections.

We will next describe the critical points of $\mathbf{cs}$ algebraically using the holonomy correspondence between flat connections and representations of the fundamental group. A variant of this classical correspondence which applies to the situation at hand was described in [34, Section 3.2] using projective $SU(2)$ representations. We will review these first, see [34, Section 3.1] for details.

2.1.2 Projective representations

Let G be a finitely presented group and view the center of $SU(2)$ as $\mathbb{Z}/2 = \{\pm 1\}$. A map $\rho : G \to SU(2)$ is called a projective representation if

$$c(g,h) = \rho(gh)\rho(h)^{-1}\rho(g)^{-1} \in \mathbb{Z}/2 \quad \text{for all } \ g, h \in G.$$

The function $c : G \times G \to \mathbb{Z}/2$ is a 2-cocycle on G defining a cohomology class $[c] \in H^2(G;\mathbb{Z}/2)$. This class has the following interpretation. The composition of $\rho : G \to SU(2)$ with $\mathrm{Ad} : SU(2) \to SO(3)$ is a representation $\mathrm{Ad}\,\rho : G \to SO(3)$. As such, it induces a continuous map $BG \to BSO(3)$ which is unique up to homotopy. The pull back of the universal Stiefel–Whitney class $w_2 \in H^2(BSO(3);\mathbb{Z}/2)$ via this map is our class $[c] = w_2(\mathrm{Ad}\,\rho) \in H^2(G;\mathbb{Z}/2)$. It serves as an obstruction to lifting $\mathrm{Ad}\,\rho : G \to SO(3)$ to an $SU(2)$ representation.

Let $\mathcal{PR}_c(G;SU(2))$ be the space of conjugacy classes of projective representations $\rho : G \to SU(2)$ whose associated cocycle is c. The topology on $\mathcal{PR}_c(G;SU(2))$ is supplied by the algebraic set structure. One can easily see that $\mathcal{PR}_c(G;SU(2))$ is determined uniquely up to homeomorphism by the cohomology class of c. The group $H^1(G;\mathbb{Z}/2) = \mathrm{Hom}(G,\mathbb{Z}/2)$ acts on $\mathcal{PR}_c(G;SU(2))$ by sending ρ to $\chi \cdot \rho$ for any $\chi \in \mathrm{Hom}(G,\mathbb{Z}/2)$. The orbits of this action are in a bijective correspondence with the conjugacy classes of representations $G \to SO(3)$ whose second Stiefel–Whitney class equals $[c]$. The bijection is given by taking the adjoint representation.

Projective representations $\rho : G \to SU(2)$ can also be described in terms of a presentation $G = F/R$. Consider a homomorphism $\gamma : R \to \mathbb{Z}/2$ defined by its values $\gamma(r) = \pm 1$ on the relators $r \in R$ and by the condition that it is constant on the orbits of the adjoint action of F on R. Also, choose a set-theoretic section $s : G \longrightarrow F$ in the exact sequence

$$1 \longrightarrow R \xrightarrow{\ i\ } F \xrightarrow{\ \pi\ } G \longrightarrow 1$$

and denote by $r : G \times G \longrightarrow R$ the function defined by the formula $s(gh) = r(g,h)s(g)s(h)$.

2.1.1 Proposition. *A choice of a section $s : G \to F$ establishes a bijective correspondence between the conjugacy classes of projective representations $\rho : G \to SU(2)$ with the cocycle $c(g,h) = \gamma(r(g,h))$, and the conjugacy classes of homomorphisms $\sigma : F \to SU(2)$ such that $i^*\sigma = \gamma$. A different choice of s results in a cohomologous cocycle.*

Proof. We begin by checking that $c(g,h) = \gamma(r(g,h))$ is a cocycle. For any $g, h, k \in G$, we have

$$s(ghk) = r(gh,k)s(gh)s(k) = r(gh,k)r(g,h)s(g)s(h)s(k),$$
$$s(ghk) = r(g,hk)s(g)s(hk) = r(g,hk)s(g)r(h,k)s(h)s(k),$$

which results in $r(gh,k)r(g,h) = r(g,hk)s(g)r(h,k)s(g)^{-1}$. Since the homomorphism γ is constant on the orbits of the adjoint action of F on R, its application to the above equality gives the cocycle condition $c(gh,k)c(g,h) = c(g,hk)c(h,k)$ as desired.

Now, given a homomorphism $\sigma : F \to SU(2)$ such that $i^*\sigma = \gamma$, define $\rho : G \to SU(2)$ by the formula $\rho(g) = \sigma(s(g))$. Then $\rho(gh) = \sigma(s(gh)) = \sigma(r(g,h)s(g)s(h)) = \gamma(r(g,h))\sigma(s(g))\sigma(s(h)) = c(g,h)\rho(g)\rho(h)$, hence ρ is a projective representation with cocycle c. It is clear that conjugate representations σ define conjugate projective representations ρ, and that a different choice of s leads to a cohomologous cocycle c.

The inverse correspondence is defined as follows. Given a projective representation $\rho : G \to SU(2)$, write elements of F in the form $r \cdot s(g)$, with $r \in R$ and $g \in G$, and define $\sigma : F \to SU(2)$ by the formula $\sigma(r \cdot s(g)) = \gamma(r)\rho(g)$. That σ is a homomorphism can be checked by a straightforward calculation using the fact that $c(g,h) = \gamma(r(g,h))$. □

2.1.2 Example. Let $G = \pi_1(M)$ be the fundamental group of a manifold M obtained by 0–surgery on a knot K in an integral homology sphere Σ. The group $\pi_1(M)$ is obtained from $\pi_1(K)$ by imposing the relation $\ell = 1$, where ℓ is a longitude of K. Therefore, $\pi_1(M)$ admits a presentation $\pi_1(M) = F/R$ with ℓ being one of the relators. Let $\gamma(\ell) = -1$ and $\gamma(r) = 1$ for the rest of the relators $r \in R$. It has been known since Floer [13] that the action of $H^1(M;\mathbb{Z}/2) = \mathbb{Z}/2$ on the set of conjugacy classes of projective representations $\sigma : F \to SU(2)$ with $i^*\sigma = \gamma$ is free, providing a two-to-one correspondence between this set and the set of the conjugacy classes of representations $\pi_1(M) \to SO(3)$ with non-trivial $w_2 \in H^2(M;\mathbb{Z}/2) = \mathbb{Z}/2$.

2.1.3 Holonomy correspondence

We will now apply the general theory of Section 2.1.2 to the group $G = \pi_1(X)$, where X is the exterior of a two-component link $\mathcal{L}$ in an integral homology sphere Σ. We begin with the following simple observation.

2.1.3 Lemma. *Unless the link $\mathcal{L}$ is split, $H^2(X;\mathbb{Z}/2) = H^2(\pi_1(X);\mathbb{Z}/2) = \mathbb{Z}/2$. For split links, $I_*(\Sigma,\mathcal{L}) = 0$.*

Proof. For a split link $\mathcal{L}$, the splitting sphere generates the group $H_2(X;\mathbb{Z}) = \mathbb{Z}$. Since there are no flat connections on this sphere with non-trivial $w_2(P)$ the group $I_*(\Sigma, \mathcal{L})$ must vanish. For a non-split link, the claimed equality follows from the Hopf exact sequence

$$\pi_2(X) \longrightarrow H_2(X) \longrightarrow H_2(\pi_1(X)) \longrightarrow 0$$

and the vanishing of the Hurewicz homomorphism $\pi_2(X) \to H_2(X)$. □

From now on, we will assume that the link $\mathcal{L} \subset \Sigma$ is not split. The holonomy correspondence of [34, Section 3.1] identifies the critical point set of the functional (2.1) with the set $\mathcal{PR}_c(X, SU(2))$ of the conjugacy classes of projective representations $\rho : \pi_1(X) \to SU(2)$, for any choice of cocycle c such that $0 \neq [c] = w_2(P) \in H^2(X;\mathbb{Z}/2) = \mathbb{Z}/2$. Note that this identification commutes with the $H^1(X;\mathbb{Z}/2)$ action, and the orbits of this action on $\mathcal{PR}_c(X, SU(2))$ are in a bijective correspondence with the conjugacy classes of representations $\operatorname{Ad}\rho : \pi_1(X) \to SO(3)$ having $w_2(\operatorname{Ad}\rho) \neq 0$.

2.1.4 Lemma. *Any representation* $\operatorname{Ad}\rho : \pi_1(X) \to SO(3)$ *with* $w_2(\operatorname{Ad}\rho) \neq 0$ *is irreducible, that is, its image is not contained in a copy of* $SO(2) \subset SO(3)$.

Proof. The restriction to ρ to either boundary torus of X has non-trivial second Stiefel–Whitney class, which implies that it does not lift to an $SU(2)$ representation. However, any reducible representation $\pi_1(T^2) \to SO(3)$ admits an $SU(2)$ lift, therefore, the image of ρ cannot be contained in a copy of $SO(2) \subset SO(3)$. □

2.1.4 Floer gradings

Given flat orbifold connections ρ and σ in the orbifold bundle $P \to \Sigma$, consider an arbitrary orbifold connection A in the pull back bundle on the product $\mathbb{R}\times\Sigma$ matching ρ and σ near the negative and positive ends. Equip $\mathbb{R}\times\Sigma$ with the orbifold product metric and consider the ASD operator

$$\mathcal{D}_A(\rho,\sigma) = d_A^* \oplus -d_A^+ : \Omega^1(\mathbb{R}\times\Sigma, \operatorname{ad} P) \to (\Omega^0 \oplus \Omega^2_+)(\mathbb{R}\times\Sigma, \operatorname{ad} P) \tag{2.2}$$

completed in the orbifold Sobolev L^2 norms as in [22, Section 3.1]. Since ρ and σ are irreducible, this operator will be Fredholm if we further assume that ρ and σ are non-degenerate as the critical points of the Chern–Simons functional (2.1). Define the relative Floer grading as

$$\operatorname{gr}(\rho,\sigma) \;=\; \operatorname{ind}\mathcal{D}_A(\rho,\sigma) \pmod 4. \tag{2.3}$$

2.1.5 Perturbations

The critical points of the Chern–Simons functional need not be non-degenerate, therefore, the Chern–Simons functional has to be perturbed. The perturbations used in [22, Section 3.4] are the standard Wilson loop perturbations along loops in Σ disjoint from the link $\mathcal{L}$. There are sufficiently many such perturbations to guarantee the non-degeneracy of the critical points of the perturbed Chern–Simons functional as well as the transversality properties for the moduli spaces of trajectories of its gradient flow. This allows to define the boundary operator and to complete the definition of $I_*(\Sigma,\mathcal{L})$.

2.2 Equivariant gauge theory

In this section, we survey some equivariant gauge theory on the double branched cover $M \to \Sigma$ of a homology sphere Σ with branch set a two-component link $\mathcal{L}$. It will be used in the forthcoming sections to make headway in computing the link homology $I_*(\Sigma, \mathcal{L})$.

2.2.1 Topological preliminaries

Let Σ be an integral homology 3-sphere and $\mathcal{L} = \ell_1 \cup \ell_2$ a link of two components in Σ. The link exterior $X = \Sigma - \operatorname{int} N(\mathcal{L})$ is a manifold whose boundary consists of two tori, with $H_1(X;\mathbb{Z}) = \mathbb{Z}^2$ spanned by the meridians μ_1 and μ_2 of the link components. The homomorphism $\pi_1(X) \to \mathbb{Z}/2$ sending μ_1 and μ_2 to the generator of $\mathbb{Z}/2$ gives rise to a regular double cover $\widetilde{X} \to X$, and also to a double branched cover $\pi : M \to \Sigma$ with branching set $\mathcal{L}$ and the covering translation $\tau : M \to M$. Denote by $\Delta(t)$ the one-variable Alexander polynomial of $\mathcal{L}$.

2.2.1 Proposition. *The first Betti number of M is one if $\Delta(-1) = 0$ and zero otherwise. In the latter case, $H_1(M;\mathbb{Z})$ is a finite group of order $|\Delta(-1)|$. The induced involution $\tau_* : H_1(M) \to H_1(M)$ is multiplication by -1.*

Proof. This is essentially proved in Kawauchi [19, Section 5.5]. The statement about τ_* follows from an isomorphism of $\mathbb{Z}[t, t^{-1}]$ modules $H_1(M) = H_1(E)/(1+t)H_1(E)$, where E is the infinite cyclic cover of X, proved in [19, Theorem 5.5.1]. A completely different proof for the special case of double branched covers of S^3 with branch set a knot can be found in Ruberman [32, Lemma 5.5]. □

2.2.2 Proposition. *Let M be the double branched cover of an integral homology sphere with branch set a two-component link $\mathcal{L} = \ell_1 \cup \ell_2$. Then $H_i(M;\mathbb{Z}/2) = H^i(M;\mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$ if $i = 0,1,2,3$, and is zero otherwise. The cup-product $H^1(M;\mathbb{Z}/2) \times H^1(M;\mathbb{Z}/2) \to H^2(M;\mathbb{Z}/2)$ is given by the linking number* $\ell k(\ell_1, \ell_2) \pmod 2$.

The proof of Proposition 2.2.2 will be postponed until Section 2.7 for the sake of exposition.

An important example of $\mathcal{L}$ to consider is that of the two-component link $k^\natural$ obtained by connect summing a knot $k \subset S^3$ with the Hopf link. The double branched cover $M \to S^3$ in this case is the connected sum $M = Y \# \mathbb{RP}^3$, where Y is the double branched cover of k. Proposition 2.2.2 easily follows because $H_*(Y;\mathbb{Z}/2) = H_*(S^3;\mathbb{Z}/2)$.

2.2.2 The orbifold exact sequence

It will be convenient to view $\Sigma = M/\tau$ as an orbifold with the singular set $\mathcal{L}$. To be precise, the regular double cover $\widetilde{X}$ is a 3-manifold whose boundary consists of two tori, and

$$M = \widetilde{X} \cup_h N(\mathcal{L}),$$

where the gluing homeomorphism $h : \partial\widetilde{X} \to \partial N(\mathcal{L})$ identifies $\pi^{-1}(\mu_i)$ with the meridian μ_i for $i = 1,2$. The involution $\tau : M \to M$ acts by meridional rotation on $N(\mathcal{L})$, thereby fixing the link $\mathcal{L}$, and by covering translation on $\widetilde{X}$. Define the orbifold fundamental group

$$\pi_1^V(\Sigma, \mathcal{L}) = \pi_1(X) \,/\, \langle {\mu_1}^2 = {\mu_2}^2 = 1\rangle.$$

Then the homotopy exact sequence of the covering $\widetilde{X} \to X$ gives rise to a split short exact sequence, called the orbifold exact sequence,

$$1 \longrightarrow \pi_1(M) \xrightarrow{\pi_*} \pi_1^V(\Sigma, \mathcal{L}) \xrightarrow{j} \mathbb{Z}/2 \longrightarrow 1 \tag{2.4}$$

The homomorphism j maps the meridians μ_1, μ_2 onto the generator of $\mathbb{Z}/2$ and one obtains a splitting by sending this generator to either μ_1 or μ_2.

2.2.3 Pulled back bundles

Let $P \to \Sigma$ be the orbifold $SO(3)$ bundle used in the definition of $I_*(\Sigma, \mathcal{L})$ in Section 2.1. It pulls back to an orbifold $SO(3)$ bundle $Q \to M$ because the projection map $\pi : M \to \Sigma$ is regular in the sense of Chen–Ruan [6]. The bundle Q is in fact smooth because orbifold connections on P, with order-two holonomy along the meridians of $\mathcal{L}$, lift to connections in Q with trivial holonomy along the meridians of the two-component link $\tilde{\mathcal{L}} = \pi^{-1}(\mathcal{L})$.

2.2.3 Proposition. *The bundle $Q \to M$ is non-trivial.*

The rest of this section is dedicated to the proof of this proposition. We will accomplish it by showing the non-vanishing of $w_2(Q) \in H^2(M; \mathbb{Z}/2) = \mathbb{Z}/2$. Our argument will split into two cases, corresponding to the parity of the linking number between the components of $\mathcal{L}$.

Suppose that $\ell k(\ell_1, \ell_2)$ is even and consider the regular double cover $\pi : M - \widetilde{\mathcal{L}} \to \Sigma - \mathcal{L}$. It gives rise to the Gysin exact sequence

$$\rightarrow H^1(\Sigma - \mathcal{L};\mathbb{Z}/2) \xrightarrow{\cup w_1} H^2(\Sigma - \mathcal{L};\mathbb{Z}/2) \xrightarrow{\pi^*} H^2(M - \widetilde{\mathcal{L}};\mathbb{Z}/2) \rightarrow$$
$$\rightarrow H^2(\Sigma - \mathcal{L};\mathbb{Z}/2) \xrightarrow{\cup w_1} H^3(\Sigma - \mathcal{L};\mathbb{Z}/2) \longrightarrow \cdots$$

where $\cup\, w_1$ means taking the cup-product with the first Stiefel–Whitney class of the cover. The cup-product on $H^*(\Sigma - \mathcal{L};\mathbb{Z}/2)$ can be determined from the following commutative diagram

$$\begin{array}{ccc} H_2(\Sigma,\mathcal{L};\mathbb{Z}/2) \times H_2(\Sigma,\mathcal{L};\mathbb{Z}/2) & \xrightarrow{\ \cdot\ } & H_1(\Sigma,\mathcal{L};\mathbb{Z}/2) \\ \text{PD}\uparrow & & \uparrow\text{PD} \\ H^1(\Sigma - \mathcal{L};\mathbb{Z}/2) \times H^1(\Sigma - \mathcal{L};\mathbb{Z}/2) & \xrightarrow{\ \cup\ } & H^2(\Sigma - \mathcal{L};\mathbb{Z}/2) \end{array}$$

where PD stands for the Poincaré duality isomorphism and the dot in the upper row for the intersection product. Note that Seifert surfaces of knots ℓ_1 and ℓ_2 generate $H_2(\Sigma,\mathcal{L};\mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and any arc in Σ with one endpoint on ℓ_1 and the other on ℓ_2 generates $H_1(\Sigma,\mathcal{L};\mathbb{Z}/2) = \mathbb{Z}/2$. An easy calculation shows that, with respect to these generators, the intersection product is given by the matrix

$$\begin{pmatrix} 0 & \ell k(\ell_1,\ell_2) \\ \ell k(\ell_1,\ell_2) & 0 \end{pmatrix}$$

Since $\ell k(\ell_1,\ell_2)$ is even, this gives a trivial cup product structure on the link complement $\Sigma - \mathcal{L}$. Therefore, the map $\cup\, w_1$ in the Gysin sequence is zero and the map $\pi^* : H^2(\Sigma - \mathcal{L};\mathbb{Z}/2) \to H^2(M - \widetilde{\mathcal{L}};\mathbb{Z}/2)$ is injective. Since $w_2(P) \in H^2(\Sigma - \mathcal{L};\mathbb{Z}/2)$ is non-zero we conclude that $\pi^*(w_2(P)) \neq 0$. This implies that $w_2(Q) \neq 0$ because

$Q = \pi^* P$ over $M - \widetilde{\mathcal{L}}$.

Now suppose that $\ell k(\ell_1, \ell_2)$ is odd. The above calculation implies that the second Stiefel–Whitney class of $\pi^* P$ vanishes in $H^2(M - \widetilde{\mathcal{L}}; \mathbb{Z}/2)$. We will prove, however, that $w_2(Q) \in H^2(M; \mathbb{Z}/2)$ is non-zero, by showing that Q carries a flat connection with non-zero w_2.

Note that the orbifold bundle P carries a flat $SO(3)$ connection whose holonomy is a representation $\alpha : \pi_1^V(\Sigma, \mathcal{L}) \longrightarrow SO(3)$ of the orbifold fundamental group

$$\pi_1^V(\Sigma, \mathcal{L}) = \pi_1(X)/\langle {\mu_1}^2 = {\mu_2}^2 = 1\rangle$$

sending the two meridians to $\operatorname{Ad} i$ and $\operatorname{Ad} j$. This flat connection pulls back to a flat connection on Q with holonomy $\pi^*\alpha : \pi_1(M) \to SO(3)$. We wish to compute the second Stiefel–Whitney class of $\pi^*\alpha$.

2.2.4 Lemma. *The representation $\pi^*\alpha : \pi_1(M) \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is non-trivial.*

Proof. Our proof will rely on the orbifold exact sequence (2.4). Assume that $\pi^*\alpha$ is trivial. Then $\pi_1(M) \subset \ker \alpha$ hence α factors through a homomorphism $\pi_1^V(\Sigma, \mathcal{L})/\pi_1(M) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Since $\pi_1^V(\Sigma, \mathcal{L})/\pi_1(M) = \mathbb{Z}/2$ we obtain a contradiction with the surjectivity of α. $\square$

Since the group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ is abelian, the representation $\pi^*\alpha : \pi_1(M) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ factors through a homomorphism $H_1(M) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ which is uniquely determined by its two components ξ, $\eta \in \operatorname{Hom}(H_1(M), \mathbb{Z}/2) = H^1(M; \mathbb{Z}/2) = \mathbb{Z}/2$, see Proposition 2.2.2. A calculation identical to that in [34, Proposition 4.3] shows that $w_2(\pi^*\alpha) = \xi^2 + \xi\eta + \eta^2$ (note that, unlike in [34], the classes ξ^2 and η^2 need not vanish). Since ξ and η cannot be both trivial by Lemma 2.2.4, we may assume without loss of generality that $\xi \neq 0$. If $\eta = 0$ then $w_2(\pi^*\alpha) = \xi^2$. If $\eta \neq 0$ then $\xi = \eta$ due to the fact that $H^1(M;\mathbb{Z}/2) = \mathbb{Z}/2$, and therefore again $w_2(\pi^*\alpha) = \xi^2$. Since $\ell k(\ell_1, \ell_2)$ is odd, it follows from Proposition 2.2.2 that $w_2(\pi^*\alpha) \neq 0$.

2.2.4 Pulled back representations

Assuming that $\mathcal{L} \subset \Sigma$ is non-split, in Section 2.1.3, we identified the critical point set of the Chern–Simons functional (2.1) with the space $\mathcal{PR}_c(X, SU(2))$ of the conjugacy classes of projective representations $\pi_1(X) \to SU(2)$, for any choice of cocycle c not cohomologous to zero. We further identified the quotient of $\mathcal{PR}_c(X, SU(2))$ by the natural $H^1(X;\mathbb{Z}/2)$ action with the subspace $\mathcal{R}_w(X; SO(3))$ of the $SO(3)$ character variety of $\pi_1(X)$ cut out by the condition $w_2 \neq 0$. The latter condition implies that both meridians μ_1 and μ_2 are represented by $SO(3)$ matrices of order two, which leads to a natural identification of this subspace with

$$\mathcal{R}_\omega(\Sigma, \mathcal{L}; SO(3)) = \{\, \rho : \pi_1^V(\Sigma, \mathcal{L}) \to SO(3)) \mid w_2(\rho) \neq 0\,\}/\operatorname{Ad} SO(3),$$

where the condition $w_2(\rho) \neq 0$ applies to the representation ρ restricted to X. To summarize, the group $H^1(X;\mathbb{Z}/2)$ acts on the space $\mathcal{PR}_c(X, SU(2))$ with the quotient map

$$\mathcal{PR}_c(X, SU(2)) \longrightarrow \mathcal{R}_\omega(\Sigma, \mathcal{L}; SO(3)).$$

We wish to study the space $\mathcal{R}_\omega(\Sigma, \mathcal{L}; SO(3))$ using equivariant representations on the double branched cover $M \to \Sigma$.

2.2.5 Lemma. *Let* $\rho : \pi_1^V(\Sigma, \mathcal{L}) \to SO(3)$ *be a representation and* $\pi^*\rho : \pi_1(M) \to SO(3)$ *its pull back via the homomorphism* π_* *of the orbifold exact sequence* (2.4). *Then there exists an element* $u \in SO(3)$ *of order two such that* $\tau^*(\pi^*\rho) = u\cdot(\pi^*\rho)\cdot u^{-1}$.

Proof. Let $\tilde{X} \to X$ be the regular double cover as in Section 2.2.2. Choose a point b in one of the boundary tori of $\widetilde{X}$ and consider the commutative diagram

$$\begin{array}{ccccc}
\pi_1(\widetilde{X}, b) & \xrightarrow{\tau_*} & \pi_1(\widetilde{X}, \tau(b)) & \xrightarrow{\psi_f} & \pi_1(\widetilde{X}, b) \\
& \searrow^{\pi_*} & \swarrow_{\pi_*} & & \downarrow{\scriptstyle \pi_*} \\
& \pi_1(X, \pi(b)) & & \xrightarrow{\varphi} & \pi_1(X, \pi(b))
\end{array}$$

whose maps ψ_f and φ are defined as follows. Given a path $f : [0,1] \to X$ from $\tau(b)$ to b, take its inverse $\overline{f}(s) = f(1-s)$ and define the map ψ_f by the formula $\psi_f(\beta) = f \cdot \beta \cdot \overline{f}$. Since $\pi(b) = \pi(\tau(b))$, the path f projects to a loop in X based at $\pi(b)$, and the map φ is the conjugation by that loop. In fact, one can choose the path f to project onto the meridian μ_i of the boundary torus on which $\pi(b)$ lies so that $\varphi(x) = \mu_i \cdot x \cdot \mu_i^{-1}$. After filling in the solid tori, we obtain the commutative diagram

$$\begin{array}{ccc}
\pi_1(M) & \xrightarrow{\tau_*} & \pi_1(M) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
\pi_1^V(\Sigma, \mathcal{L}) & \xrightarrow{\varphi} & \pi_1^V(\Sigma, \mathcal{L})
\end{array}$$

which tells us that, for any $\rho : \pi_1^V(\Sigma, \mathcal{L}) \to SO(3)$, the pull back representation $\pi^*\rho$ has the property that $\tau^*(\pi^*\rho) = u \cdot (\pi^*\rho) \cdot u^{-1}$ with $u = \rho(\mu_i)$ of order two. $\square$

2.2.6 Example. Let $\mathcal{L} \subset S^3$ be the Hopf link then $M = \mathbb{R}\mathrm{P}^3$ and the orbifold exact sequence (2.4) takes the form

$$1 \longrightarrow \mathbb{Z}/2 \xrightarrow{\pi_*} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{j} \mathbb{Z}/2 \longrightarrow 1$$

with the two copies of $\mathbb{Z}/2$ in the middle group generated by the meridians μ_1 and μ_2. Define $\rho : \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to SO(3)$ on the generators by $\rho(\mu_1) = \mathrm{Ad}\, i$ and $\rho(\mu_2) = \mathrm{Ad}\, j$; up to conjugation, this is the only representation $\mathbb{Z}_2 \oplus \mathbb{Z}/2 \to SO(3)$ with $w_2(\rho) \neq 0$. The pull back representation $\pi^*\rho : \mathbb{Z}/2 \to SO(3)$ sends the generator to $\mathrm{Ad}\, i \cdot \mathrm{Ad}\, j = \mathrm{Ad}\, k$. Since $\tau^*(\pi^*\rho) = \pi^*\rho$, the identity $\tau^*(\pi^*\rho) = u \cdot (\pi^*\rho) \cdot u^{-1}$ holds for multiple choices of u including the second order u of the form $u = \mathrm{Ad}\, q$, where q is any unit quaternion such that $-qk = kq$.

Given a double branched cover $\pi : M \to \Sigma$ with branch set $\mathcal{L}$ and the covering translation $\tau : M \to M$, define

$$\mathcal{R}_\omega(M; SO(3)) \;=\; \{\, \beta : \pi_1 M \to SO(3) \mid w_2(\beta) \neq 0 \,\} / \,\mathrm{Ad}\, SO(3).$$

Since $w_2(\tau^*\beta) = w_2(\beta) \in H^2(M; \mathbb{Z}/2) = \mathbb{Z}/2$, the pull back of representations via τ gives rise to a well defined involution

$$\tau^* : \mathcal{R}_\omega(M; SO(3)) \longrightarrow \mathcal{R}_\omega(M; SO(3)). \tag{2.5}$$

Its fixed point set $\mathrm{Fix}\,(\tau^*)$ consists of the conjugacy classes of representations $\beta : \pi_1 M \to SO(3)$ such that $w_2(\beta) \neq 0$ and there exists an element $u \in SO(3)$ having the property that $\tau^*\beta = u \cdot \beta \cdot u^{-1}$. Consider the sub-variety

$$\mathcal{R}^\tau_w(M; SO(3)) \ \subset \ \text{Fix}\,(\tau^*) \tag{2.6}$$

defined by the condition that the conjugating element u has order two. It is well defined because all elements of order two in $SO(3)$ are conjugate to each other. The following proposition is the main result of this section.

2.2.7 Proposition. *The homomorphism $\pi_* : \pi_1(M) \to \pi_1^V(\Sigma, \mathcal{L})$ of the orbifold exact sequence* (2.4) *induces via the pull back a homeomorphism*

$$\pi^* : \mathcal{R}_\omega(\Sigma, \mathcal{L}; SO(3)) \longrightarrow \mathcal{R}^\tau_\omega(M; SO(3)).$$

Proof. Orbifold representations $\pi_1^V(\Sigma, \mathcal{L}) \to SO(3)$ with non-trivial w_2 pull back to representations $\pi_1(M) \to SO(3)$ with non-trivial w_2, see Section 2.2.3. In addition, these pull back representations are equivariant in the sense of Lemma 2.2.5. Therefore, the map $\pi^* : \mathcal{R}_\omega(\Sigma, \mathcal{L}; SO(3)) \longrightarrow \mathcal{R}^\tau_\omega(M; SO(3))$ is well defined. To finish the proof, we will construct an inverse of π^*. Given $\beta : \pi_1 M \to SO(3)$ whose conjugacy class belongs to $\mathcal{R}^\tau_\omega(M; SO(3))$, there exists an element $u \in SO(3)$ of order two such that $\tau^*\beta = u \cdot \beta \cdot u^{-1}$. The pair (β, u) then defines an $SO(3)$ representation of $\pi_1^V(\Sigma, \mathcal{L}) = \pi_1(M) \rtimes \mathbb{Z}/2$ by the formula $\rho(x, t^\ell) = \beta(x) \cdot u^\ell$, where $x \in \pi_1(M)$ and t is the generator of $\mathbb{Z}/2$. $\square$

2.3 Knot homology: the generators

We will now use the equivariant theory of Section 2.2 to better understand the chain complex $IC^\natural(k)$ which computes the singular instanton knot homology $I^\natural(k) = I_*(S^3, k^\natural)$ of Kronheimer and Mrowka [22]. In this section, we describe the conjugacy classes of projective $SU(2)$ representations on the exterior of $k^\natural$ with non-trivial $[c]$ and separate them into the orbits of the canonical $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ action.

2.3.1 Projective representations

Given a knot $k \subset S^3$, denote by $K = S^3 - N(k)$ its exterior and by $K^\natural = S^3 - N(k^\natural)$ the exterior of the two-component link $k^\natural = k \cup \ell$, obtained by connect summing k with the Hopf link. The Wirtinger presentation

$$\pi_1(K) = \langle a_1, a_2, \cdots, a_n \mid r_1, \ldots, r_m \rangle$$

with meridians a_i and relators r_j gives rise to the Wirtinger presentation

$$\pi_1(K^\natural) = \langle a_1, a_2, \cdots, a_n, b \mid r_1, \ldots, r_m,\ [a_1, b] = 1 \rangle,$$

where b stands for the meridian of the component ℓ. Since the link $k^\natural$ is not split, it follows from Lemma 2.1.3 that $H^2(\pi_1(K^\natural); \mathbb{Z}/2) = H^2(K^\natural; \mathbb{Z}/2) = \mathbb{Z}/2$. The generator of the latter group evaluates non-trivially on both boundary components of $K^\natural$, which makes it Poincaré dual to any arc connecting these two boundary components. It follows from Proposition 2.1.1 that the projective representations with non-trivial $[c]$ which we are interested in are precisely the homomorphisms $\rho : F \to SU(2)$ of the free group F generated by the meridians $a_1, \ldots, a_n, b$ such that

$$\rho(r_1) = \ldots = \rho(r_n) = 1 \quad \text{and} \quad \rho([a_1, b]) = -1.$$

Representations ρ are uniquely determined by the $SU(2)$ matrices $A_i = \rho(a_i)$ and $B = \rho(b)$ subject to the above relations, and the space $\mathcal{PR}_c(K^\natural, SU(2))$ consists of all such tuples $(A_1, \ldots, A_n; B)$ up to conjugation.

Observe that the relation $A_1 B = -BA_1$ implies that, up to conjugation, $A_1 = i$ and $B = j$. Since the Wirtinger relations $r_1 = 1, \cdots, r_m = 1$ are of the form $a_i a_j a_i^{-1} = a_k$, all the matrices A_i must have zero trace. In particular, the matrices $A_1 = \ldots = A_n = i$ and $B = j$ satisfy all of the relations, thereby giving rise to the special projective representation $\alpha = (i, i, \ldots, i; j)$. On the other hand, if we assume that not all A_i commute with each other, we have an entire circle of projective representations,

$$(i, e^{i\varphi} A_2 \, e^{-i\varphi}, \cdots, e^{i\varphi} A_n \, e^{-i\varphi}; j). \tag{2.7}$$

It is parameterized by $e^{2i\varphi} \in S^1$ due to the fact that the center of $SU(2)$ is the stabilizer of the adjoint action of $SU(2)$ on itself. Note that two tuples like (2.7) are conjugate if and only if they are equal to each other. One can easily see that the formula $\psi(A_1, \ldots, A_n; B) = (A_1, \ldots, A_n)$ defines a surjective map

$$\psi : \mathcal{PR}_c(K^\natural, SU(2)) \to \mathcal{R}_0(K, SU(2)), \tag{2.8}$$

where $\mathcal{R}_0(K, SU(2))$ is the space of the conjugacy classes of traceless representations $\rho_0 : \pi_1(K) \to SU(2)$. If ρ_0 is irreducible, the fiber $C(\rho_0) = \psi^{-1}([\rho_0])$ is a circle of the form (2.7). The special projective representation α is a fiber of (2.8) in its own right over the unique (up to conjugation) reducible traceless representation $\pi_1(K) \to H_1(K) \to SU(2)$ sending all the meridians to the same traceless matrix i. Therefore, assuming that $\mathcal{R}_0(K, SU(2))$ is non-degenerate, the space $\mathcal{PR}_c(K^\natural, SU(2))$ consists of an isolated point and finitely many circles, one for each conjugacy class of irreducible representations in $\mathcal{R}_0(K, SU(2))$. The same result holds in general after perturbation.

2.3.2 The action of $H^1(K^\natural; \mathbb{Z}/2)$

The group $H^1(K^\natural; \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ generated by the duals χ_k and χ_ℓ of the meridians of the link $k^\natural = k \cup \ell$ acts on the space of projective representations $\mathcal{PR}_c(K^\natural, SU(2))$ as explained in Section 2.1.2. In terms of the tuples (2.7), the generators χ_k and χ_ℓ send $(i, e^{i\varphi} A_2 e^{-i\varphi}, \dots, e^{i\varphi} A_n e^{-i\varphi}; j)$ to

$$(-i, -e^{i\varphi} A_2 e^{-i\varphi}, \dots, -e^{i\varphi} A_n e^{-i\varphi}; j) \text{ and}$$
$$(i, e^{i\varphi} A_2 e^{-i\varphi}, \dots, e^{i\varphi} A_n e^{-i\varphi}; -j),$$

respectively. The isolated point $\alpha = (i, i, \dots, i; j)$ is a fixed point of this action since $(-i, -i, \dots, -i; j) = j \cdot (i, i, \dots, i; j) \cdot j^{-1}$ and $(i, i, \dots, i; -j) = i \cdot (i, i, \dots, i; j) \cdot i^{-1}$.

To describe the action of χ_ℓ on the circle $C(\rho_0)$ for an irreducible ρ_0 conjugate $(i, e^{i\varphi} A_2 e^{-i\varphi}, \dots, e^{i\varphi} A_n e^{-i\varphi}; -j)$ by i to obtain

$$(i, e^{i(\varphi+\pi/2)} A_2 e^{-i(\varphi+\pi/2)}, \dots, e^{i(\varphi+\pi/2)} A_n e^{-i(\varphi+\pi/2)}; j).$$

Since the circle $C(\rho_0)$ is parameterized by $e^{2i\varphi}$, we conclude that the involution χ_ℓ

acts on $C(\rho_0)$ via the antipodal map.

The action of χ_k on the circle $C(\rho_0)$ for an irreducible ρ_0 will depend on whether ρ_0 is a binary dihedral representation or not. Recall that a representation $\rho_0 : \pi_1(K) \to SU(2)$ is called *binary dihedral* if it factors through a copy of the binary dihedral subgroup $S^1 \cup j \cdot S^1 \subset SU(2)$, where S^1 stands for the circle of unit complex numbers. Equivalently, ρ_0 is binary dihedral if its adjoint representation $\mathrm{Ad}(\rho_0) : \pi_1(K) \to SO(3)$ is *dihedral* in that it factors through a copy of $O(2)$ embedded into $SO(3)$ via the map $A \to (A, \det A)$.

One can show that a representation ρ_0 is binary dihedral if and only if $\chi \cdot \rho_0$ is conjugate to ρ_0, where $\chi : \pi_1(K) \to \mathbb{Z}/2$ is the generator of $H^1(K; \mathbb{Z}/2) = \mathbb{Z}/2$. Note that χ defines an involution on $\mathcal{R}_0(K, SU(2))$ which makes the following diagram commute

$$\begin{array}{ccc} \mathcal{PR}_c(K^\natural, SU(2)) & \xrightarrow{\pi} & \mathcal{R}_0(K, SU(2)) \\ \chi_k \downarrow & & \downarrow \chi \\ \mathcal{PR}_c(K^\natural, SU(2)) & \xrightarrow{\pi} & \mathcal{R}_0(K, SU(2)). \end{array}$$

The action of χ_k can now be described as follows. If an irreducible $\rho_0 : \pi_1(K) \to SU(2)$ is not binary dihedral, the involution χ_k takes the circle $C(\rho_0)$ to the circle $C(\chi \cdot \rho_0)$. Since $\chi \cdot \rho_0$ is not conjugate to ρ_0, these two circles are disjoint from each other, and χ_k permutes them. If an irreducible $\rho_0 : \pi_1(K) \to SU(2)$ is binary dihedral, there exists $u \in SU(2)$ such that $uiu^{-1} = -i$ and $uA_iu^{-1} = -A_i$ for $i = 2, \ldots, n$. The irreducibility of ρ_0 also implies that $u^2 = -1$ so after conjugation we may assume that $u = k$. Now conjugate $\chi_k \cdot (i, e^{i\varphi}A_2e^{-i\varphi}, \ldots, e^{i\varphi}A_ne^{-i\varphi}; j) =$

$(-i, -e^{i\varphi}A_2e^{-i\varphi}, \ldots, -e^{i\varphi}A_ne^{-i\varphi}; j)$ by j to obtain

$$\begin{aligned}
&(i, j(-e^{i\varphi}A_2e^{-i\varphi})j^{-1}, \cdots, j(-e^{i\varphi}A_ne^{-i\varphi})j^{-1}; j) \\
&\quad = (i, -e^{-i\varphi}jA_2j^{-1}e^{i\varphi}, \cdots, -e^{-i\varphi}jA_nj^{-1}e^{i\varphi}; j) \\
&\qquad = (i, -(ie^{-i\varphi})\, kA_2k^{-1}(i^{-1}e^{i\varphi}), \cdots, -(ie^{-i\varphi})\, kA_nk^{-1}(i^{-1}e^{i\varphi}); j) \\
&\qquad\quad = (i, e^{i(\pi/2-\varphi)}A_2e^{-i(\pi/2-\varphi)}, \cdots, e^{i(\pi/2-\varphi)}A_ne^{-i(\pi/2-\varphi)}; j).
\end{aligned}$$

Therefore, χ_k acts on $C(\rho_0)$ by sending $e^{2i\varphi}$ to $-e^{-2i\varphi}$, which is an involution on the complex unit circle with two fixed points, i and $-i$.

Finally, observe that the quotient of $\mathcal{R}_0(K, SU(2))$ by the involution χ is precisely the space $\mathcal{R}_0(K, SO(3))$ of the conjugacy classes of representations $\operatorname{Ad}\rho_0 : \pi_1(K) \to SO(3)$. Since $H^2(K; \mathbb{Z}/2) = 0$, every $SO(3)$ representations lifts to an $SU(2)$ representations, hence $\mathcal{R}_0(K, SO(3))$ can also be described as the space of the conjugacy classes of representations $\pi_1(K) \to SO(3)$ sending the meridians to $SO(3)$ matrices of trace -1. Compose (2.8) with the projection $\mathcal{R}_0(K, SU(2)) \to \mathcal{R}_0(K, SO(3))$ to obtain a surjective map $\psi : \mathcal{PR}_c(K^\natural, SU(2)) \to \mathcal{R}_0(K, SO(3))$. The above discussion can now be summarized as follows.

2.3.1 Proposition. *The group* $H^1(K^\natural, \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *acts on the space* $\mathcal{PR}_c(K^\natural, SU(2))$ *preserving the fibers of the map* $\psi : \mathcal{PR}_c(K^\natural, SU(2)) \to \mathcal{R}_0(K, SO(3))$. *Furthermore,*

(a) for the unique reducible in $\mathcal{R}_0(K, SO(3))$*, the fiber of* ψ *consists of just one point, which is the conjugacy class of the special projective representation* α*. This point is fixed by both* χ_k *and* χ_ℓ*;*

(b) for any dihedral representation in $\mathcal{R}_0(K, SO(3))$, the fiber of ψ is a circle. The involution χ_k is a reflection of this circle with two fixed points, while χ_ℓ is the antipodal map;

(c) otherwise, the fiber of ψ consists of two circles. The involution χ_k permutes these circles, while χ_ℓ acts as the antipodal map on both.

2.3.3 Double branched covers

Next, we would like to describe the space $\mathcal{PR}_c(K^\natural, SU(2))$ using the equivariant theory of Section 2.2. We could proceed as in that section, by passing to the double branched cover $M \to S^3$ with branch set the link $k^\natural$ and working with the equivariant representations $\pi_1(M) \to SO(3)$. However, in the special case at hand, one can observe that M is simply the connected sum $Y \# \mathbb{R}\mathrm{P}^3$, where Y is the double branched cover of S^3 with branch set the knot k, hence the same information about $\mathcal{PR}_c(K^\natural, SU(2))$ can be extracted more easily by working directly with Y and using Proposition 2.3.1. The only missing step in this program is a description of $\mathcal{R}_0(K, SO(3))$ in terms of equivariant representations $\pi_1(Y) \to SO(3)$, which we will take up next.

Every representation $\rho : \pi_1(K) \to SO(3)$ which sends the meridians to $SO(3)$ matrices of trace -1, gives rise to a representation of the orbifold fundamental group $\pi_1^V(S^3, k) = \pi_1(K)/\langle \mu^2 = 1\rangle$, where we choose $\mu = a_1$ to be our meridian. The latter group can be included into the split orbifold exact sequence

$$1 \longrightarrow \pi_1(Y) \xrightarrow{\pi_*} \pi_1^V(S^3, k) \xrightarrow{j} \mathbb{Z}/2 \longrightarrow 1.$$

2.3.2 Proposition. *Let Y be the double branched cover of S^3 with branch set a knot k and let $\tau : Y \to Y$ be the covering translation. The pull back of representations via the map π_* in the orbifold exact sequence establishes a homeomorphism*

$$\pi^* : \mathcal{R}_0(K, SO(3)) \longrightarrow \mathcal{R}^\tau(Y, SO(3)),$$

where $\mathcal{R}^\tau(Y)$ is the fixed point set of the involution $\tau^* : \mathcal{R}(Y, SO(3)) \to \mathcal{R}(Y, SO(3))$. The unique reducible representation in $\mathcal{R}_0(K, SO(3))$ pulls back to the trivial representation of $\pi_1(Y)$, and the dihedral representations in $\mathcal{R}_0(K, SO(3))$ are the ones and only ones that pull back to reducible representations of $\pi_1(Y)$.

Proof. A slight modification of the argument of Proposition 2.2.7, see also [8, Proposition 3.3], establishes a homeomorphism between $\mathcal{R}_0(K, SO(3))$ and the subspace of $\mathcal{R}^\tau(Y, SO(3))$ consisting of the conjugacy classes of representations $\beta : \pi_1(Y) \to SO(3)$ such that $\tau^*\beta = u \cdot \beta \cdot u^{-1}$ for some $u \in SO(3)$ of order two. The proof of the first statement of the proposition will be complete after we show that this subspace in fact comprises the entire space $\mathcal{R}^\tau(Y, SO(3))$.

If $\beta : \pi_1(Y) \to SO(3)$ is reducible, it factors through a representation $H_1(Y) \to SO(2)$. According to Proposition 2.2.1, the involution τ_* acts on $H_1(Y)$ as multiplication by -1. Therefore, $\tau^*\beta = \beta^{-1}$, and the latter representation can obviously be conjugated to β by an element $u \in SO(3)$ of order two. If $\beta : \pi_1(Y) \to SO(3)$ is irreducible, the condition $\beta \in \mathrm{Fix}\,(\tau^*)$ implies that there exists a unique $u \in SO(3)$ such that $\tau^*\beta = u \cdot \beta \cdot u^{-1}$ and $u^2 = 1$. If $u = 1$,then $\tau^*\beta = \beta$, which implies that β is the pull back of a representation of $\pi_1^V(S^3, k)$ which sends the meridian μ to the identity matrix and hence factors through $\pi_1(S^3) = 1$. This contradicts the irreducibility of

β.

To prove the second statement of the proposition, observe that the homomorphism j in the above orbifold exact sequence, sending μ to the generator of $\mathbb{Z}/2$, is in fact the abelianization homomorphism. This implies that the unique reducible representation in $\mathcal{R}_0(K, SO(3))$ pulls back to the trivial representation of $\pi_1(Y)$. Since $\pi_1(Y)$ is the commutator subgroup of $\pi_1^V(S^3, k)$, any dihedral representation $\rho : \pi_1^V(S^3, k) \to O(2)$ must map $\pi_1(Y)$ to the commutator subgroup of $O(2)$, which happens to be $SO(2)$. This ensures that the pull back of ρ is reducible. Conversely, if the pull back of ρ is reducible, its image is contained in a copy of $SO(2)$, and the image of ρ itself in its 2-prime extension. The latter group is of course just a copy of $O(2) \subset SO(3)$. □

2.3.3 Remark. For future use note that, for any projective representation $\rho : \pi_1(K^\natural) \to SU(2)$ in $C(\rho_0)$ described by a tuple (2.7), the adjoint representation $\operatorname{Ad}\rho : \pi_1(K^\natural) \to SO(3)$ pulls back to an $SO(3)$ representation of $\pi_1(Y \# \mathbb{R}\mathrm{P}^3) = \pi_1(Y) * \mathbb{Z}/2$ of the form

$$\beta * \gamma : \ \pi_1(Y) * \mathbb{Z}/2 \to SO(3),$$

where $\beta = \pi^* \operatorname{Ad}\rho_0$ and $\gamma : \mathbb{Z}/2 \to SO(3)$ sends the generator of $\mathbb{Z}/2$ to $\operatorname{Ad} i \cdot \operatorname{Ad} j = \operatorname{Ad} k$. The representation $\beta * \gamma$ is equivariant in that $\tau^*(\beta * \gamma) = u \cdot (\beta * \gamma) \cdot u^{-1}$ with the conjugating element $u = \operatorname{Ad}\rho_0(a_1) = \operatorname{Ad} i$.

2.4 Knot homology: grading of the special generator

Given a knot $k \subset S^3$, we will continue using the notations K for its exterior and $K^\natural$ for the exterior of the two-component link $k^\natural = k \cup \ell$ obtained by connect summing k with the Hopf link H. The special projective representation $\alpha : \pi_1(K^\natural) \to SU(2)$, which sends all the meridians of k to i and the meridian of ℓ to j, is a generator in the chain complex $IC^\natural(k)$. In this section, we compute its Floer grading.

2.4.1 Theorem. *For any knot k in S^3, we have* $\operatorname{gr}(\alpha) = \operatorname{sign} k \pmod 4$.

Before we go on to prove this theorem recall that, according to [22, Proposition 4.4], the absolute Floer index of α is given by the formula

$$\operatorname{gr}(\alpha) \;=\; -\operatorname{ind}\mathcal{D}_{A'}(\alpha,\alpha) - \frac{3}{2}\,(\chi(W') + \sigma(W')) - \chi(S') \pmod 4, \tag{2.9}$$

where (W', S') is a cobordism of the pairs (S^3, H) and $(S^3, k^\natural)$ in the sense of [22, Section 4.3], and the two representations bearing the same name α are the special generators in the Floer chain complexes of the unknot and of the knot k. The operator $\mathcal{D}_{A'}(\alpha,\alpha)$ refers to the ASD operator on the non-compact manifold obtained from W' by attaching cylindrical ends to the two boundary components; this manifold is again called W'. The connection A' can be any connection on W' which is singular along the surface S' and which limits to flat connections with the holonomy α on the two ends. The index of $\mathcal{D}_{A'}(\alpha,\alpha)$ is understood as the L^2_δ index for a small $\delta > 0$.

2.4.1 Constructing the cobordism

Our calculation of the Floer index $\mathrm{gr}\,(\alpha)$ will use a specific cobordism (W', S') constructed as follows.

Let Σ be the double branched cover of S^3 with branch set the knot k. Choose a Seifert surface F' of k and push its interior slightly into the ball D^4 so that the resulting surface, which we still call F', is transversal to $\partial D^4 = S^3$. Let V be the double branched cover of D^4 with branch set F'. Then V is a smooth simply connected spin 4-manifold with boundary Σ, which admits a handle decomposition with only 0– and 2–handles, see Akbulut–Kirby [1, page 113].

Next, choose a point in the interior of the surface $F' \subset D^4$. Excising a small open 4-ball containing that point from (D^4, F') results in a manifold W_1' diffeomorphic to $I \times S^3$ together with the surface $F_1' = F' - \mathrm{int}(D^2)$ properly embedded into it, thereby providing a cobordism (W_1', F_1') from an unknot to the knot k. The double branched cover $W_1 \to W_1'$ with branch set F_1' is a cobordism from S^3 to Σ. The manifold W_1 is simply connected because it can be obtained from the simply connected manifold V by excising an open 4-ball.

Similarly, consider the manifold $W_2' = I \times S^3$ and surface $F_2' = I \times H \subset W_2'$ providing a product cobordism from the Hopf link H to itself. The double branched cover $W_2 \to W_2'$ with branch set F_2' is then a cobordism $W_2 = I \times \mathbb{R}\mathrm{P}^3$ from $\mathbb{R}\mathrm{P}^3$ to itself.

As the final step of the construction, consider a path γ_1' in the surface F_1' connecting its two boundary components. Similarly, consider a path γ_2' of the form $I \times \{p\}$ in the surface $F_2' = I \times H$. Remove tubular neighborhoods of these two paths and glue the resulting manifolds and surfaces together using an orientation reversing diffeomorphism $1 \times h : I \times S^2 \to I \times S^2$. The resulting pair (W', S') is the desired cobordism of the pairs (S^3, H) and $(S^3, k^\natural)$. One can easily see that

$$\chi(W') = \sigma(W') = 0 \quad \text{and} \quad \chi(S') = \chi(F') - 1. \tag{2.10}$$

Note that the double branched cover $W \to W'$ with branch set S' is a cobordism from $\mathbb{RP}^3$ to $\Sigma \# \mathbb{RP}^3$ which can be obtained from the cobordisms W_1 and W_2 by taking a connected sum along the paths $\gamma_1 \subset W_1$ and $\gamma_2 \subset W_2$ lifting, respectively, the paths γ_1' and γ_2'. To be precise,

$$W = W_1^\circ \cup W_2^\circ, \tag{2.11}$$

where W_1° and W_2° are obtained from W_1 and W_2 by removing tubular neighborhoods of γ_1 and γ_2. The identification in (2.11) is done along a copy of $I \times S^2$. In particular, we see that $\pi_1(W) = \mathbb{Z}/2$.

2.4.2 L^2–index

We will rely on Ruberman [33] and Taubes [42] in our index calculations. Let $\pi : W \to W'$ be the double branched cover with branch set S' constructed in the previous section, and $\tau : W \to W$ the covering translation. The non-trivial representation $\gamma : \pi_1(\mathbb{RP}^3) \to SO(3)$ and the representation $\theta * \gamma : \pi_1(\Sigma) * \pi_1(\mathbb{RP}^3) \to SO(3)$ obviously extend to a representation $\pi_1(W) \to SO(3)$, making W into a flat cobordism. This representation is equivariant with respect to τ, with the conjugating element of order two, hence it is of the form $\pi^*\rho$ for an orbifold representation $\rho : \pi_1^V(W', S') \to SO(3)$. The representation ρ restricts to the representations α on the two ends of W'.

Let A and A' be flat connections on W and W' whose holonomies are, respectively, $\pi^*\rho$ and ρ. We will use A' as the twisting connection of the operator $\mathcal{D}_{A'}(\alpha, \alpha)$. Instead of computing the index of this operator we will compute the equivariant index $\operatorname{ind} \mathcal{D}_A^\tau(\gamma, \theta * \gamma)$ of its pull back to W. The latter index equals minus the equivariant index of the elliptic complex

$$0 \longrightarrow \Omega^0(W, \operatorname{ad} P) \xrightarrow{\;-d_A\;} \Omega^1(W, \operatorname{ad} P) \xrightarrow{\;d_A^+\;} \Omega^2_+(W, \operatorname{ad} P).$$

The equivariance here is understood with respect to a lift of $\tau : W \to W$ to the bundle $\operatorname{ad} P$ which has second order on the fiber. The connection A is equivariant with respect to this lift.

The zeroth equivariant cohomology of the above elliptic complex vanishes because the lift of τ acts as minus identity on $H^0(W; \operatorname{ad} A) = \mathbb{R}$, compare with Example 2.2.6. This vanishing result can also be derived from the irreducibility of the singular connection A'.

To compute the remaining cohomology, notice that the coefficient bundle $\operatorname{ad} P$ splits into a sum of two bundles, $\operatorname{ad} P = \mathbb{R} \oplus L$, with the lift of τ acting as identity on $\mathbb{R}$ and as multiplication by -1 on L. The above elliptic complex splits correspondingly into a sum of two elliptic complexes, one with the trivial real coefficients, and the other with coefficients in L. Applying [33, Proposition 4.1] to the former complex and [33, Corollary 4.2] to the latter, we conclude that the non-equivariant cohomology of the above complex in degrees one and two is isomorphic to the reduced singular cohomology of W with coefficients in $\operatorname{ad} P$. Restricting to the equivariant part identifies the equivariant cohomology of the above complex in degrees one and two with the reduced equivariant singular cohomology of W with coefficients in $\operatorname{ad} P$. This argument reduces the index problem to computing the cohomology groups

$$H^k(W; \operatorname{ad} \pi^*\gamma) = H^k(W; \mathbb{R}) \oplus H^k(W; \mathbb{R}_-) \oplus H^k(W; \mathbb{R}_-), \quad k = 1, 2,$$

and their equivariant versions, where $\mathbb{R}_-$ stands for the real line coefficients on which $\mathbb{Z}/2$ acts as multiplication by -1.

2.4.3 Trivial coefficients

Our computation will be based on the Mayer–Vietoris exact sequence applied twice, first to compute cohomology of W_1° and W_2°, and then to compute cohomology of $W = W_1^\circ \cup W_2^\circ$. The cohomology groups of W_1° and $W_1 = W_1^\circ \cup (I \times D^3)$ are related by the following long exact sequence

$$\begin{array}{llllll} 0 \longrightarrow & H^1(W_1; \mathbb{R}) & \longrightarrow & H^1(W_1^\circ; \mathbb{R}) & \longrightarrow & 0 \longrightarrow \\ \longrightarrow & H^2(W_1; \mathbb{R}) & \longrightarrow & H^2(W_1^\circ; \mathbb{R}) & \longrightarrow & H^2(I \times S^2; \mathbb{R}) \longrightarrow \\ \xrightarrow{\delta} & H^3(W_1; \mathbb{R}) & \longrightarrow & H^3(W_1^\circ; \mathbb{R}) & \longrightarrow & 0, \end{array}$$

Since W_1 and therefore W_1° are simply connected, both $H^1(W_1; \mathbb{R})$ and $H^1(W_1^\circ; \mathbb{R})$ vanish. Applying the Poincaré–Lefschetz duality to the manifold W_1 and using the long exact sequence of the pair $(W_1, \partial W_1)$ we obtain

$$H^3(W_1;\mathbb{R}) = H_1(W_1,\partial W_1;\mathbb{R}) = \widetilde{H}_0(\partial W_1;\mathbb{R}) = \mathbb{R}.$$

Similarly, viewing W_1° as a manifold whose boundary is a connected sum of the two boundary components of W_1, we obtain

$$H^3(W_1^\circ;\mathbb{R}) = H_1(W_1^\circ,\partial W_1^\circ;\mathbb{R}) = \widetilde{H}_0(\partial W_1^\circ;\mathbb{R}) = 0.$$

Therefore, the connecting homomorphism δ in the above exact sequence must be an isomorphism, which leads to the isomorphisms

$$H^2(W_1^\circ;\mathbb{R}) = H^2(W_1;\mathbb{R}) = H^2(V;\mathbb{R}).$$

A similar long exact sequence, relating the cohomology of W_2° and $W_2 = W_2^\circ \cup (I \times D^3)$, implies that

$$H^2(W_2^\circ;\mathbb{R}) = H^2(W_2;\mathbb{R}) = H^2(\mathbb{R}\mathrm{P}^3;\mathbb{R}) = 0.$$

Since $\pi_1(W_2) = \pi_1(W_2^\circ) = \mathbb{Z}/2$, both $H^1(W_2;\mathbb{R})$ and $H^1(W_2^\circ;\mathbb{R})$ vanish. The Mayer–Vietoris exact sequence of the splitting $W = W_1^\circ \cup W_2^\circ$,

$$
\begin{aligned}
0 \longrightarrow H^1(W;\mathbb{R}) \longrightarrow H^1(W_1^\circ;\mathbb{R}) \oplus H^1(W_2^\circ;\mathbb{R}) &\longrightarrow 0 \longrightarrow \\
\longrightarrow H^2(W;\mathbb{R}) \longrightarrow H^2(W_1^\circ;\mathbb{R}) \oplus H^2(W_2^\circ;\mathbb{R}) &\longrightarrow H^2(I\times S^2;\mathbb{R}) \to \\
\longrightarrow H^3(W;\mathbb{R}) \longrightarrow H^3(W_1^\circ;\mathbb{R}) \oplus H^3(W_2^\circ;\mathbb{R}) &\longrightarrow 0
\end{aligned}
$$

together with the isomorphisms $H^3(W;\mathbb{R}) = H_1(W,\partial W;\mathbb{R}) = \widetilde{H}_0(\partial W;\mathbb{R}) = \mathbb{R}$ and $\pi_1(W) = \mathbb{Z}/2$, implies that

$$H^1(W;\mathbb{R}) = 0 \quad \text{and} \quad H^2(W;\mathbb{R}) = H^2(V;\mathbb{R}).$$

2.4.4 Twisted coefficients

We will now do a similar calculation using the Mayer–Vietoris sequence of $W = W_1^\circ \cup W_2^\circ$ with twisted coefficients. Since W_1° is simply connected, the twisted coefficients $\mathbb{R}_-$ pull back to the trivial $\mathbb{R}$–coefficients over W_1° and the cohomology calculations from the previous section are unchanged. A direct calculation using homotopy equivalences $W_2 \simeq \mathbb{R}\mathrm{P}^3$ and $W_2^\circ \simeq \mathbb{R}\mathrm{P}^2$ shows that

$$H^1(W_2^\circ;\mathbb{R}_-) = 0 \quad \text{and} \quad H^2(W_2^\circ;\mathbb{R}_-) = \mathbb{R}.$$

The latter isomorphism is induced by the inclusion $I \times S^2 \to W_2^\circ$, which can be easily seen from the Mayer–Vietoris exact sequence of $W_2 = W_2^\circ \cup (I \times D^3)$. Now, consider the Mayer–Vietoris exact sequence of the splitting $W = W_1^\circ \cup W_2^\circ$ with twisted $\mathbb{R}$–coefficients,

$$0 \longrightarrow H^1(W;\mathbb{R}_-) \to H^1(W_1^\circ;\mathbb{R}) \oplus H^1(W_2^\circ;\mathbb{R}_-) \longrightarrow 0 \longrightarrow$$
$$\longrightarrow H^2(W;\mathbb{R}_-) \to H^2(W_1^\circ;\mathbb{R}) \oplus H^2(W_2^\circ;\mathbb{R}_-) \to H^2(I \times S^2;\mathbb{R}) \longrightarrow$$
$$\longrightarrow H^3(W;\mathbb{R}_-) \to H^3(W_1^\circ;\mathbb{R}) \oplus H^3(W_2^\circ;\mathbb{R}_-) \longrightarrow 0.$$

Keeping in mind that the map $H^2(W_1^\circ;\mathbb{R}) \to H^2(I \times S^2;\mathbb{R})$ in this sequence is zero and the map $H^2(W_2^\circ;\mathbb{R}_-) \to H^2(I \times S^2;\mathbb{R})$ is an isomorphism $\mathbb{R} \to \mathbb{R}$, we conclude that

$$H^1(W;\mathbb{R}_-) = 0 \quad \text{and} \quad H^2(W;\mathbb{R}_-) = H_2(V;\mathbb{R}).$$

2.4.5 Equivariant cohomology

Combining results of the previous two sections we obtain $H^1(W;\operatorname{ad} P) = 0$ and $H^2(W;\operatorname{ad} P) = H^2(V;\mathbb{R}^3)$. The action of τ is compatible with these isomorphisms, from which we immediately conclude that

$$H^1_\tau(W;\operatorname{ad} P) = 0$$

and $H^2_\tau(W;\operatorname{ad} P)$ is the fixed point set of the map $H^2(V;\mathbb{R}^3) \to H^2(V;\mathbb{R}^3)$ obtained by twisting $\tau^* : H^2(V;\mathbb{R}) \to H^2(V;\mathbb{R})$ by the action on the coefficients $\mathbb{R}^3 \to \mathbb{R}^3$. The involution τ^* is minus the identity, which follows from the usual transfer argument applied to the covering $V \to D^4$, while the action on the coefficients is given by an $SO(3)$ operator of second order. Such an operator must have a single eigenvalue 1 and a double eigenvalue -1, which leads us to the conclusion that $\operatorname{rk} H^2_\tau(W;\operatorname{ad} P) =$

$2 \cdot b_2(V)$. Similarly,

$$\operatorname{rk} H^2_{\tau,+}(W; \operatorname{ad} P) = 2 \cdot b_2^+(V).$$

2.4.6 Proof of Theorem 2.4.1

It follows from the discussion in Section 2.4.2 and the calculation in Section 2.4.5 that

$$\operatorname{ind} \mathcal{D}_{A'}(\alpha, \alpha) \;=\; \operatorname{rk} H^1_\tau(W; \operatorname{ad} P) - \operatorname{rk} H^2_{+,\tau}(W; \operatorname{ad} P) = -2 \cdot b_2^+(V).$$

Taking into account (2.9) and (2.10), we obtain the formula

$$\operatorname{gr}(\alpha) \;=\; 2 \cdot b_2^+(V) \;-\; \chi(F') + 1 \pmod 4.$$

To simplify it, let us compute $\chi(V)$ in two different ways: $\chi(V) = 1 + b_2^+(V) + b_2^-(V)$ by definition, and $\chi(V) = 2\chi(D^4) - \chi(F') = 2 - \chi(F')$ using the fact that V is a double branched cover of D^4 with branch set F'. Combining these formulas with the knot signature formula of Viro [43], we obtain the desired result,

$$\operatorname{gr}(\alpha) \;=\; -\operatorname{sign} V \;=\; -\operatorname{sign} k \;=\; \operatorname{sign} k \pmod 4.$$

2.5 Computations for some classes of knots

Proposition 2.3.1 identified the critical points of the Chern–Simons functional with the fibers of the map $\psi : \mathcal{PR}_c(K^\natural, SU(2)) \to \mathcal{R}_0(K, SO(3))$. Assuming that the space $\mathcal{R}_0(K, SO(3))$ is non-degenerate, all of these fibers with the exception of the special generator α are Morse–Bott circles. The actual generators of the chain complex of $I^\natural(k)$ are then obtained by perturbing each Morse–Bott circle into two points [17]. Our calculation will depend on whether an irreducible trace-free representation $\rho_0 : \pi_1 K \to SO(3)$ giving rise to the Morse–Bott circle $C(\rho_0)$ is dihedral or not. The two cases will be considered separately starting with the easier case when ρ_0 is not dihedral. If $\mathcal{R}_0(K, SO(3))$ fails to be non-degenerate, similar results hold after additional perturbations.

2.5.1 Non-dihedral representations

Let $\rho_0 : \pi_1 K \to SO(3)$ be an irreducible trace-free representation which is not dihedral, and assume that it is non-degenerate. Proposition 2.3.1 (c) then tells us that the fiber $C(\rho_0)$ consists of two circles. Perturbing each of these circles into two isolated points contributes four generators the the chain complex.

This completes the calculation of the generators of the Floer chain complex $IC^\natural(k)$ for an important special class of knots $k \subset S^3$ with $\Delta(-1) = 1$, where $\Delta(t)$ is the Alexander polynomial of k normalized so that $\Delta(t) = \Delta(t^{-1})$ and $\Delta(1) = 1$. These are precisely the knots $k \subset S^3$ whose double branched covers Y are integral homology spheres, and which are known to have no dihedral representations in $\mathcal{R}_0(K, SO(3))$; see [21, Theorem 10] or [8, Proposition 3.4]. Also note that $\operatorname{sign} k = 0 \pmod 8$ for all such knots because $1 = \Delta(-1) = \det(i \cdot Q)$, where Q is the (even) quadratic form of the knot.

2.5.1 Example. Let p and q be positive integers which are odd and relatively prime. The double branched cover of the right handed (p, q)–torus knot $T_{p,q}$ is the Brieskorn homology sphere $\Sigma(2, p, q)$. According to Fintushel–Stern [11, Proposition 2.5], all irreducible $SO(3)$ representations of the fundamental group of $\Sigma(2,p,q)$ are non-degenerate and, up to conjugacy, there are $a = -\operatorname{sign}(T_{p,q})/4$ of them. All of these representations are equivariant [8, Section 4.2] hence each of them contributes four generators to the chain complex of $I^\natural(T_{p,q})$. Since $\operatorname{sign}(T_{p,q}) = 0 \mod 4$, the special generator resides in degree zero, and we conclude that the rank of the chain complex $IC^\natural(T_{p,q})$ is $4a+1$.

2.5.2 Example. Let p, q, and r be pairwise relatively prime positive integers, and view the Brieskorn homology sphere $\Sigma(p,q,r)$ as the link of singularity of the complex polynomial $x^p + y^q + z^r = 0$. The involution induced by the complex conjugation on the link makes $\Sigma(p,q,r)$ into a double branched cover of S^3 with branch set a Montesinos knot $k(p,q,r)$, see for instance [37, Section 7]. According to Fintushel–Stern [11, Proposition 2.5], all irreducible $SO(3)$ representations of the fundamental group of $\Sigma(p,q,r)$ are non-degenerate, and there are $b = -2\,\lambda(\Sigma(p,q,r))$ of them, where $\lambda(\Sigma(p,q,r))$ is the Casson invariant of $\Sigma(p,q,r)$. These representations are all equivariant [37, Proposition 8] hence each of them contributes four generators to the Floer chain complex of $I^\natural(k(p,q,r))$. Since $\operatorname{sign} k(p,q,r) = 0 \pmod 4$, the special generator has degree zero, and the rank of the chain complex is $IC^\natural(k(p,q,r))$ is $4b+1$.

For example, $\Sigma(2,3,7)$ is a double branched cover of S^3 whose branch set $k(2,3,7)$ is the pretzel knot $P(-2,3,7)$. Since $\lambda(\Sigma(2,3,7)) = -1$, we conclude that the rank of the chain complex is $IC^\natural(P(-2,3,7))$ is 9. This is consistent with the calculation in [15, Section 5].

One can show that the same formula holds for all Brieskorn homology spheres $\Sigma(a_1,\dots,a_n)$ and the corresponding Montesinos knots $k(a_1,\dots,a_n)$ using the τ–

equivariant perturbations of [39] modeled after the perturbations of Kirk and Klassen [20]. Note that the action of $H^1(K;\mathbb{Z}/2)$ on the conjugacy classes of projective representations is free hence it causes no equivariant transversally issues.

2.5.2 Dihedral representations

Let $\rho_0 : \pi_1 K \to SO(3)$ be an irreducible trace-free representation which is dihedral, and assume that it is non-degenerate. Proposition 2.3.1 (c) then tells us that the fiber $C(\rho_0)$ consists of one circle. After perturbation, this circle contributes two generators to the chain complex.

2.5.3 Two-bridge knots

Let p be an odd positive integer and k a two-bridge knot of type $-p/q$ in the 3-sphere. Its double branched cover Y is the lens space $L(p,q)$ oriented as the $(-p/q)$–surgery on an unknot in S^3. One can easily check that all representations $\beta : \pi_1(Y) \to SO(3)$ are equivariant.

For example, the figure-eight knot k is the two-bridge knot of type $-5/3$. Its double branched cover is the lens space $L(5,3)$ whose fundamental group has no irreducible representations and has two non-trivial reducible representations, up to conjugacy. These two representations contribute the 4 generators to the chain complex and therefore the rank of the chain complex $IC^\natural(k)$ is 5. The Khovanov homology of the mirror image of k has rank 5, hence we conclude from the Kronheimer–Mrowka spectral sequence that the generators of the chain complex are also the generators of $I^\natural(k)$.

2.5.4 General torus knots

Let p and q be positive relatively prime integers. The double branched cover Y of a torus knot $T_{p,q}$ is an integral homology sphere if and only if both p and q are odd, which is the case we studied in Example 2.5.1. In this section, we will assume that p is odd and $q = 2r$ is even. Then Y can be viewed as the link of singularity at zero of the complex polynomial $x^2 + y^p + z^{2r} = 0$, with the covering translation given by the formula $\tau(x, y, z) = (-x, y, z)$. Neumann and Raymond [31] showed that Y admits a fixed point free circle action making it into a Seifert fibration over S^2 with the Seifert invariants $\{(a_1, b_1), \ldots, (a_n, b_n)\} = \{(1, b_1), (p, b_2), (p, b_2), (r, b_3)\}$, where $b_1 \cdot pr + 2b_2 \cdot r + b_3 \cdot p = 1$. The involution τ is a part of the circle action, which implies that all reducible representations $\beta : \pi_1(Y) \to SO(3)$ are equivariant.

Note that $\operatorname{sign}(T_{p,q}) = (p-1)(q-1) \bmod 4$ for all relatively prime p and q, even or odd, see for instance [3, Proposition 4.1].

2.5.3 Example. We will illustrate this calculation for the torus knot $T_{3,4}$. The Seifert invariants of the manifold Y are $\{(1, -1), (3, 1), (3, 1), (2, 1)\}$ and its fundamental group has presentation

$$\pi_1(Y) = \langle x_1, x_2, x_3, x_4, h \mid h \text{ central},\ x_1 = h,\ x_2^3 = h^{-1}, \\ x_3^3 = h^{-1},\ x_4^2 = h^{-1},\ x_1\, x_2\, x_3\, x_4 = 1 \rangle$$

It admits one non-trivial reducible representation β with $\beta(x_1) = \beta(x_4) = 1$, $\beta(x_2) = \operatorname{Ad}(\exp(2\pi i/3))$ and $\beta(x_3) = \operatorname{Ad}(\exp(-2\pi i/3))$, which contributes 2 generators to the chain complex. One can easily see that $\pi_1(Y)$ admits exactly one irreducible representation, therefore, the rank of the chain complex $IC^\natural(T_{3,4})$ is 7.

2.5.5 General Montesinos knots

Let $(a_1, b_1), \dots, (a_n, b_n)$ be pairs of integers such that, for each i, the integers a_i and b_i are relatively prime and a_i is positive. Burde and Zieschang [5, Chapter 7] associated with these pairs a Montesinos link $K((a_1, b_1), \dots, (a_n, b_n))$ and showed that its double branched cover is a Seifert fibered manifold Y with unnormalized Seifert invariants $(a_1, b_1), \dots, (a_n, b_n)$. In particular,

$$\pi_1(Y) = \langle\, x_1, \dots x_n, h \mid h \text{ central},\; x_i^{a_i} = h^{-b_i},\; x_1 \cdots x_n = 1 \,\rangle,$$

with the covering translation $\tau : Y \to Y$ acting on the fundamental group by the rule

$$\tau_*(h) = h^{-1},\;\; \tau_*(x_i) = x_1 \cdots x_{i-1} x_i^{-1} x_{i-1}^{-1} \cdots x_1^{-1},\; i = 1, \dots, n,$$

see Burde–Zieschang [5, Proposition 12.30]. Two-bridge and pretzel knots and links are special cases of Montesinos knots and links. In this section, we will only be interested in Montesinos knots.

Let k be a Montesinos knot $K((a_1, b_1), \dots, (a_n, b_n))$ and Y the double branch cover of S^3 with branch set k. The manifold Y need not be an integral homology sphere; in fact, one can easily see that its first homology is a finite abelian group of the order

$$|H_1(Y;\mathbb{Z})| \;=\; \left(\sum_{i=1}^{n}\; b_i/a_i\right) \cdot a_1 \cdots a_n.$$

Note that this integer is always odd because Y is a $\mathbb{Z}/2$ homology sphere.

All reducible representations $\beta : \pi_1(Y) \to SO(3)$ are equivariant because the involution $\tau_* : H_1(Y) \to H_1(Y)$ acts as multiplication by -1, see Proposition 2.2.1. There are no irreducible representations for $n \leq 2$. If $n = 3$, all irreducible representations are non-degenerate and equivariant, which can be shown using a minor modification of the arguments of [11, Proposition 2.5] and [37, Proposition 30]. For $n \geq 4$, one encounters positive dimensional manifolds of representations; the action of τ^* on these manifolds can be described as in [39], together with equivariant perturbations making them non-degenerate. This discussion followed by Propositions 2.3.1 and 2.3.2 identifies the generators of the chain complex $IC^\natural(k)$ for all Montesinos knots in terms of representations for Seifert fibered manifolds, which are well known. An independent calculation of the generators of $IC^\natural(k)$ for pretzel knots k with $n = 3$ can be found in Zentner [44].

2.6 Floer homology of other two-component links

This section deals with general two-component links $\mathcal{L} = \ell_1 \cup \ell_2$ and not just the links $\mathcal{L} = k^\natural$ used in the definition of the knot Floer homology $I^\natural(k)$. After computing the Euler characteristic of $I_*(\Sigma, \mathcal{L})$, we explicitly compute the generators of the Floer chain groups for some links $\mathcal{L}$ with particularly simple double branched covers.

2.6.1 Euler characteristic

Let $\mathcal{L} = \ell_1 \cup \ell_2$ be a two-component link in an integral homology sphere Σ. The linking number $\ell k(\ell_1, \ell_2)$ is well defined up to a sign by choosing an arbitrary orientation on $\mathcal{L}$.

2.6.1 Theorem. *The Euler characteristic of the Floer homology $I_*(\Sigma, \mathcal{L})$ of a two-component link $\mathcal{L} = \ell_1 \cup \ell_2$ equals $\pm\,\ell k(\ell_1, \ell_2)$.*

Proof. The Floer excision principle can be used as in [22] to establish an isomorphism between $I_*(\Sigma, \mathcal{L})$ and the sutured Floer homology of $\mathcal{L}$. The latter is the Floer homology of the 3-manifold X_φ obtained by identifying the two boundary components of $S^3 - \operatorname{int} N(\mathcal{L})$ via an orientation reversing homeomorphism $\varphi : T^2 \to T^2$. According to [16, Lemma 2.1], the homeomorphism φ can be chosen so that X_φ has integral homology of $S^1 \times S^2$. The result then follows from [16, Theorem 2.3] which asserts that the Euler characteristic of the sutured Floer homology of $\mathcal{L}$ equals $\pm\,\ell k(\ell_1, \ell_2)$. □

Theorem 2.6.1 implies in particular that the Euler characteristic of $I^\natural(k)$ equals ± 1, which is the linking number of the two components of the link $k^\natural$. This also follows from the fact that the critical point set of the orbifold Chern–Simons functional used to define $I^\natural(k)$ consists of an isolated point and finitely many isolated circles, possibly after a perturbation. An absolute grading on $I^\natural(k)$ was fixed in [22] so that the grading of the isolated point is even; this is consistent with our Theorem 2.4.1 because $\operatorname{sign} k$ is always even. The Euler characteristic of $I^\natural(k)$ then equals $+1$. We do not know how to fix an absolute grading on $I_*(\Sigma, \mathcal{L})$ for a general two-component link $\mathcal{L}$.

2.6.2 Pretzel link $P(2, -3, -6)$

This is the two-component link $\mathcal{L}$ whose double branched cover is the Seifert fibered manifold M with unnormalized Seifert invariants $(2, 1)$, $(3, -1)$, and $(6, -1)$, see for instance [40, Section 4]. In particular,

$$\pi_1(M) = \langle\, x, y, z, h \mid h \text{ central},\ x^2 = h^{-1},\ y^3 = h,\ z^6 = h,\ xyz = 1 \,\rangle,$$

with the covering translation $\tau : M \to M$ acting on the fundamental group by the rule

$$\tau_*(h) = h^{-1},\ \ \tau_*(x) = x^{-1},\ \ \tau_*(y) = xy^{-1}x^{-1},\ \ \tau_*(z) = xyz^{-1}y^{-1}x^{-1},$$

see Burde–Zieschang [5, Proposition 12.30]. The manifold M has integral homology of $S^1 \times S^2$. In fact, it can be obtained by 0–surgery on the right-handed trefoil so that $\pi_1(M) = \pi_1(K)/\langle \ell \rangle$, where K is the exterior of the trefoil and ℓ is its longitude. The relation $\ell = 1$ shows up as the relation $z^6 = h$ in the above presentation of $\pi_1(M)$.

We will use this surgery presentation of M to describe representations of $\pi_1(M) \to SO(3)$ with non-trivial $w_2 \in H^2(M; \mathbb{Z}/2) = \mathbb{Z}/2$. According to Example 2.1.2, the conjugacy classes of such representations are in one-to-two correspondence with the conjugacy classes of representations $\rho : \pi_1(K) \to SU(2)$ such that $\rho(\ell) = -1$. In the terminology of Section 2.1.2, these ρ are projective representations $\rho : \pi_1(M) \to SU(2)$, and the group $H^1(M; \mathbb{Z}/2) = \mathbb{Z}/2$ acts on them freely providing the claimed one-to-two correspondence. Therefore, we wish to find all the $SU(2)$ matrices $\rho(h)$, $\rho(x)$, $\rho(y)$, and $\rho(z)$ such that

$$\rho(x)^2 = \rho(h)^{-1},\ \ \rho(y)^3 = \rho(h),\ \ \rho(z)^6 = -\rho(h),\ \ \rho(x)\rho(y)\rho(z) = 1,$$

and $\rho(h)$ commutes with $\rho(x)$, $\rho(y)$, and $\rho(z)$. Since ρ is irreducible, we conclude as in Fintushel–Stern [11, Section 2] that $\rho(h) = -1$ and that $\rho(x)$ is conjugate to i, $\rho(y)$ is conjugate to $e^{\pi i/3}$, and $\rho(z)$ is conjugate to either $e^{\pi i/3}$ or $e^{2\pi i/3}$. These give rise to two conjugacy classes of projective representations $\rho : \pi_1(M) \to SU(2)$ corresponding to a single conjugacy class of representations $\operatorname{Ad}\rho : \pi_1(M) \to SO(3)$.

The arguments of [11, Proposition 2.5] and [37, Proposition 8] can be easily adapted to conclude that the representation $\operatorname{Ad}\rho$ is non-degenerate and equivariant. It gives rise to a single $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ orbit of generators in $IC_*(S^3, \mathcal{L})$, and therefore the rank of the chain complex is 4.

2.6.2 Remark. A more precise result can be obtained using the isomorphism between $I_*(S^3, \mathcal{L})$ and the sutured Floer homology of $\mathcal{L}$ defined in [23]. The latter is the Floer homology of the manifold X_φ obtained by identifying the two boundary components of $X = S^3 - \operatorname{int} N(\mathcal{L})$ via an orientation reversing homeomorphism $\varphi : T^2 \to T^2$. A surgery description of X_φ can be found in [16]; computing its Floer homology is then an exercise in applying the Floer exact triangle to this surgery description. The resulting Floer homology groups are free abelian of the ranks $(2, 0, 2, 0)$ up to cyclic permutation.

2.7 Homology of double branched covers

In this section, we will compute the groups $H_*(M; \mathbb{Z}/2)$ using the transfer homomorphism approach of [26].

The transfer homomorphisms can be defined in the following two equivalent ways, see for instance [10, Section 3]. For each singular simplex $\sigma : \Delta \to \Sigma$, choose a lift $\tilde{\sigma} : \Delta \to M$ and define the chain map $\pi_! : C_*(\Sigma) \longrightarrow C_*(M)$ by the formula

$\pi_!(\sigma) = \tilde{\sigma} + \tau \circ \tilde{\sigma}$. This map is obviously independent of the choice of $\tilde{\sigma}$, and it induces homomorphisms $\pi_! : H_*(\Sigma) \longrightarrow H_*(M)$ and $\pi^! : H^*(M) \longrightarrow H^*(\Sigma)$ in homology and cohomology with arbitrary coefficients, called transfer homomorphisms. Another way to define $\pi_!$ is as the map that makes the following digram commute,

$$\begin{array}{ccc} H_*(M) & \xleftarrow{\text{PD}} & H^*(M) \\ \uparrow \pi_! & & \uparrow \pi^* \\ H_*(\Sigma) & \xleftarrow{\text{PD}} & H^*(\Sigma) \end{array}$$

where PD stands for the Poincaré duality isomorphism, and similarly for $\pi^!$.

From now on, all chain complexes and (co)homology will be assumed to have $\mathbb{Z}/2$ coefficients. It is then immediate from the definition of $\pi_! : C_*(\Sigma) \longrightarrow C_*(M)$ that $\ker \pi_! = C_*(\mathcal{L})$ and that we have a short exact sequence of chain complexes

$$0 \longrightarrow C_*(\Sigma, \mathcal{L}) \xrightarrow{\pi_!} C_*(M) \xrightarrow{\pi_*} C_*(\Sigma) \longrightarrow 0$$

This exact sequence induces long exact sequences in homology

$$\begin{array}{l} 0 \longrightarrow H_3(\Sigma, \mathcal{L}) \xrightarrow{\pi_!} H_3(M) \longrightarrow H_3(\Sigma) \longrightarrow \\ \longrightarrow H_2(\Sigma, \mathcal{L}) \xrightarrow{\pi_!} H_2(M) \longrightarrow H_2(\Sigma) \longrightarrow \\ \longrightarrow H_1(\Sigma, \mathcal{L}) \xrightarrow{\pi_!} H_1(M) \longrightarrow H_1(\Sigma) \longrightarrow 0 \end{array}$$

and in cohomology

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^1(\Sigma) & \longrightarrow & H^1(M) & \xrightarrow{\pi^!} & H^1(\Sigma,\mathcal{L}) & \longrightarrow & \\ \longrightarrow & H^2(\Sigma) & \longrightarrow & H^2(M) & \xrightarrow{\pi^!} & H^2(\Sigma,\mathcal{L}) & \longrightarrow & \\ \longrightarrow & H^3(\Sigma) & \longrightarrow & H^3(M) & \xrightarrow{\pi^!} & H^3(\Sigma,\mathcal{L}) & \longrightarrow & 0 \end{array}$$

Combining these with the long exact sequence of the pair $(\Sigma, \mathcal{L})$ we obtain the following result.

2.7.1 Proposition. *Let $\pi : M \longrightarrow \Sigma$ be a double branched cover over an integral homology sphere Σ with branching set a two-component link $\mathcal{L}$. Then $H_i(M;\mathbb{Z}/2) = H^i(M;\mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$ if $i = 0, 1, 2, 3$, and is zero otherwise.*

2.7.1 The cup-product on $H^*(M;\mathbb{Z}/2)$

This section is devoted to the proof of the following result. We continue working with $\mathbb{Z}/2$ coefficients.

2.7.2 Proposition. *The cup-product $H^1(M) \times H^1(M) \to H^2(M)$ is the bilinear form $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$ with the matrix $\ell k(\ell_1, \ell_2) \pmod 2$.*

Proof. We will reduce the cup-product calculation to intersection theory using the commutative diagram

$$\begin{array}{ccc} H_2(M) \times H_2(M) & \xrightarrow{\quad \cdot \quad} & H_1(M) \\ \text{PD} \uparrow & & \uparrow \text{PD} \\ H^1(M) \times H^1(M) & \xrightarrow{\quad \cup \quad} & H^2(M) \end{array}$$

where PD stands for the Poincaré duality isomorphisms and $\cdot$ for the intersection product. The transfer homomorphism $\pi_! : H_*(\Sigma, \mathcal{L}) \to H_*(M)$ will give us explicit generators of $H_1(M)$ and $H_2(M)$ that we need to proceed with this approach.

We begin with the group $H_1(M)$. Note that $H_1(\Sigma, \mathcal{L}) = \mathbb{Z}/2$ is generated by the homology class $[w]$ of any embedded arc $w \subset \Sigma$ whose endpoints belong to two different components of $\mathcal{L}$. The transfer homomorphism $\pi_! : H_1(\Sigma, \mathcal{L}) \to H_1(M)$ maps the homology class of w to that of the circle $\pi^{-1}(w)$. Since $\pi_!$ is an isomorphism, we conclude that the circle $\pi^{-1}(w)$ represents a generator of $H_1(M)$.

To describe a generator of $H_2(M)$, observe that $H_2(\Sigma, \mathcal{L}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is generated by the homology classes of Seifert surfaces S_1 and S_2 of the knots ℓ_1 and ℓ_2. We will assume that S_1 and S_2 intersect transversely in a finite number of circles and arcs, and note that $S_1 \cap S_2$ is homologous to $\ell k(\ell_1, \ell_2) \cdot w$. We claim that the closed orientable surfaces $\pi^{-1}(S_1)$ and $\pi^{-1}(S_2)$, representing the homology classes $\pi_!([S_1])$ and $\pi_!([S_2])$, are homologous to each other and generate $H_2(M)$. To see this, we will appeal to Theorem 2 of [26], which supplies us with the commutative diagram with an exact row,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_3(\Sigma) & \xrightarrow{d_*} & H_2(\Sigma, \mathcal{L}) & \xrightarrow{\pi_!} & H_2(M) & \longrightarrow & 0 \\ & & & \searrow^{f} & \downarrow{\partial_*} & & & & \\ & & & & H_1(\mathcal{L}) & & & & \end{array}$$

where $f([\Sigma]) = [\ell_1] + [\ell_2]$ and ∂_* is the connecting homomorphism in the long exact sequence of the pair $(\Sigma, \mathcal{L})$. One can easily see that ∂_* is an isomorphism. Since $\partial_*([S_1] + [S_2]) = [\ell_1] + [\ell_2] = f([\Sigma])$ we conclude that $[S_1] + [S_2] \in \operatorname{im} d_* = \ker \pi_!$ and hence $\pi_!\,([S_1]) = \pi_!\,([S_2])$ is a generator of $H_2(M)$.

The calculation of the intersection form $H_2(M) \times H_2(M) \to H_1(M)$ is now completed as follows:

$$\begin{aligned} [\pi^{-1}(S_1)] \cdot [\pi^{-1}(S_2)] &= [\pi^{-1}(S_1) \cap \pi^{-1}(S_2)] \\ &= [\pi^{-1}(S_1 \cap S_2)] = \ell k(\ell_1, \ell_2) \cdot [\pi^{-1}(w)]. \end{aligned}$$

$\square$

2.7.3 Remark. Let $\beta \in H^1(M)$ be a generator and assume that $\ell k(\ell_1, \ell_2)$ is odd. Then $\beta \cup \beta \in H^2(M)$ is non-trivial, and a straightforward argument with the Poincaré duality shows that $\beta \cup \beta \cup \beta$ generates $H^3(M)$. If $\ell k(\ell_1, \ell_2)$ is even then all cup-products are of course zero. This gives a complete description of the cohomology ring $H^*(M)$.

2.7.2 An example

The real projective space $\mathbb{R}\mathrm{P}^3$ is a double branched cover over the Hopf link in S^3 with linking number ± 1. Choose Seifert surfaces S_1 and S_2 to be the obvious disks intersecting in a single interval w. Then $\pi^{-1}(S_1)$ and $\pi^{-1}(S_2)$ are two copies of $\mathbb{R}\mathrm{P}^2$, each represented as a double branched cover of a disk with branching set a disjoint union of a circle and a point. These two copies of $\mathbb{R}\mathrm{P}^2$ intersect in the circle $\pi^{-1}(w)$ thereby recovering the familiar cup-product structure on $H^*(\mathbb{R}\mathrm{P}^3; \mathbb{Z}/2)$.

Chapter 3

Lescop's Invariant and Gauge Theory

3.1 Floer homology of admissible bundles

Let M be a closed oriented connected 3-manifold and $P \to M$ a principal bundle such that one of the following conditions holds:

(1) M is an integral homology sphere and P is a trivial $SU(2)$-bundle, or

(2) $b_1(M) \geq 1$ and P is a $U(2)$-bundle whose first Chern class $c_1(P)$ has an odd pairing with some integral homology class in $H_2(M)$. Note that the second Stiefel-Whitney class $w_2(\mathrm{ad}(P)) \in H^2(M;\mathbb{Z}/2)$ of the associated $SO(3)$-bundle $\mathrm{ad}(P)$ is then not zero as a map $H_2(M) \to \mathbb{Z}/2$.

Both the bundle P and its adjoint bundle $\mathrm{ad}(P)$ will be referred to as admissible bundles.

Given an admissible bundle P, consider the space $\mathcal{C}$ of $SO(3)$-connections in $\mathrm{ad}(P)$. This space is acted upon by the group $\mathcal{G}$ of determinant one gauge transformations of P. The instanton Floer homology $I_*(M, P)$ is the Floer homology arising from the Chern-Simons function of the space $\mathcal{C}/\mathcal{G}$; see Donaldson [9].

The groups $I_*(M, P)$ depend on the choice of $\mathrm{ad}(P)$ but not P. They have an absolute $\mathbb{Z}/2$-grading defined as in Section 5.6 of [9]. These groups also admit a relative $\mathbb{Z}/8$-grading, which becomes an absolute $\mathbb{Z}/8$-grading if M is an integral homology sphere. Note that our setup is consistent with that of Kronheimer and Mrowka [24], Section 7.1. Using their notations, $I_*(M, P) = I_*(M)_w$, where $w = \det(P)$ is the determinant bundle of P. Before we proceed with the calculations we will establish a few notations used throughout this chapter.

Let $\mathcal{L} = \ell_1 \cup \ell_2 \cdots \cup \ell_n$ be a framed link in a rational homology sphere Σ with framings $\gamma = (\gamma_1, \cdots, \gamma_n)$. Then denote by $\Sigma_{\gamma_1,\cdots,\gamma_n}(\ell_1 \cup \cdots \cup \ell_n)$ or $\Sigma_\gamma(\mathcal{L})$ the manifold obtained by surgery on $\mathcal{L}$ with the given framings. In particular, $\Sigma_0(\mathcal{L})$ will denote the manifold obtained by surgery on 0-framed link $\mathcal{L}$.

If Σ is a rational homology sphere and k is an arbitrary knot in Σ, then there exists an integer q such that $q \cdot k$ represents a trivial class in $H_1(\Sigma)$. For any disjoint knot ℓ in Σ, the linking number $\ell k_\Sigma(q \cdot k, \ell)$ is defined as the intersection number of a Seifert surface of $q \cdot k$ with ℓ, and one further defines

$$\ell k_\Sigma(k, \ell) = \frac{1}{q} \cdot \ell k_\Sigma(q \cdot k, \ell) \;\in\; \mathbb{Q}.$$

Two main ingredients that go into our calculation of the Euler characteristic of $I_*(M, P)$ are as follows. The first one is a special surgery presentation of M as in [27], Lemma 5.1.1.

3.1.1 Lemma. *Let M be a closed connected oriented 3-manifold with $b_1(M) = n$, and*

$\mathrm{ad}(P)$ *be any admissible* $SO(3)$*-bundle over* M*. Then there exists a rational homology sphere* Σ *and an algebraically split* n*-component link* $\mathcal{L} \subset \Sigma$ *such that* $M = \Sigma_0(\mathcal{L})$*, each component of* $\mathcal{L}$ *is null-homologous in* Σ*,* $|H_1(\Sigma)| = |\operatorname{Tor}(H_1(M))|$*, and the restriction of* $\mathrm{ad}(P)$ *to the exterior of* $\mathcal{L} \subset \Sigma$ *is trivial.*

Proof. The proof will be very similar to the proof of Lemma 5.1.1 in [27], the only difference being that we pick a specific basis in $H_1(M)/\operatorname{Tor}$, which depends on the choice of our admissible bundle $\mathrm{ad}(P)$.

Given an admissible $SO(3)$-bundle $\mathrm{ad}(P)$ over M, its second Stiefel-Whitney class $w_2(\mathrm{ad}(P))$ evaluates non-trivially on the fundamental class of some surface in M. We can define a link $C = \{C_1, \cdots C_n\}$ in M which represents a basis of $H_1(M)/\operatorname{Tor}$. The link C is chosen so that we pick a link dual to $w_2(\mathrm{ad}(P)$ and it is completed to a basis of $H_1(M)/\operatorname{Tor}$. Then $\mathrm{ad}(P)$, when restricted to the exterior of C in M, is trivial.

By Poincaré duality, we can pick embedded oriented surfaces $S_1, \cdots, S_n$, where S_i transversally intersects C_i at exactly one point and S_i is disjoint from C_j for $i \neq j$. Let S_i' denote the complement of meridional disk of C_i in S_i. Then the meridian μ_i of C_i bounds a surface S_i' in $M - C$ and therefore the inclusion from $H_1(M - C; \mathbb{Z})$ to $H_1(M; \mathbb{Z})$ is an isomorphism.

Let m_i be a closed curve on the boundary of the tubular neighborhood $N(C_i)$ of C_i which runs parallel to C_i. Let Σ be the manifold obtained by performing surgery on C_i with framing m_i in M. An immediate consequence of this surgery is that $H_1(\Sigma) = H_1(M)/\langle m_i \rangle = \operatorname{Tor}(H_1(M))$, and therefore Σ is a rational homology sphere.

Denote by ℓ_i the core of the surgery in Σ and by $\mathcal{L}$ the link $\ell_1 \cup \cdots \cup \ell_n$. Then M is obtained by performing surgery on 0-framed surgery on $\mathcal{L}$. Each component ℓ_i of $\mathcal{L}$ is null homologous in Σ since it bounds a Seifert surface S_i' in $\Sigma - \mathcal{L}$ and

$\ell k_\Sigma(\ell_i, \ell_j) = 0$ for $i \neq j$. Furthermore, $\Sigma - N(\mathcal{L})$ is homeomorphic to $M - N(C)$, and therefore the restriction of $\mathrm{ad}(P)$ to $\Sigma - N(\mathcal{L})$ is trivial. □

The second ingredient is a long exact sequence known as the Floer exact triangle; see Theorem 2.5 of [4, Part II]. Let Σ be a rational homology sphere and $\mathcal{L} = \ell_1 \cup \ell_2 \cup \cdots \cup \ell_n$ be an algebraically split link in Σ. Denote by $\mathcal{L}'$ the link $\ell_1 \cup \cdots \cup \ell_{n-1}$. If $n = 1$, we will require that Σ be an integral homology sphere. Then we have the following Floer exact triangle:

$$\begin{array}{ccc} & I_*(\Sigma_{0,0}(\mathcal{L}' \cup \ell_n)) & \\ {}^{-1}\swarrow & & \nwarrow \\ I_*\,(\Sigma_0(\mathcal{L}')) & \longrightarrow & I_*\,(\Sigma_{0,-1}(\mathcal{L}' \cup \ell_n)) \end{array}$$

The admissible bundles P, which are omitted from the notations, have $c_1(P)$ pair non-trivially with the natural homology classes obtained by capping off a Seifert surface of ℓ_j by a meridional disk of the surgery. In addition, the three admissible bundles that show up in the Floer exact triangle match when restricted to the exterior of the link $\mathcal{L}$ in Σ. If $n = 1$, the bundles P over Σ and $\Sigma_{-1}(\ell_1)$ are trivial $SU(2)$-bundles.

It should be pointed out that we will not use the complete strength of the Floer exact triangle; all we will derive from it is the following relation on Euler characteristics:

$$\chi(I_*\,(\Sigma_{0,0}(\mathcal{L}' \cup \ell_n)) = \chi(I_*\,(\Sigma_{0,-1}(\mathcal{L}' \cup \ell_n)) - \chi(I_*\,(\Sigma_0(\mathcal{L}'))$$

The obvious observation that

$$b_1(\Sigma_{0,0}(\mathcal{L}' \cup \ell_n)) - 1 = b_1(\Sigma_{0,-1}(\mathcal{L}' \cup \ell_n)) = b_1(\Sigma_0(\mathcal{L}'))$$

allows us to proceed via induction on $b_1(M)$.

3.2 Case of $b_1(M) = 0$

If M is an integral homology sphere, then $\lambda_L(M) = \lambda(M)$, which is the Casson invariant of M; see Section 1.5 of [27]. On the Floer homology side, we work with the trivial $SU(2)$-bundle P over M, and denote the instanton Floer homology by $I_*(M, P)$. According to Taubes [41], we have $\chi(I_*(M, P)) = 2 \cdot \lambda(M)$. Therefore, $\chi(I_*(M, P)) = 2 \cdot \lambda_L(M)$. □

3.3 Case of $b_1(M) = 1$

Let $k \subset \Sigma$ be a null-homologous knot in a rational homology sphere Σ. Choose a Seifert surface F of k, and denote by V its Seifert matrix with respect to a basis of $H_1(F)$. The Laurent polynomial

$$\Delta_{k\subset\Sigma}(t) = |H_1(\Sigma)| \cdot \det\left(t^{1/2}\, V - t^{-1/2}\, V^{\top}\right)$$

is called the Alexander polynomial of $k \subset \Sigma$. Note that $\Delta_{k\subset\Sigma}(t) = \Delta_{k\subset\Sigma}(t^{-1})$ and $\Delta_{k\subset\Sigma}(1) = |H_1(\Sigma)| > 0$.

Given a closed oriented connected 3-manifold M with $b_1(M) = 1$, according to

Lemma 3.1.1, there exists a null-homologous knot $\ell_1 \subset \Sigma$ in a rational homology sphere Σ such that $M = \Sigma_0(\ell_1)$. The Lescop invariant of M is then equal to

$$\lambda_L(M) = \frac{1}{2}\Delta''_{\ell_1\subset\Sigma}(1) - \frac{1}{12}\,|\operatorname{Tor}(H_1(M))|, \tag{3.1}$$

see [27], Section 1.5. It is independent of the choice of surgery presentation of M.

In the special case when Σ is an integral homology sphere, it follows from the Floer exact triangle and Casson's surgery formula [2] that

$$\begin{aligned}\chi(I_*(M,P)) &= \chi(I_*(\Sigma_{-1}(\ell_1)) - \chi(I_*(\Sigma)) = 2\lambda(\Sigma_{-1}(\ell_1)) - 2\lambda(\Sigma)\\ &= 2\left(-\frac{1}{2}\Delta''_{\ell_1\subset\Sigma}(1)\right) = -\Delta''_{\ell_1\subset\Sigma}(1).\end{aligned}$$

Therefore,

$$\chi(I_*(M,P)) = -2\lambda_L(M) - \frac{1}{6}\,|\operatorname{Tor}(H_1(M))| \tag{3.2}$$

as claimed, for the unique admissible $SO(3)$-bundle P over M. The general case is handled similarly using the following result.

3.3.1 Proposition. *Let $M = \Sigma_0(\ell_1)$, where ℓ_1 is a null-homologous knot in a rational homology sphere Σ. Then $\chi(I_*(M,P)) = -\Delta''_{\ell_1\subset\Sigma}(1)$ for any admissible bundle* $\operatorname{ad}(P)$ *over M whose restriction to the exterior of $\ell_1 \subset \Sigma$ is trivial.*

Proof. If $H_1(\Sigma)$ has non-trivial torsion, the starting point for our calculation will be the result from [29] which, with our normalization, reads

$$\frac{1}{2}\chi(I_*(M,P)) = -\lambda_L(\Sigma_1(\ell_1)) + \lambda_L(\Sigma).$$

We wish to identify the right hand side of this equation with $-\frac{1}{2}\Delta''_{\ell_1\subset\Sigma}(1)$. By Lescop [27],

$$\lambda_L(\Sigma_1(\ell_1)) = \frac{|H_1(\Sigma_1(\ell_1))|}{|H_1(\Sigma)|}\cdot\lambda_L(\Sigma) + \mathbb{F}_\Sigma(\ell_1),$$

where $\mathbb{F}_\Sigma(\ell_1)$ is defined by equation 1.4.8 in [27]. Since $|H_1(\Sigma_1(\ell_1))| = |H_1(\Sigma)|$, we conclude that $\lambda_L(\Sigma_1(\ell_1)) = \lambda_L(\Sigma) + \mathbb{F}_\Sigma(\ell_1)$ and therefore

$$\frac{1}{2}\chi(I_*(M,P)) = -\mathbb{F}_\Sigma(\ell_1).$$

A straightforward calculation of $\mathbb{F}_\Sigma(\ell_1)$ shows that

$$\mathbb{F}_\Sigma(\ell_1) = \frac{1}{2}\Delta''_{\ell_1\subset\Sigma}(1),$$

which leads to the desired formula. Therefore, for our choice of admissible bundle P over M,

$$\chi(I_*(M,P)) = -2\lambda_L(M) - \frac{1}{6}\,|\mathrm{Tor}(H_1(M))|.$$

□

3.3.2 Remark. In fact, the above theorem holds for any admissible bundle P over M since $\lambda_L(M)$ is independent of the choice of surgery presentation.

3.4 Case of $b_1(M) = 2$

Let $\mathcal{L} = \ell_1 \cup \ell_2$ be an oriented two-component link in a rational homology sphere Σ such that ℓ_1, ℓ_2 are null-homologous in Σ and $\ell k_\Sigma(\ell_1,\ell_2) = 0$. There exist Seifert surfaces F_1 and F_2 of ℓ_1 and ℓ_2, respectively, such that $F_1\cap\ell_2 = \emptyset$ and $F_2\cap\ell_1 = \emptyset$. If the surfaces intersect, they may be assumed to intersect in a circle c, see [7]. The self linking number of c with respect to either F_1 or F_2 is called the Sato-Levine invariant [36] and is denoted by $s(\ell_1 \cup \ell_2 \subset \Sigma)$. To be precise, $s(\ell_1 \cup \ell_2 \subset \Sigma) = \ell k_\Sigma(c, c^+)$, where c^+ is a positive push-off of c with respect to either F_1 or F_2. If the surfaces do not intersect then $s(\ell_1 \cup \ell_2 \subset \Sigma)$ =0.

Given a closed oriented connected 3-manifold M with $b_1(M) = 2$, according to Lemma 3.1.1, there exists an algebraically split link $\ell_1 \cup \ell_2 \subset \Sigma$ in a rational homology sphere Σ such that $M = \Sigma_{0,0}(\ell_1 \cup \ell_2)$. According to Lescop [27], Section 5.1,

$$\lambda_L(M) = -|\operatorname{Tor}(H_1(M))| \cdot s(\ell_1 \cup \ell_2 \subset \Sigma); \tag{3.3}$$

it is independent of the choice of surgery presentation of M.

Before we proceed with the main theorem of the section, we will establish a fact which we will need for the calculations.

3.4.1 Lemma. *Let Σ be a rational homology sphere, $\ell_2 \subset \Sigma$ a null-homologous knot and k_1, k_2 knots in $\Sigma \setminus N(\ell_2)$. Then*

$$\ell k_{\Sigma_{-1}(\ell_2)}(k_1, k_2) = \ell k_\Sigma(k_1, k_2) + \ell k_\Sigma(k_1, \ell_2) \cdot \ell k_\Sigma(k_2, \ell_2). \tag{3.4}$$

Proof. If k_1, k_2 are null-homologous in $\Sigma \setminus N(\ell_2)$, the proof proceeds exactly as in [18, Lemma 1.2]. If k_1 and k_2 are arbitrary knots in $\Sigma \setminus N(\ell_2)$, there exist non-zero integers q_1, q_2 such that $q_1 \cdot k_1$ and $q_2 \cdot k_2$ are null-homologous in $\Sigma \setminus N(\ell_2)$ and therefore in $\Sigma_{-1}(\ell_2)$. Then

$$\ell k_{\Sigma_{-1}(\ell_2)}(q_1\cdot k_1, q_2\cdot k_2) = \ell k_{\Sigma}(q_1\cdot k_1, q_2\cdot k_2) + \ell k_{\Sigma}(q_1\cdot k_1, \ell_2)\cdot \ell k_{\Sigma}(q_2\cdot k_2, \ell_2)$$

and the result follows by dividing both sides by $q_1 \cdot q_2$. □

3.4.2 Proposition. *Let $M = \Sigma_{0,0}(\ell_1 \cup \ell_2)$, where Σ is a rational homology sphere and $\ell_1, \ell_2 \subset \Sigma$ are null-homologous knots such that $\ell k_{\Sigma}(\ell_1, \ell_2) = 0$. Then*

$$\chi(I_*(M,P)) = 2\cdot\lambda_L(M) \tag{3.5}$$

for any admissible bundle $\mathrm{ad}(P)$ *over M whose restriction to the exterior of $\ell_1 \cup \ell_2 \subset \Sigma$ is trivial.*

Proof. Let P be a $U(2)$-bundle over M such that $w_2(\mathrm{ad}(P))$ evaluates non-trivially on both homology classes obtained by capping off Seifert surfaces of ℓ_1 and ℓ_2 by meridional disks of the surgery. It follows from the induction hypothesis that,

$$\begin{aligned}\chi(I_*(\Sigma_{0,0}(\ell_1\cup\ell_2)) &= \chi(I_*(\Sigma_{0,-1}(\ell_1\cup\ell_2)) - \chi(I_*(\Sigma_0(\ell_1))\\ &= -\Delta''_{\ell_1\subset\Sigma_{-1}(\ell_2)}(1) + \Delta''_{\ell_1\subset\Sigma}(1).\end{aligned}$$

By a similar reasoning as in [18], we can choose Seifert surfaces F_1 and F_2 of ℓ_1 and ℓ_2, respectively, such that $F_1 \cap F_2$ is either empty or a single ribbon intersection, and furthermore, $F_1 \cap F_2$ does not separate F_1 or F_2.

If $F_1 \cap F_2$ is non-empty then it is a single ribbon intersection as in Figure 3.1. The intersection $F_1 \cap \partial F_2 \subset \ell_2$ consists of two points that separate ℓ_2 into two arcs,

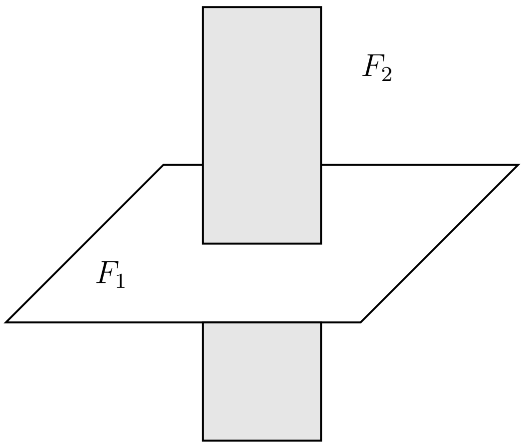

Figure 3.1: Single Ribbon Intersection

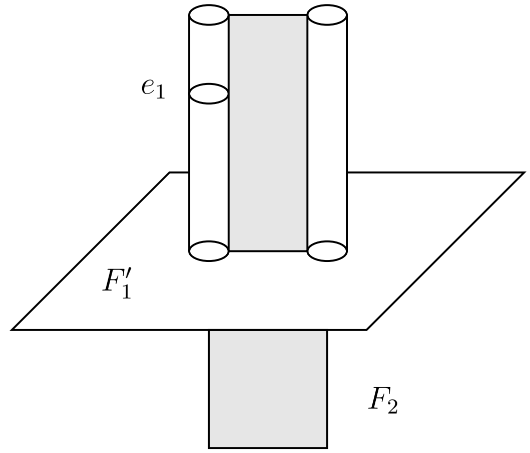

Figure 3.2: Stabilization of F_1

ℓ_2' and ℓ_2''. Stabilize the surface F_1 by adding a tube with core ℓ_2' and call this new modified surface F_1', see Figure 3.2. If F_1 had genus g, the genus of F_1' will be $g+1$.

The intersection $F_1' \cap F_2$ is a closed loop c which represents a primitive homology class in $H_1(F_1')$. Complete c to a basis $\{e_1, c, e_3, ..., e_{2g+2}\}$ of $H_1(F_1')$, where e_1 is a meridional curve of ℓ_2 and $\{e_3, ..., e_{2g+2}\}$ is a basis for $H_1(F_1)$. In addition, the basis of $H_1(F_1)$ is chosen so that $e_m \cap F_2 = \emptyset$ and hence $\ell k_\Sigma(e_m, \ell_2) = 0$ for $m \geq 3$. Moreover, it is obvious that $\ell k_\Sigma(e_1, \ell_2) = \pm 1$, and that $\ell k_\Sigma(c, \ell_2) = \ell k_\Sigma(c^+, \ell_2) = 0$, where c^+ is a positive push-off of c with respect to F_2. To summarize, we have the matrix

$$E = \begin{bmatrix} \ell k_\Sigma(e_1, \ell_2) \\ \ell k_\Sigma(c, \ell_2) \\ \vdots \\ \ell k_\Sigma(e_{2g+2}, \ell_2) \end{bmatrix} = \begin{bmatrix} \pm 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This matrix accounts for the difference in the Seifert matrices of ℓ_1 when viewed as a knot in Σ and $\Sigma_{-1}(\ell_2)$. To be precise, it follows from Lemma 3.4.1 and Hoste

[18],

$$V(\ell_1 \subset \Sigma_{-1}(\ell_2)) = V(\ell_1 \subset \Sigma) + E \cdot E^\top. \tag{3.6}$$

We would like next to determine the Seifert matrix $V(\ell_1 \subset \Sigma)$ with respect to the basis $\{e_1, c, e_3, \ldots e_{2g+2}\}$ in $H_1(F_1')$. As e_1 bounds a meridional disk D, it is null-homologous in Σ. Since e_1^+ is disjoint from D, we have $\ell k_\Sigma(e_1, e_1^+) = 0$. For $m \geq 3$, $\ell k_\Sigma(e_1, e_m^+) = 0$, since the curve e_m lies on the surface F_1 and therefore its push-off e_m^+ can be isotoped to make it disjoint from D . Similarly $\ell k_\Sigma(e_m, e_1^+) = 0$ for $m \geq 3$. Next, $\ell k_\Sigma(c, e_1^+) = \pm 1$ and since c^+ does not intersect the meridional disk bounded by e_1, $\ell k_\Sigma(e_1, c^+) = 0$. To finish the calculation, observe that $V(\ell_1 \subset \Sigma) - V(\ell_1 \subset \Sigma)^\top = -I$, where where I is the intersection form on $H_1(F_1'; \mathbb{Q})$ given by $I(v, w) = v \cdot w$. For our choice of basis, $c \cdot e_m = 0$ and therefore $\ell k_\Sigma(c^+, e_m) - \ell k_\Sigma(c, e_m^+) = 0$. For $m \geq 3$, denote $\ell k_\Sigma(c^+, e_m) = \ell k_\Sigma(c, e_m^+) = a_{m-2}$. Finally, $\ell k_\Sigma(c, c^+)= s$ by the definition of $s = s(\ell_1 \cup \ell_2 \subset \Sigma)$. Therefore, we obtain the matrix

$$V(\ell_1 \subset \Sigma) = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & \cdots & 0 \\ \pm 1 & s & a_1 & \cdots & a_{2g} \\ \hline 0 & a_1 & & & \\ \vdots & \vdots & & & \\ & & & W & \\ 0 & a_{2g} & & & \end{array}\right]$$

where W is the Seifert matrix of ℓ_1 with respect to the basis $e_3, \ldots e_{2g} \in H_1(F_1)$. Using Hoste's formula (3.6), we obtain

$$V(\ell_1 \subset \Sigma_{-1}(\ell_2)) = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & \cdots & 0 \\ \pm 1 & s & a_1 & \cdots & a_{2g} \\ \hline 0 & a_1 & & & \\ \vdots & \vdots & & W & \\ 0 & a_{2g} & & & \end{array}\right]$$

Now we are ready to compute the Alexander polynomials. Denote $V(\ell_1 \subset \Sigma_{-1}(\ell_2))$ by V and let $z = t^{1/2} - t^{-1/2}$. Then

$$t^{1/2}V - t^{-1/2}V^\top = \left[\begin{array}{cc|ccc} z & \mp t^{-1/2} & 0 & \cdots & 0 \\ \pm t^{1/2} & sz & a_1 z & \cdots & a_{2g} z \\ \hline 0 & a_1 z & & & \\ \vdots & \vdots & & t^{1/2}W - t^{-1/2}W^\top & \\ 0 & a_{2g} z & & & \end{array}\right]$$

Hence $\det(t^{1/2}V - t^{-1/2}V^\top) =$

$$= z \left|\begin{array}{c|ccc} sz & a_1 z & \cdots & a_{2g} z \\ \hline a_1 z & & & \\ \vdots & & t^{1/2}W - t^{-1/2}W^\top & \\ a_{2g} z & & & \end{array}\right| \mp t^{1/2} \left|\begin{array}{c|ccc} \mp t^{-\frac{1}{2}} & 0 & \cdots & 0 \\ \hline a_1 z & & & \\ \vdots & & t^{1/2}W - t^{-1/2}W^\top & \\ a_{2g} z & & & \end{array}\right|$$

$$= z^3 \left| \begin{array}{c|ccc} sz^{-1} & a_1 & \cdots & a_{2g} \\ \hline a_1 & & & \\ \vdots & & & \\ & & t^{1/2}W - t^{-1/2}W^\top & \\ a_{2g} & & & \end{array} \right| + \det(t^{1/2}W - t^{-1/2}W^\top)$$

$$= s\, z^2 \cdot \det(t^{1/2}W - t^{-1/2}W^\top) + z^3 \cdot f(t) + \det(t^{1/2}W - t^{-1/2}W^\top)$$

for some function $f(t)$, which is a polynomial in $t^{1/2}$ and $t^{-1/2}$. Therefore,

$$\frac{\Delta_{\ell_1 \subset \Sigma_{-1}(\ell_2)}(t)}{|H_1(\Sigma_{-1}(\ell_2))|} = \frac{sz^2 \Delta_{\ell_1 \subset \Sigma}(t)}{|H_1(\Sigma)|} + z^3 f(t) + \frac{\Delta_{\ell_1 \subset \Sigma}(t)}{|H_1(\Sigma)|}.$$

Next we differentiate twice and set $t = 1$. An easy calculation taking into account that $|H_1(\Sigma_{-1}(\ell_2))| = |H_1(\Sigma)|$, $z(1) = 0$ and $z'(1) = 1$ leads to the formula

$$\begin{aligned} \Delta''_{\ell_1 \subset \Sigma_{-1}(\ell_2)}(1) &= 2s \cdot \Delta_{\ell_1 \subset \Sigma}(1) + \Delta''_{\ell_1 \subset \Sigma}(1) \\ &= 2s \cdot |H_1(\Sigma)| + \Delta''_{\ell_1 \subset \Sigma}(1). \end{aligned}$$

Since $|H_1(\Sigma)| = |\operatorname{Tor}(H_1(M))|$,

$$\chi(I_*\,(M, P)) = -2s(\ell_1 \cup \ell_2 \subset \Sigma) \cdot |\operatorname{Tor}(H_1(M))| = 2 \cdot \lambda_L(M).$$

This is independent of the choice of admissible bundle which restricts to a trivial bundle on the exterior of $\ell_1 \cup \ell_2 \subset \Sigma$. $\square$

3.4.3 Remark. We can show that Proposition 3.4.2 holds for an arbitrary admissible bundle as follows.

The above calculation holds for a specific bundle which has $w_2(\mathrm{ad}(P)) = (1,1)$ in the natural basis of $\mathrm{Hom}(H_2(M), \mathbb{Z}/2)$. We will now prove the result for admissible bundles with $w_2(\mathrm{ad}(P)) = (0,1)$ and $w_2(\mathrm{ad}(P)) = (1,0)$ using the fact that $\lambda_L(M) = -|\operatorname{Tor}(H_1(M))| \cdot s(\ell_1 \cup \ell_2 \subset \Sigma)$ is an invariant of the manifold M and therefore is independent of the surgery presentation.

Without loss of generality, let us assume that $w_2(\mathrm{ad}(P)) = (1,0)$. After sliding ℓ_1 over ℓ_2, we will obtain a new surgery presentation for M, namely, $M = \Sigma_{0,0}(\ell_1 \cup \ell_\sharp)$, where $\ell_\sharp$ is the new knot obtained by sliding ℓ_1 over ℓ_2. Note that $\ell_\sharp$ bounds a Seifert surface which is a band sum of F_1 and F_2 and also that $\ell k_\Sigma(\ell_1, \ell_\sharp) = 0$. In the new basis, $w_2(\mathrm{ad}(P)) = (1,1)$, hence $\chi(I_*(M,P)) = s(\ell_1 \cup \ell_\sharp \subset \Sigma)$ by the above argument. The independence of surgery presentation then implies that $s(\ell_1 \cup \ell_\sharp \subset \Sigma)$ $= s(\ell_1 \cup \ell_2 \subset \Sigma)$ and therefore $\chi(I_*(M,P))$ is independent of the choice of admissible bundle.

3.4.4 Example. Given a two component link $\mathcal{L} = \ell_1 \cup \ell_2$ in an integral homology sphere Σ, Harper and Saveliev [16] defined its Floer homology $I_*(\Sigma, \mathcal{L})$ as follows. The link exterior $X = \Sigma \setminus \mathrm{int}\; N(\mathcal{L})$ is a compact manifold whose boundary consists of two 2-tori. Furl it up by gluing the boundary tori together via an orientation reversing diffeomorphism $\varphi : T^2 \longrightarrow T^2$ chosen so that the resulting closed manifold X_φ has homology of $S^1 \times S^2$. Then $I_*(\Sigma, \mathcal{L}) = I_*(X_\varphi)$. By Floer's excision principle, $I_*(\Sigma, \mathcal{L})$ is independent of the choice of φ.

The manifold X_φ can be chosen so that it has the following surgery description, see [16]. Attach a band from one component of $\mathcal{L}$ to the other, and call the resulting knot k_1. Let k_2 be a small unknotted circle going once around the band with linking number zero. Then X_φ is the manifold obtained from Σ by performing surgery on the link $k_1 \cup k_2$, with k_1 framed by ± 1 and k_2 framed by 0. A quick argument with Floer exact triangle shows that $I_*(X_\varphi) = I_*(Y, P)$, where Y is the manifold obtained by surgery on the link $k_1 \cup k_2$ with both components framed by zero [16, Section 4]. Since the link $k_1 \cup k_2$ is algebraically split, it follows from equations (3.3) and (3.5) that

$$\chi(I_*(\Sigma, \mathcal{L})) = \chi(I_*(Y, P)) = -2s(k_1 \cup k_2).$$

A straightforward calculation then shows that $s(k_1 \cup k_2) = \pm \ell k_\Sigma(\ell_1, \ell_2)$, which recovers the formula $\chi(I_*(\Sigma, \mathcal{L})) = \pm 2 \cdot \ell k_\Sigma(\ell_1, \ell_2)$ of [16].

3.5 Case of $b_1(M) = 3$

Let $\mathcal{L} = \ell_1 \cup \ell_2 \cup \ell_3$ be an algebraically split oriented three-component link in a rational homology sphere Σ such that each component of $\mathcal{L}$ is null-homologous. Let F_1, F_2, F_3 be Seifert surfaces of the knots ℓ_1, ℓ_2, ℓ_3, respectively, chosen so that $F_i \cap \ell_j = \emptyset$ for $i \neq j$. Define the Milnor triple linking number $\mu(\ell_1, \ell_2, \ell_3)$ as a signed count of points in the intersection $F_1 \cap F_2 \cap F_3$.

Given a closed oriented connected 3-manifold with $b_1(M) = 3$, by Lemma 3.1.1, there exists an algebraically split link $\mathcal{L} = \ell_1 \cup \ell_2 \cup \ell_3$ in a rational homology sphere Σ such that $M = \Sigma_0(\mathcal{L})$ and the components of $\mathcal{L}$ are all null-homologous. According to Lescop [27],

$$\lambda_L(M) = |\operatorname{Tor}(H_1(M))| \cdot (\mu(\ell_1, \ell_2, \ell_3))^2. \tag{3.7}$$

Again, $\lambda_L(M)$ is independent of the choice of surgery presentation as above.

3.5.1 Proposition. *Let $M = \Sigma_0(\mathcal{L})$, where Σ is a rational homology sphere, $\mathcal{L} = \ell_1 \cup \ell_2 \cup \ell_3$ is a link in Σ such that each component of $\mathcal{L}$ is null-homologous and $\ell k_\Sigma(\ell_i, \ell_j)$ =0 for $i \neq j$. Then*

$$\chi(I_*(M,P)) = -2 \cdot \lambda_L(M) \tag{3.8}$$

for any admissible bundle $\mathrm{ad}(P)$ *over M whose restriction to the exterior of $\ell_1 \cup \ell_2 \cup \ell_3 \subset \Sigma$ is trivial.*

Proof. Let P be a $U(2)$-bundle over M such that $w_2(\mathrm{ad}(P))$ evaluates non-trivially on all three homology classes obtained by capping off Seifert surfaces of ℓ_1, ℓ_2 and ℓ_3 by meridional disks of the surgery. It follows from the induction hypothesis that,

$$\begin{aligned}
\chi(I_*(M,P))) &= \chi(I_*(\Sigma_{0,0,-1}(\ell_1 \cup \ell_2 \cup \ell_3)) - \chi(I_*(\Sigma_{0,0}(\ell_1 \cup \ell_2)) \\
&= -2s(\ell_1 \cup \ell_2 \subset (\Sigma_{-1}(\ell_3)) \cdot |H_1(\Sigma_{-1}(\ell_3))| + 2s(\ell_1 \cup \ell_2 \subset \Sigma) \cdot |H_1(\Sigma)| \\
&= -2 \cdot \ell k_{\Sigma_{-1}(\ell_3)}(c, c^+) \cdot |H_1(\Sigma_{-1}(\ell_3))| + 2 \cdot \ell k_\Sigma(c, c^+) \cdot |H_1(\Sigma)|
\end{aligned}$$

where c is the intersection circle $c = F_1 \cap F_2$ of the Seifert surfaces F_1 and F_2 chosen to intersect in a circle. Since $|H_1(\Sigma_{-1}(\ell_3))| = |H_1(\Sigma)|$, we can conclude that

$$\chi(I_*(M,P))) = -2 \cdot \left[\ell k_{\Sigma_{-1}(\ell_3)}(c, c^+) - \ell k_\Sigma(c, c^+)\right] \cdot |H_1(\Sigma)|$$

We wish to identify the right hand side of this equation with $-2(\mu(\ell_1, \ell_2, \ell_3))^2 \cdot$

$|\operatorname{Tor}(H_1(M))|$. By (3.4),

$$\ell k_{\Sigma_{-1}(\ell_3)}(c, c^+) - \ell k_\Sigma(c, c^+) = \ell k_\Sigma(c, \ell_3) \cdot \ell k_\Sigma(c^+, \ell_3),$$

therefore,

$$\begin{aligned}\chi(I_*(M,P))) &= -2 \cdot \left[\ell k_\Sigma(c, \ell_3) \cdot \ell k_\Sigma(c^+, \ell_3)\right] \cdot |H_1(\Sigma)| \\ &= -2 \cdot \ell k(c, \ell_3)^2 \cdot |\operatorname{Tor}(H_1(M))|.\end{aligned}$$

Since $\ell k_\Sigma(c, \ell_3) = c \cdot F_3 = \mu(\ell_1, \ell_2, \ell_3)$, the result follows. □

3.5.2 Remark. By the same reasoning as in the remark after Proposition 3.4.2, the statement of Proposition 3.5.1 holds for any admissible bundle over M.

It is worth mentioning that Ruberman and Saveliev [34] showed that $1/2 \cdot \chi(I_*(M,P)) = \lambda_L(M) \mod 2$ for all M with $H_*(M) = H_*(T^3)$ using techniques different from ours.

3.5.3 Example. Given a knot $\ell \subset S^3$, Kronheimer and Mrowka [23] defined its reduced singular instanton knot homology $I^\natural(\ell)$ as follows. Take the knot exterior $S^3 \setminus N(\ell)$ and construct a closed 3-manifold Y by attaching, along the boundary, the manifold $F \times S^1$, where F is a punctured 2-torus. The attaching is done so that the curve $\partial F \times \{\text{point}\}$ matches the canonical longitude of ℓ, and the curve $\{\text{point}\} \times S^1$ matches its meridian. Then

$$I_*(Y,P) = I_*^\natural(\ell) \oplus I_*^\natural(\ell) \tag{3.9}$$

for a particular choice of admissible bundle P over Y; see [22, Proposition 5.7]. One can easily see that Y is homeomorphic to the manifold obtained by $(0,0,0)$-surgery on $\ell \# \mathrm{Br}$, where Br stands for the Borromean rings. Since the link $\ell \# \mathrm{Br}$ is algebraically split with $\mu(\ell \# \mathrm{Br}) = 1$, it follows from equations (3.7), (3.8) and (3.9) that

$$\chi(I_*^{\natural}(\ell)) \;=\; \frac{1}{2} \cdot \chi(I_*(Y,P)) = -1.$$

3.6 Case of $b_1(M) \geq 4$

For all closed oriented connected 3-manifolds M with $b_1(M) = n \geq 4$, the Lescop invariant $\lambda_L(M)$ is known to vanish.

3.6.1 Proposition. *Let $M = \Sigma_0(\mathcal{L})$, where Σ is a rational homology sphere, $\mathcal{L} = \ell_1 \cup \cdots \cup \ell_n$ is a link in Σ such that each component of $\mathcal{L}$ is null-homologous and $\ell k_\Sigma(\ell_i, \ell_j) = 0$ for $i \neq j$. Then $\chi(I_*(M,P)) = 0$ for any admissible bundle* $\mathrm{ad}(P)$ *over M whose restriction to the exterior of $\mathcal{L} = \ell_1 \cup \cdots \cup \ell_n$ in Σ is trivial.*

Proof. Let P denote the $U(2)$-bundle over M such that $w_2(\mathrm{ad}(P))$ evaluates non-trivially on the homology classes obtained by capping off Seifert surfaces of the components ℓ_i, $i = 1, \ldots, n$, by meridional disks of the surgery. It follows from the induction hypothesis that, if $n = 4$,

$$\begin{aligned}
\chi(I_*(M,P)) = \chi(I_*(\Sigma_{0,0,0,-1}(\ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4)) \\
- \chi(I_*(\Sigma_{0,0,0}(\ell_1 \cup \ell_2 \cup \ell_3)) \\
= -2 \cdot \ell k_{\Sigma_{-1}(\ell_4)}(c, \ell_3)^2 \cdot |H_1(\Sigma_{-1}(\ell_4))| + 2 \cdot \ell k_\Sigma(c, \ell_3)^2 \cdot |H_1(\Sigma)|
\end{aligned}$$

As $|H_1(\Sigma_{-1}(\ell_4))| = |H_1(\Sigma)|$, the result now follows because

$$\ell k_{\Sigma_{-1}(\ell_4)}(c,\ell_3) - \ell k_\Sigma(c,\ell_3) = \ell k_\Sigma(c,\ell_4)\cdot \ell k_\Sigma(\ell_3,\ell_4) = 0,$$

using equation (3.4) and the fact that $\ell k(\ell_3,\ell_4) = 0$.

If $n \geq 5$, choose an admissible bundle $\mathrm{ad}(P)$ whose restriction to the exterior of $\mathcal{L} \subset \Sigma$ is trivial, and proceed by induction. Suppose that $\chi(I_*(M',P)) = 0$ for all M' with $b_1(M') = k$, where $4 \leq k \leq n-1$.

Let $M = \Sigma_0(\mathcal{L})$, where Σ is a rational homology sphere, $\mathcal{L} = \ell_1 \cup \cdots \cup \ell_n$ is a link in Σ such that each component of $\mathcal{L}$ is null-homologous and $\ell k_\Sigma(\ell_i,\ell_j)$ =0 for $i \neq j$. Denote by $\mathcal{L}'$ the link $\ell_1 \cup \cdots \cup \ell_{n-1}$. Then it follows from the induction hypothesis that

$$\chi(I_*(M,P)) = \chi(I_*(\Sigma_{0,-1}(\mathcal{L}'\cup\ell_n)) - \chi(I_*(\Sigma_0(\mathcal{L}'))$$

Since

$$b_1(\Sigma_{0,0}(\mathcal{L}'\cup\ell_n)) - 1 = b_1(\Sigma_{0,-1}(\mathcal{L}'\cup\ell_n)) = b_1(\Sigma_0(\mathcal{L}')) \geq 4$$

we conclude that $\chi(I_*(M,P)) = 0$. □

3.6.2 Remark. As before, the result holds for all admissible bundles over M.

3.7 An example

The following example illustrates the appearance of the factor $|\operatorname{Tor} H_1(M)|$ in the Lescop invariant from a gauge-theoretic viewpoint.

Let Y be a closed oriented 3-manifold with torsion-free first homology of rank at least one, and consider the manifold $M = Y \# L(p,q)$ obtained by connect summing Y with a lens space $L(p,q)$. It follows from the connected sum formula in Theorem T4 [27, Page 13] that

$$\lambda_L(M) = p \cdot \lambda_L(Y) = |\operatorname{Tor} H_1(M)| \cdot \lambda_L(Y)$$

but we wish to explain the factor $|\operatorname{Tor} H_1(M)|$ from a gauge-theoretic viewpoint.

Let P be an admissible bundle over M obtained by matching an admissible bundle Q over Y with a trivial bundle over $L(p,q)$. As in [34, Section 3.2], the holonomy map provides a bijective correspondence between gauge equivalence classes of projectively flat connections in P and conjugacy classes of projective representations

$$\alpha : \pi_1(Y \# L(p,q)) \to SU(2)$$

with the Stiefel–Whitney class $w_2(P)$. Since $\pi_1(Y \# L(p,q)) = \pi_1(Y) * \pi_1(L(p,q))$ is a free product, all such α will be of the form $\alpha = \beta * \gamma$, where $\beta : \pi_1(Y) \to SU(2)$ is a projective representation with the Stiefel–Whitney class $w_2(Q)$, and $\gamma : \pi_1(L(p,q)) \to SU(2)$ is a representation. We will assume for the sake of simplicity that the character variety of $\pi_1(Y)$ is non-degenerate; the general case can be handled using perturbations. Note that since β is irreducible, each pair of conjugacy classes $[\beta], [\gamma]$ gives rise to a family of $[\alpha]$ parameterized by $SU(2)/\operatorname{Stab}(\gamma)$.

We will next examine the $SU(2)$-character variety of $\pi_1(L(p,q))$. Since the group $\pi_1(L(p,q)) = \mathbb{Z}/p$ is abelian, one may assume after conjugation that the image of a representation $\gamma : \pi_1(L(p,q)) \to SU(2)$ is a unit complex number. Fix a generator $1 \in \pi_1(L(p,q))$, then such representations γ correspond to the roots of unity $\gamma(1) = \exp(2\pi i n/p)$, with $0 \le n \le p-1$. The number of conjugacy classes of γ and the size of $\mathrm{Stab}(\gamma)$ depend on the parity of p as follows.

When p is odd, the trivial representation θ corresponding to $n = 0$ is the only central representation. Its stabilizer equals $SU(2)$ hence it gives rise to a single point in the character variety of $\pi_1(M)$ for each $[\beta]$. Other representations γ are non-central and, since $\cos(2\pi n/p) = \cos(2\pi(p-n)/p)$, there are $(p-1)/2$ conjugacy classes of them enumerated by $\mathrm{tr}(\gamma(1)) = 2\cos(2\pi n/p)$, $1 \le n \le (p-1)/2$. Since each of these γ has stabilizer $U(1)$, it gives rise to a copy of $SU(2)/U(1) = S^2$ in the character variety of $\pi_1(M)$ for each $[\beta]$.

When p is even, there are two central representations, $\pm\theta$, each giving rise to two points in the character variety of $\pi_1(M)$ for each $[\beta]$. Like in the odd case, each of the remaining $(p-2)/2$ conjugacy classes corresponding to $\mathrm{tr}(\gamma(1)) = 2\cos(2\pi n/p)$ gives rise to a 2-sphere's worth of representations in the character variety of $\pi_1(M)$ for each $[\beta]$.

One can easily see that the 2-spheres in the character variety of $\pi_1(M)$ described above are non-degenerate in the Morse–Bott sense. Therefore, each of them contributes $\pm\chi(S^2) = \pm 2$ to the Euler characteristic of the instanton Floer homology of M. The latter follows for instance from [38, Theorem 5.1] which compares the Wilson loop perturbations of Floer [12] to Morse-type perturbations. The signs of these contributions can be figured out by either computing the Floer indices of $\beta * \gamma$ or by using an ad hoc argument equating the Euler characteristic of instanton homology to the Lescop invariant.

The final outcome is that each $[\beta]$ in the character variety of $\pi_1(Y)$ contributes $1+2\cdot(p-1)/2 = p$ to the Euler characteristic $\chi(I_*\,(M,P)$ if p is odd, and it contributes $2+2\cdot(p-2)/2 = p$ if p is even. In both cases, this results in the desired formula $\chi(I_*\,(M,P)) = p\cdot\chi(I_*\,(Y,Q)) = |\operatorname{Tor} H_1(M)|\cdot\chi(I_*\,(Y,Q))$.

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