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On Geometric Problems Involving Schwarzschild Manifolds

Armando J. Cabrera Pacheco
University of Miami, cabrera@math.miami.edu

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ON GEOMETRIC PROBLEMS INVOLVING SCHWARZSCHILD MANIFOLDS

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Armando José Cabrera Pacheco

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Armando José Cabrera Pacheco

Approved:

Pengzi Miao, Ph.D.
Associate Professor of Mathematics

Gregory J. Galloway, Ph.D.
Professor of Mathematics

Lev Kapitanski, Ph.D.
Professor of Mathematics

Guillermo Prado, Ph.D.
Dean of the Graduate School

Orlando Alvarez, Ph.D.
Professor of Physics

CABRERA PACHECO, ARMANDO JOSE (Ph.D., Mathematics)
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A spatial Schwarzschild manifold of mass m is an asymptotically flat manifold that represents the $\{t = 0\}$ spacelike slice of the Schwarzschild spacetime, a spacetime that models the gravitational field surrounding a spherically symmetric non-rotating massive body. The Riemannian Penrose inequality establishes an upper bound for the ADM mass of an asymptotically flat manifold with non-negative scalar curvature in terms of the volume of its outermost apparent horizon; remarkably, this inequality is rigid and equality is achieved for Schwarzschild manifolds. Our main results are that any metric g of positive scalar curvature on the 3-sphere can be realized as the induced metric of the outermost apparent horizon of a 4-dimensional asymptotically flat manifold with non-negative scalar curvature, whose ADM mass can be arranged to be arbitrarily close to the optimal value determined by the Riemannian Penrose inequality. Along the same lines, any metric g of positive scalar curvature on the n-sphere, with $n \geq 4$, such that it isometrically embeds into the $(n+1)$-Euclidean space as a star-shaped surface, can be realized as the induced metric on the outermost apparent horizon of an $(n+1)$-dimensional asymptotically flat manifold with non-negative scalar curvature, whose ADM mass can be made to be arbitrarily close to the optimal value.

In addition to the extension problems above, motivated by various definitions of quasi-local masses, we study an isometric embedding problem of 2-spheres into

Schwarzschild manifolds. We prove that if g is a Riemannian metric on the 2-sphere which is close to the standard metric, then it admits an isometric embedding into any spatial Schwarzschild manifold with small mass.

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Chapter 1

Preliminaries

1.1 Introduction

A physically meaningful *spacetime* is a time orientable *Lorentzian* 4-dimensional manifold $(\overline{M}, \overline{g})$ satisfying the *Einstein Equations*

$$\overline{\mathrm{Ric}} - \frac{1}{2}\overline{R}\,\overline{g} = 8\pi T, \tag{1.1}$$

where $\overline{\mathrm{Ric}}$ and $\overline{R}$ are the Ricci tensor and the scalar curvature of $(\overline{M}, \overline{g})$, respectively, and T is the energy-momentum tensor, modeling the matter content of the system. These equations relate the geometry of the spacetime with the matter contained in it.

A *spacelike* hypersurface M in a spacetime $(\overline{M}, \overline{g})$ (which is a *Riemannian* manifold) can be considered as the initial value of the "Cauchy" problem associated with (1.1). An *initial data set* is a triple (M, g, K), where (M, g) is a Riemannian manifold and K is a symmetric $(0,2)$-tensor.

In this setting, (M,g) and K act as the initial position and velocity of the system; when the initial data set is evolved by (1.1), g and K play the role of the induced metric and the second fundamental form of M inside $\overline{M}$, respectively.

One basic fact about initial data sets is that they cannot be freely specified. Some geometric restrictions arise from (1.1) via the Gauss-Codazzi-Mainardi equations, called the *constraint equations*:

$$R_g + (\mathrm{tr}_g K)^2 - \|K\|_g^2 = 16\pi\mu, \tag{1.2}$$

$$\mathrm{div}\,(K - \mathrm{tr}_g K g) = 8\pi J, \tag{1.3}$$

where R_g is the scalar curvature of (M,g), $\mu = T(n,n)$ is the energy density and $J = T(X,n)$, for any tangent vector X to M, is the momentum density. Here n denotes the future-directed unit normal to (M,g). A diagram of an initial data set is shown in Figure 1.1.

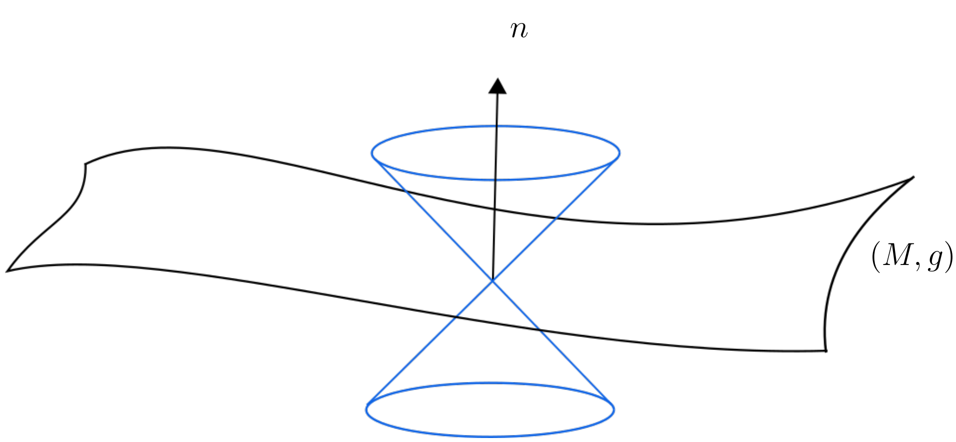

Figure 1.1: The Riemannian manifold (M,g) sits inside the spacetime $(\overline{M},\overline{g})$ as a spacelike hypersurface.

In a *time-symmetric* scenario (meaning that $K \equiv 0$) and assuming the *dominant energy condition* $\mu \geq \|J\|_g$, the constraint equations (1.2) and (1.3) imply that

$$R_g \geq 0. \tag{1.4}$$

When describing isolated gravitational systems like stars or black holes, one is interested in *asymptotically flat spacetimes*, that is, spacetimes that away from the energy source approach the flat spacetime, i.e., the Minkowski spacetime. At the level of time-symmetric initial data sets, this implies that the manifold (M, g) approaches the Euclidean space at infinity.

Given their great importance in the development of mathematical relativity, we focus on *asymptotically flat Riemannian manifolds* (see Definition 6) with non-negative scalar curvature. These manifolds have the remarkable property of having a well-defined notion of *total mass*, called *ADM mass* [1] (c.f. Definition 7), denoted by m_{ADM}.

A basic yet extremely important example in mathematical relativity of an asymptotically flat time-symmetric initial data set, is the 3-dimensional spatial Schwarzschild manifold of mass m, $(\mathbb{R}^3 \setminus B_{\frac{m}{2}}(0), g_m)$, that arises from considering the $\{t = 0\}$ spacelike slice of the Schwarzschild spacetime; this spacetime models the gravitational field around a spherically symmetric non-rotating massive object. Spatial Schwarzschild manifolds will play an essential role in this work.

Schoen and Yau in [35], and Witten in [41], proved that indeed the total mass of an asymptotically flat Riemannian manifold with non-negative scalar curvature is non-negative. Later, Huisken and Ilmanen in [22], and Bray in [4], established the so called *Riemannian Penrose inequality*, which in the language of general relativity can be interpreted as a lower bound on the energy contributed by a collection of black holes; in mathematical terms, it states that for an asymptotically flat Riemannian manifold with non-negative scalar curvature (M^3, g),

$$m_{\text{ADM}} \geq \sqrt{\frac{A}{16\pi}}, \tag{1.5}$$

where A is the area of the outer-minimizing *horizon* in M (i.e., an outermost minimal surface). Furthermore, (1.5) is rigid: equality is achieved if and only if (M, g) is isometric to the spatial Schwarzschild space of mass m on the corresponding exterior regions.

Recently, Mantoulidis and Schoen in [25] carried out an elegant geometric construction of 3-dimensional asymptotically flat spaces with non-negative scalar curvature whose horizons have prescribed geometry. Moreover, the total mass of these extensions can be arranged to be arbitrarily close to the optimal value determined by (1.5). This can be understood as a statement establishing the instability of the Riemannian Penrose inequality.

This dissertation can be divided in two parts. The first part is related to the geometry of initial data sets in higher dimensions, that is, Riemannian manifolds with non-negative scalar curvature of dimension $n+1 \geq 4$; we extend to higher dimensions the techniques in [25] and use them to obtain a certain type of asymptotically flat extensions with non-negative scalar curvature of given metrics with positive scalar curvature on S^n [7]. The second part deals with an isometric embedding problem of 2-spheres into Schwarzschild manifolds [8], motivated by the definitions of well-known notions of quasi-local masses.

Prescribing the geometry on the horizon of a higher dimensional black hole initial data.

In recent years, the interest on higher dimensional black hole geometry has been growing. Galloway and Schoen [14] generalized Hawking's black hole topology theorem [20] to higher dimensions. Bray and Lee [5] proved the Riemannian Penrose inequality in higher dimensions ($n < 7$) (see Theorem 2.1.1).

Motivated by the above results, we establish some higher dimensional analogues of the Mantoulidis-Schoen construction in [25]. Let ω_n denote the volume of the standard n-sphere S^n. Our main results are:

Theorem I ([7])**.** Let g be a metric with positive scalar curvature on S^3. Given any $m > 0$ such that $m > \frac{1}{2}\left(\mathrm{vol}\,(g)/\omega_3\right)^{\frac{2}{3}}$, there exists an asymptotically flat 4-dimensional manifold M^4 with non-negative scalar curvature such that

(i) ∂M^4 is isometric to (S^3, g) and it is minimal,

(ii) M^4, outside a compact set, is isometric to a spatial Schwarzschild manifold of mass m, and

(iii) M^4 is foliated by mean convex 3-spheres which eventually coincide with the rotationally symmetric 3-spheres in the spatial Schwarzschild manifold.

Theorem II ([7])**.** Given any $n \geq 4$, let g be a metric with positive scalar curvature on S^n. Suppose (S^n, g) isometrically embeds into the Euclidean space $\mathbb{R}^{n+1}$ as a star-shaped hypersurface. Given any $m > 0$ such that $m > \frac{1}{2}\left(\mathrm{vol}\,(g)/\omega_n\right)^{\frac{n-1}{n}}$, there exists an asymptotically flat $(n+1)$-dimensional manifold M^{n+1} with non-negative scalar curvature such that

(i) ∂M^{n+1} is isometric to (S^n, g) and it is minimal,

(ii) M^{n+1}, outside a compact set, is isometric to a spatial Schwarzschild manifold of mass m, and

(iii) M^{n+1} is foliated by mean convex n-spheres which eventually coincide with the rotationally symmetric n-spheres in the spatial Schwarzschild manifold.

These results can be interpreted as statements demonstrating the *instability* of the Riemannian Penrose inequality in higher dimensions, in the sense that the ADM mass of the extensions is arbitrarily close to the optimal value determined by the Riemannian Penrose inequality, while the intrinsic geometry on the outermost apparent horizon is "far away" from being rotationally symmetric.

To prove Theorems I and II, we adapt the strategy developed in [25] for $n = 2$ to higher dimensions, and replace their use of the Uniformization Theorem with applications of the path-connectedness property of the space of Riemannian metrics of positive scalar curvature in S^3 and a type of inverse curvature flow on star-shaped surfaces in Euclidean spaces for dimensions $n \geq 4$. We can briefly summarize the main ideas as follows:

(i) Consider $\mathcal{C} \subset \mathrm{Scal}^+(S^n)$, where $\mathrm{Scal}^+(S^n)$ is the set of metrics on S^n with positive scalar curvature. Given any $g \in \mathcal{C}$, construct a *collar extension* of g, by which we mean a metric of positive scalar curvature on the product $[0,1] \times S^n$ such that the bottom boundary $\{0\} \times S^n$, having g as the induced metric, is outer-minimizing while the top boundary $\{1\} \times S^n$ is a round sphere.

(ii) Pick any $m > 0$ arbitrarily close to $\frac{1}{2}\left(\mathrm{vol}\,(g)/\omega_n\right)^{\frac{n-1}{n}}$. Consider an $(n+1)$-dimensional spatial Schwarzschild manifold of mass m (which is scalar flat), deform it to have positive scalar curvature in a small region near the horizon, and then smoothly glue it to the above collar extension.

The key ingredient of our proof lies in the first step. The construction of the collar extensions depends on the existence of a volume-preserving smooth path of metrics connecting the initial metric g in $\mathcal{C}$ with a round metric, preserving the positivity of the scalar curvature.

For Theorem I, we use a result of Marques [26] regarding deformations of metrics with positive scalar curvature on 3-dimensional closed manifolds, to connect any given metric g of positive scalar curvature on S^3 to a round metric via a continuous path of metrics with positive scalar curvature. By a mollifying procedure, we obtain a smooth path of metrics with positive scalar curvature connecting g to a round metric.

For Theorem II, we apply the results of Urbas [37] and Gerhardt [15] on inverse curvature flows of star-shaped surfaces in Euclidean spaces. We use a *normalized* H/R *flow,* where H and R denote the mean curvature and scalar curvature of the surface, respectively. Since the initial surface has positive scalar curvature, the definition of the flow implies that $R > 0$ is preserved throughout the flow, and moreover, a rescaled surface converges to a round metric when $t \to \infty$.

Isometric embeddings into Schwarzschild manifolds.

Even though the concept of total mass of an asymptotically flat spacetime is well understood, it has been a challenging problem in general relativity to give a suitable definition of *quasi-local mass*, i.e., how much energy (or mass) is present in a given region of a spacetime. Using the Hamiltonian formulation of general relativity, Brown and York defined one of the most promising notions of quasi-local mass [6], known as the Brown-York mass. One crucial fact used in their definition is the existence of *isometric embeddings* of 2-spheres with positive Gaussian curvature into the Euclidean space $\mathbb{R}^3$, i.e., the Weyl problem [40], solved independently by Nirenberg [28] and Pogorelov [33].

In [38, 39] Wang and Yau defined a new quasi-local mass generalizing the Brown-York mass. In particular, they proved an existence theorem (cf. [39]) that provides isometric embeddings of a space-like 2-surface into the Minkowski spacetime $\mathbb{R}^{3,1}$.

The definitions of the Brown-York and the Wang-Yau masses suggest that results concerning isometric embeddings of a 2-surface into a model space play an important role in quasi-local mass related problems. Motivated by the previous discussion, we consider isometric embeddings of (S^2, g) into spatial Schwarzschild manifolds. Specifically, we prove:

Theorem III ([8])**.** Given $\alpha \in (0,1)$, there exist $\varepsilon > 0$ and $\delta = \delta(\alpha) > 0$, such that given any metric g' on S^2 with $|g' - \sigma|_{2,\alpha} < \delta$ and any Schwarzschild manifold $(\mathbb{R}^3 \setminus \{0\}, g_m)$ with $|m| < \varepsilon$, there exists an isometric embedding of (S^2, g') into $(\mathbb{R}^3 \setminus \{0\}, g_m)$. Here, σ denotes the standard metric on S^2.

Theorem IV ([8])**.** Let Σ be an embedded, convex surface in $\mathbb{R}^3$. Given any positive function u on $\mathbb{R}^3$, let $g_u = u^4 g_{_E}$, where $g_{_E}$ denotes the standard Euclidean metric on $\mathbb{R}^3$. Suppose the following conditions hold:

(i) g_u and $g_{_E}$ induce the same metric h on Σ, and

(ii) the mean curvatures of Σ in $(\mathbb{R}^3, g_u)$ and $(\mathbb{R}^3, g_{_E})$ agree.

Then, for any Riemannian metric $\tilde{h}$ on Σ that is sufficiently close to h on Σ, there exists an isometric embedding of $(\Sigma, \tilde{h})$ in $(\mathbb{R}^3, g_u)$.

Conditions (i) and (ii) in Theorem IV coincide with the geometric boundary conditions in Bartnik's static metric extension conjecture [3].

The proofs of Theorems III and IV consist of applications of the contraction mapping theorem, inspired by Nirenberg's proof of the Weyl's problem in [28].

This work is organized as follows: in Chapter 1 we provide some background and describe some notation that is used throughout this dissertation. Chapter 2 describes more precisely Mantoulidis and Schoen's construction and provides its adaptation to higher dimensions in a general form, to later explain how to use Marques' result and the mentioned geometric flow to obtain Theorems I and II. The last chapter consists of a brief description of Nirenberg's approach and the proofs of Theorems III and IV.

1.2 Differential Geometry

We begin by recalling some concepts from differential geometry and taking the opportunity to describe some notation and sign conventions that will be used throughout this dissertation. In particular, we describe concepts pertinent to Riemannian geometry. We do not intend to give a comprehensive exposition, and familiarity with the topics discussed here is assumed. For a more detailed exposition of this subject the reader is referred to the classical references, for example [9], [23] and [32].

1.2.1 Riemannian manifolds and curvature

A *Riemannian manifold* (M, g) of dimension n is an n-dimensional smooth manifold M with a non-degenerate, positive definite and symmetric $(0,2)$-tensor g. The tensor g assigns to each tangent space of M (denoted by T_pM for p in M) a positive definite inner product, thus providing a notion of distance in M. The symmetric tensor g is referred to as the *metric* of the Riemannian manifold. We say that a Riemannian manifold is *closed* if it is connected, compact and boundary-less.

In differential geometry, one is interested in properties of manifolds that are preserved under *isometries*, i.e., diffeomorphisms that preserve the metric. Usually, these properties are described in terms of various notions of curvature, whose definitions depend on the *Levi-Civita connection* $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the set of smooth vector fields on a manifold M. We denote $\nabla(X, Y) = \nabla_X Y$ for all X and Y in $\mathfrak{X}(M)$. In general, a linear connection on a manifold allows one to take covariant derivatives of tensor fields. In a Riemannian manifold (M, g), the Levi-Civita connection is the unique linear connection which is symmetric and compatible with the metric g:

$$[X, Y] = \nabla_X Y - \nabla_Y X \quad \text{(symmetry)}, \tag{1.6}$$

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(X, \nabla_X Z) \quad \text{(compatibility)}, \tag{1.7}$$

for all X, Y and Z in $\mathfrak{X}(M)$. Here, $[\cdot, \cdot]$ denotes the so called *Lie Bracket*, defined as $[X, Y] = XY - YX$, for any two vector fields X and Y.

Given a local system of coordinates of M, $(x^1, ..., x^n)$, with coordinate vectors $\{\partial_1, ..., \partial_n\}$ (which form a basis for the tangent space), we can express the action of the connection in terms of this coordinate basis, namely,

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k, \tag{1.8}$$

where Γ_{ij}^k are smooth functions on the coordinate neighborhood, called the *Christoffel symbols*, given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{km} \{\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\}, \tag{1.9}$$

where $g_{ij} = g(\partial_i, \partial_j)$.

Here we are using (and we will keep doing so) the *Einstein summation convention*; every time two indices appear repeated in an expression, a summation on that index is understood, unless stated otherwise.

The first notion of curvature arises from the *Riemann curvature morphism*, which consists of a map $R(\cdot,\cdot)\cdot : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.10}$$

for all X, Y and Z in $\mathfrak{X}(M)$. From this morphism we can obtain a $(0,4)$-tensor, called the *curvature tensor*, defined as

$$Rm(X,Y,Z,W) = g(R(X,Y)Z, W), \tag{1.11}$$

for all X, Y, Z and W in $\mathfrak{X}(M)$.

We can express the Riemann curvature morphism in local coordinates as

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^{l} \partial_l, \tag{1.12}$$

while the curvature tensor reads as

$$Rm = R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l, \tag{1.13}$$

where $R_{ijkl} = Rm(\partial_i, \partial_j, \partial_k, \partial_l)$ or, equivalently, $R_{ijkl} = g_{lm} R_{ijk}{}^{m}$.

Since working with a $(0,4)$-tensor can be rather complicated, we can look at other important quantities that also give information about the geometric properties of the

Riemannian manifold. The *Ricci tensor*, Ric, is a $(0,2)$-tensor obtained by taking trace (with respect to g) of the Riemann curvature morphism, that is to say,

$$\mathrm{Ric}(X,Y) = \mathrm{trace}(Z \mapsto R(Z,X)Y). \tag{1.14}$$

In local coordinates its components $\mathrm{Ric}_{ij} = R_{ij}$ are given by

$$R_{ij} = R_{kij}{}^{k} = g^{kl} R_{kijl}. \tag{1.15}$$

The *scalar curvature* is obtained by taking trace of the Ricci tensor, that is,

$$R = \mathrm{tr}_g \mathrm{Ric} = g^{ij} R_{ij}. \tag{1.16}$$

Definition 1. We say that a Riemannian manifold (M,g) is positive scalar if at every p in M, the scalar curvature R is positive; we denote the set of such smooth metrics as $\mathrm{Scal}^{+}(M)$.

If $\Pi \subset T_pM$ is a 2-plane, the *sectional curvature* of Π is defined by

$$K(\Pi) = \frac{Rm(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \tag{1.17}$$

where $\{X,Y\}$ span the plane Π. This definition is independent of the basis. In particular, it follows from the definition of scalar curvature that when $n=2$, $R=2K$, where K is the sectional curvature of (M,g) (also called *Gaussian curvature*).

Definition 2. We say that a Riemannian manifold (M,g) has constant sectional curvature if the sectional curvature of every 2-plane is the same. That is, there exists a number C, such that at any p in M and $\Pi \subset T_pM$, $K(\Pi) = C$.

Remark 1.2.1. Let M be a complete, simply connected Riemannian n-dimensional manifold with constant curvature C. Then, M is isometric to one of the following model spaces: $\mathbb{R}^n$, a round n-sphere of radius $C^{-1/2}$ or a hyperbolic space of radius $C^{-1/2}$ (see [23], Theorem 11.12).

Integration of functions over a manifold (M, g) is possible through the *volume form*, dV_g. If $\{E_1, ..., E_n\}$ is any local oriented frame,

$$dV_g = \sqrt{\det(g_{ij})}\, \varphi^1 \wedge ... \wedge \varphi^n, \tag{1.18}$$

where $g_{ij} = g(E_i, E_j)$ and $\{\varphi^i\}$ is the dual coframe. In particular, the *volume* of a manifold, denoted by $\mathrm{vol}\,(g)$, is given by the integral of the constant function $f = 1$ over M:

$$\mathrm{vol}\,(g) = \int_{S^n} dV_g. \tag{1.19}$$

Example 1.2.2. Consider the sphere $S^n = \{x \in \mathbb{R}^{n+1} \,|\, |x| = 1\}$, where $|\cdot|$ denotes the Euclidean norm. Endow S^n with the induced Euclidean metric from $\mathbb{R}^{n+1}$, we will refer to this metric as the standard metric on S^n and we will denote it by g_* (whenever the dimension is understood). Thus, (S^n, g_*) is a simply connected manifold with constant positive sectional curvature $K = 1$ and scalar curvature $R = n(n-1)$. We denote by ω_n its volume. For example, for $n = 2$, $K = 1$, $R = 2$ and $\omega_2 = 4\pi$.

Let T be a $(0, p)$-tensor. We define the *divergence* of T as the $(p-1)$-tensor, given by

$$\mathrm{div}\, T(X_1, ..., X_{p-1}) = \mathrm{tr}_g^{YZ} \nabla_Z T(X, ..., X_{p-1}, Y). \tag{1.20}$$

In particular, if T is a $(0,2)$-tensor, its components in coordinates are given by

$$(\operatorname{div} T)_i = g^{jk} T_{ij;k}, \tag{1.21}$$

and if ω is a one form,

$$\operatorname{div} \omega = g^{ij} \omega_{i;j}, \tag{1.22}$$

where ; denotes covariant derivative.

For a vector field X, for simplicity, the definition of the divergence is given in local coordinates as

$$\operatorname{div} X = \nabla_{\partial_i} X^i = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} (\sqrt{|g|} X^i), \tag{1.23}$$

where $X = X^i \partial_i$. This allows us to define the *Laplacian* of a smooth function f, Δf, as

$$\Delta f = \operatorname{div} (\nabla f). \tag{1.24}$$

Here ∇f is the *gradient* of f, that is, the unique vector field on M such that $g(\nabla f, X) = df(X)$. In coordinates, $\nabla f = \nabla^i f \partial_i = g^{ij} \nabla_{\partial_j} f \partial_i = g^{ij} f_{,j} \partial_i = g^{ij} \partial_j f \partial_i$, and hence,

$$\Delta f = \nabla_{\partial_i} \nabla^i f = g^{ij} \nabla_{\partial_i} \nabla_{\partial_j} f = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x^j} \right), \tag{1.25}$$

where $|g| = \det(g)$.

We can use the metric to define an inner product of tensors. Let S and T be $(0,p)$-tensors, their product is given in coordinates by

$$\langle S, T \rangle = g^{i_1 j_1} ... g^{i_p j_p} S_{i_1 ... i_p} T_{j_1 ... j_p}. \tag{1.26}$$

Hence, the *norm* of a $(0,p)$-tensor T is defined by

$$\langle T, T\rangle = |T|^2 = g^{i_1 j_1} ... g^{i_p j_p} T_{i_1 ... i_p} T_{j_1 ... j_p}. \tag{1.27}$$

In particular, if T is a $(0,2)$-tensor, we have

$$|T|^2 = g^{ik} g^{jl} T_{ij} T_{kl}. \tag{1.28}$$

Direct computations show how some important geometric quantities behave under a scaling, which we summarize in the following lemma.

Lemma 1.2.3. *Let $\tilde{g} = \lambda g$, where $\lambda > 0$ is a real number. The following identities hold:*

$$\tilde{g}^{ij} = \lambda^{-1} g^{ij}, \tag{1.29}$$

$$\widetilde{Ric} = Ric, \tag{1.30}$$

$$\tilde{R} = \lambda^{-1} R, \tag{1.31}$$

$$d\tilde{V} = \lambda^{n/2} dV, \ \text{and} \tag{1.32}$$

$$\tilde{\Delta} f = \lambda^{-1} \Delta f. \tag{1.33}$$

1.2.2 The structure equations

Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold and (M,g) be a submanifold of codimension 1, i.e., a hypersurface. Recall that the *second fundamental form* of (M,g) in $(\tilde{M}, \tilde{g})$ is the map $\mathrm{II} : T_pM \times T_pM \longrightarrow (T_pM)^{\perp}$, defined by

$$\mathrm{II}(X, Y) = (\tilde{\nabla}_X Y)^{\perp} = \langle \tilde{\nabla}_X Y, \nu \rangle \nu, \tag{1.34}$$

where $\langle \cdot, \cdot \rangle$ denotes the product in $\tilde{M}$ and ν denotes a unit vector in the orthogonal complement of T_pM, $(T_pM)^{\perp}$. We define the *scalar second fundamental form* (which is a $(0, 2)$-tensor) as $K(X, Y) = -\langle \tilde{\nabla}_X Y, \nu \rangle$, i.e., $\mathrm{II}(X, Y) = -K(X, Y)\nu$.

The *mean curvature* H is defined as the trace of the scalar second fundamental from with respect to g, i.e., $H = \mathrm{tr}_g K$. Note that under this sign convention, the mean curvature of the standard round sphere, with respect to the outward unit vector, is positive. We say that (M, g) is *mean convex* if $H \geq 0$ for all p in M.

Remark 1.2.4. For simplicity, sometimes we will refer to K as the second fundamental form.

It is very useful to know how the geometry of the ambient manifold and the geometry of the submanifold are related. One relation is given by the well known *Gauss Equation*, which relates the Riemann tensor of the ambient space and the Riemann tensor of the submanifold by means of evaluating the Riemann tensor of the ambient on vector fields tangent to M:

$$\tilde{Rm}(X, Y, Z, W) = Rm(X, Y, Z, W) - \langle \mathrm{II}(X, W), \mathrm{II}(Y, Z) \rangle + \langle \mathrm{II}(X, Z), \mathrm{II}(Y, W) \rangle, \tag{1.35}$$

for any X, Y, Z and W in T_pM.

A straightforward computation consisting of taking traces two times in (1.35), yields the following relation:

$$\tilde{R} - 2\tilde{\mathrm{Ric}}(\nu, \nu) = R - H^2 + |K|^2. \tag{1.36}$$

We will refer to (1.36) also as the Gauss equation.

The *Codazzi equation* relates the normal component of the Riemann tensor of the ambient space to the covariant derivative of the second fundamental form:

$$\langle \tilde{R}(X, Y)Z, \nu \rangle = (\nabla_Y K)(X, Z) - (\nabla_X K)(Y, Z). \tag{1.37}$$

1.2.3 Conformal metrics

Let (M, g) be a Riemannian manifold.

Definition 3. A metric $\tilde{g}$ is said to be conformal to g if and only if there exists a diffeomorphism $f : M \to M$ and a positive function ρ in $C^\infty(M)$ such that

$$\tilde{g} = \rho f^* g, \tag{1.38}$$

where f^*g denotes the pullback of g by f. If f is the identity, we say that $\tilde{g}$ is a conformal deformation of g.

Definition 4. We define the conformal class of g, $[g]$, as the set of metrics that are conformal deformations of g, that is,

$$[g] = \{\rho g \,|\, \rho \in C^\infty(M)\, ,\, \rho > 0\}. \tag{1.39}$$

Direct computations provide a formula for the scalar curvature under a conformal change $\tilde{g} = \rho g$ (see [36]):

$$\tilde{R} = \rho^{-1} R_g - (n-1)\rho^{-2}\Delta_g \rho - \frac{1}{4}(n-1)(n-6)\rho^{-3}|\nabla^g \rho|^2. \tag{1.40}$$

There are two special cases (in which we can get rid of the gradient).

(1) For $n = 2$. Let $\rho = e^{2u}$, where u is a smooth function on M, then

$$\tilde{R} = e^{-2u}(R_g - 2\Delta_g u). \tag{1.41}$$

(2) For $n > 2$. Let $\rho = u^{\frac{4}{n-2}}$, where $u > 0$ is a smooth function on M, then

$$\tilde{R} = u^{-\frac{n+2}{n-2}} \left(R_g u - \frac{4(n-1)}{n-2} \Delta_g u \right), \tag{1.42}$$

or equivalently,

$$u^{\frac{n+2}{n-2}} \tilde{R} = u R_g - \frac{4(n-1)}{n-2} \Delta_g u. \tag{1.43}$$

We will restrict our discussion of conformal metrics to the manifolds $\mathbb{R}^n$ and S^n.

Definition 5. The Riemannian manifold $(\mathbb{R}^n, g_u)$ is called conformally flat if

$$g_u = u^{\frac{4}{n-2}} g_E, \tag{1.44}$$

where $u > 0$ is a smooth function on $\mathbb{R}^n$ and g_E is the Euclidean metric on $\mathbb{R}^n$.

From (1.43), the scalar curvature of a conformally flat manifold $(\mathbb{R}^n, g)$ $(n \geq 3)$ is given by

$$R_{g_u} = -\frac{4(n-1)}{n-2} \frac{\Delta u}{u^{\frac{n+2}{n-2}}}, \tag{1.45}$$

where Δ is the usual Laplacian of $\mathbb{R}^n$.

Suppose that Σ is an $(n-1)$-dimensional closed submanifold of $\mathbb{R}^n$ $(n \geq 3)$. Note that Σ carries two metrics, the induced by the Euclidean metric g_E, and the induced by g_u. Let ν and ν_u be the outward unit normal vectors to Σ with respect to g_E and g_u, respectively. We can relate the mean curvatures of Σ with respect to g_E and g_u.

Lemma 1.2.5. *The mean curvatures H_E and H_u of Σ with respect to g_E and g_u, respectively, are related by*

$$H_u = \frac{2(n-1)}{n-2} u^{-\frac{n}{n-2}} g_E(\nabla u, \nu) + u^{-\frac{2}{n-2}} H_E, \tag{1.46}$$

where ∇u denotes the usual gradient in $\mathbb{R}^n$.

In particular, for $n = 3$, we have

$$H_u = 4u^{-3} g_E(\nabla u, \nu) + u^{-2} H_E. \tag{1.47}$$

Proof. Locally, we can find coordinates of $\mathbb{R}^n$ such that $\{\partial_1, ..., \partial_{n-1}\}$ are coordinate vectors of Σ and $\partial_n = \nu$. Note that $\nu_u = u^{-2}\nu$.

We begin by computing the Christoffel symbols of $(\mathbb{R}^n, g_u)$, ${}^u\Gamma_{ij}^k$, in terms of the Christoffel symbols with respect to g_E, ${}^E\Gamma_{ij}^k$. By (1.9) we have

$${}^u\Gamma_{ij}^k = \frac{2}{n-2} u^{-1} g^{km}(u_j g_{im} + u_i g_{jm} - u_m g_{ij}) +^E \Gamma_{ij}^k. \tag{1.48}$$

It follows that the second fundamental form of Σ with respect to g_u, ${}^uK_{ij}$, is given by

$${}^uK_{ij} = -g_u({}^u\Gamma_{ij}^k \partial_k, \nu_u) \tag{1.49}$$

$$= u^{\frac{2}{n-2}} \left[\frac{2}{n-2} u^{-1} u_n g_{ij} \right] + u^{\frac{2}{n-2}} \, {}^{E}K_{ij}. \tag{1.50}$$

Hence, we have

$$H_u = g_u^{ij} \, {}^{u}K_{ij} = \frac{2(n-1)}{n-2} u^{\frac{-n}{n-2}} g_E(\nabla u, \nu) + u^{-\frac{2}{n-2}} \, {}^{E}H. \tag{1.51}$$

□

We finish this section by noting that if we consider the conformal class of the standard metric on S^n, $[g_*]$, and recall that $R_* = n(n-1)$, then for any $g \in [g_*]$, if $n = 2$ and $\rho = e^{2u}$, (1.41) becomes

$$\tilde{R} = 2e^{-2u}(1 - \Delta_* u), \tag{1.52}$$

and for $n \geq 3$ and $\rho = u^{\frac{4}{n-2}}$, (1.43) becomes

$$\frac{n-2}{4(n-1)} u^{\frac{n+2}{n-2}} R_g = \frac{n(n-2)}{4} u - \Delta_* u. \tag{1.53}$$

1.2.4 Families of metrics

Let $\{g_t\}$ be a smooth 1-parameter family of smooth metrics on a manifold M. It will be of our interest to know the rate of change of the volume form dV_{g_t}. Note that at a given point in M, the derivative of volume forms with respect to t gives another top-dimensional differential form, then we know that $\frac{d}{dt} dV_{g_t} = \lambda(t) dV_{g_t}$ for some real number $\lambda(t)$. We recall the well-known matrix algebra formula to compute the derivative of the determinant of a differentiable path of invertible matrices $A(t)$, given by

$$\frac{d}{dt}\det A(t) = \det A(t)\cdot \mathrm{tr}(A^{-1}(t)A'(t)). \tag{1.54}$$

Thus we have,

$$\frac{d}{dt}dV_{g_t} = \frac{1}{2}\frac{(\det g(t))'}{\sqrt{\det g(t)}}\,dx^1\wedge\ldots\wedge dx^n \tag{1.55}$$

$$= \frac{1}{2}\mathrm{tr}(g^{-1}(t)\dot{g}(t))dV_{g_t}. \tag{1.56}$$

Since for $(0,2)$-tensors on (M,g), $\langle h_{ij}, k_{ij}\rangle = g^{im}g^{jn}h_{ij}k_{mn}$, we obtain:

$$\frac{d}{dt}dV_{g_t} = \frac{1}{2}\langle g_t, \dot{g}_t\rangle dV_{g_t} = \frac{1}{2}\mathrm{tr}_{g_t}\dot{g}_t\,dV_{g_t}. \tag{1.57}$$

Given a family of metrics $\{g_t\}$ we can define another family $\{h_t\}$ by considering the pullback

$$h_t = \phi_t^*(g_t), \tag{1.58}$$

where $\{\phi_t\}$ is the integral flow of a t-dependent vector field X_t. A well known formula allows us to compute $\frac{d}{dt}\phi_t^*(g_t)$ as follows:

$$\dot{h}_t = \frac{d}{dt}\phi_t^*(g_t) = \phi_t^*\left(\frac{\partial g}{\partial t}\right) + \phi_t^*(\mathcal{L}_X g_t), \tag{1.59}$$

where, $\mathcal{L}_X g$ is the *Lie derivative*, defined as

$$\mathcal{L}_Y g(y) = \frac{d}{dt}\phi_t^*(g)(y)\bigg|_{t=0}, \tag{1.60}$$

for a vector field Y in (M,g). Here ϕ_t is the flow generated by Y.

A direct computation gives the coordinate expression of the Lie derivative of the metric g, which is given by

$$(\mathcal{L}_Y g)_{ij}(y) = \frac{\partial Y^k}{\partial y^i}(y) g_{kj}(y) + \frac{\partial Y^l}{\partial y^j}(y) g_{il}(y) + Y^p(y)\frac{\partial g_{ij}}{\partial y^p}(y). \tag{1.61}$$

In later calculations we will need the following two lemmas:

Lemma 1.2.6. *Let g and h be $(0,2)$-tensors on M and $\phi : M \longrightarrow M$ be a diffeomorphism. Then,*

$$tr_{\phi^* g}\phi^* h = \phi^*(tr_g h). \tag{1.62}$$

Proof. Let $\phi(x) = (y^1(x), \ldots, y^n(x)) = y$, be the local expression of ϕ.

$$\begin{aligned} \mathrm{tr}_{\phi^* g}\phi^* h(x) &= (\phi^* g)^{ij}(\phi^* h)_{ij}(x) && (1.63)\\ &= \frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^l} g^{kl}(y)\frac{\partial y^m}{\partial x^i}\frac{\partial y^n}{\partial x^j} h_{mn}(y) && (1.64)\\ &= g^{kl}(y) h_{kl}(y) && (1.65)\\ &= \phi^*(\mathrm{tr}_g h). && (1.66)\end{aligned}$$

□

Lemma 1.2.7. *Let (M, g) be a Riemannian manifold. Then*

$$tr_g(\mathcal{L}_X g) = 2\,div_g X. \tag{1.67}$$

Proof. In coordinates (see (1.23)),

$$\mathrm{div}_g X = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^k}(\sqrt{|g|}X^k) \tag{1.68}$$

$$= \frac{1}{2}\frac{1}{|g|}\frac{\partial}{\partial x^k}(|g|)X^k + \frac{\partial X^k}{\partial x^k} \tag{1.69}$$

$$= \frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial x^k}X^k + \frac{\partial X^k}{\partial x^k}. \tag{1.70}$$

Using (1.61), we have

$$\operatorname{tr}_g(\mathcal{L}_X g) = g^{ij}(\mathcal{L}_X g)_{ij} \tag{1.71}$$

$$= g^{ij}\left[\frac{\partial X^k}{\partial x^i}g_{kj} + \frac{\partial X^l}{\partial x^j}g_{il} + X^k\frac{\partial g_{ij}}{\partial x^k}\right] \tag{1.72}$$

$$= 2\frac{\partial X^k}{\partial x^k} + X^k g^{ij}\frac{\partial g_{ij}}{\partial x^k} \tag{1.73}$$

$$= 2\operatorname{div}_g X. \tag{1.74}$$

□

Let us consider now a special setting to derive some formulas that will be used later. Let Σ be a manifold of dimension n, and consider the $(n+1)$-dimensional manifold $I \times \Sigma$.

Let v be a positive function on $I \times \Sigma$ and $\{h_t\}_{0\leq t\leq 1}$ be a smooth family of smooth metrics on Σ. Consider the metric $g = v^2dt^2 + h_t$ on $I \times \Sigma$ and define a "background metric" $g_b = dt^2 + h_t$. We are interested in deriving some relations between geometric quantities of these metrics. Let $\{\partial_t, \partial_1, ..., \partial_n\}$ be a coordinate basis such that $g_b(\partial_i, \partial_j) = (h_t)_{ij} = g(\partial_i, \partial_j)$. For simplicity, we will drop the sub-index t in h_t, but we will keep in mind that h depends on t. Note that $\nu_b = \partial_t$ and $\nu = v^{-1}\partial_t$ are unit normal vectors to Σ with respect to g_b and g, respectively. We will attach the letter

"b" to the geometric quantities with respect to g_b. We have in coordinates,

$$\Gamma^t_{ij} = \frac{1}{2}g^{tm}\{\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\} = -\frac{1}{2}v^{-2}\partial_t h_{ij}, \tag{1.75}$$

and

$${}^b\Gamma^t_{ij} = \frac{1}{2}g_b^{tm}\{\partial_i g_{b\,jm} + \partial_j g_{b\,im} - \partial_m g_{b\,ij}\} = -\frac{1}{2}\partial_t h_{ij}. \tag{1.76}$$

Thus, we have

$$K_b(\partial_i, \partial_j) = -g_b(\nabla_{\partial_i}\partial_j, \nu_b) = \frac{1}{2}\partial_t h_{ij}, \tag{1.77}$$

and

$$K(\partial_i, \partial_j) = -g(\nabla_{e_i}e_j, \nu) = v^{-1}\frac{1}{2}\partial_t h_{ij} = v^{-1}K_b(\partial_i, \partial_j), \tag{1.78}$$

and therefore,

$$K = v^{-1}K_b, \tag{1.79}$$

and in consequence, the mean curvatures H and H_b are related by

$$H = v^{-1}H_b. \tag{1.80}$$

We want to compare the scalar curvatures of g and g_b. To do this, we will use the Gauss Equation (1.36):

$$R - 2\mathrm{Ric}(n, n) = R_h - H^2 + |K|^2, \tag{1.81}$$

or equivalently,

$$R = R_h + 2\mathrm{Ric}(n, n) - H^2 + |K|^2. \tag{1.82}$$

We will also use the second variation formula (C.18), which in this setting reads as

$$\partial_t H = -\Delta_h v - (\mathrm{Ric}(n,n) + |K|^2)v. \tag{1.83}$$

Combining (1.82) and (1.83), we have

$$\begin{aligned} R &= R_h - H^2 - |K|^2 + 2\left(\mathrm{Ric}(n,n) + |K|^2\right) && (1.84)\\ &= R_h - H^2 - |K|^2 - 2\left(v^{-1}\partial_t H + v^{-1}\Delta_h v\right) && (1.85)\\ &= R_h - 2v^{-1}\Delta_h v - 2v^{-1}\partial_t H - H^2 - |K|^2. && (1.86)\end{aligned}$$

We can express R in terms of the background metric using (1.80),

$$\partial_t(H) = \partial_t(v^{-1}H_b) = -v^2\partial_t v H_b + v^{-1}\partial_t H_b, \tag{1.87}$$

hence

$$\begin{aligned} R &= R_h - 2v^{-1}\Delta_h v - 2v^{-1}\left(-v^{-2}\partial_t v H_b + v^{-1}\partial_t H_b\right) - v^{-2}H_b^2 - v^{-2}|K_b|^2 && (1.88)\\ &= R_h - 2v^{-1}\Delta_h v + 2v^{-3}\partial_t v H_b - v^{-2}(2\partial_t H_b + H_b^2 + |K_b|^2) && (1.89)\\ &= 2v^{-1}\left[-\Delta_h v + \frac{1}{2}R_h v\right] + 2v^{-3}\partial_t v H_b - v^{-2}(2\partial_t H_b + H_b^2 + |K_b|^2). && (1.90)\end{aligned}$$

Remark 1.2.8. For a metric of the form $g = v(t,x)^2dt^2 + h_t$, the mean curvature of a cross-section $\{t\} \times \Sigma$ with respect to the unit normal $v^{-1}\partial_t$, is given by

$$H = \frac{1}{2v(t,x)}\mathrm{tr}_h \dot{h}. \tag{1.91}$$

Lemma 1.2.9. *Let* $g = v(t,x)^2dt^2 + h_t$ *be a metric on* $I \times \Sigma$*, where* $\{h_t\}_{0\leq t\leq 1}$ *is* *a family of Riemannian metrics on the n-dimensional manifold Σ. Then the scalar curvature R of g is given by*

$$R = 2v^{-1}\left[-\Delta_h v + \frac{1}{2}R_h v\right] + v^{-2}\left(2v^{-1}\partial_t v tr_h \dot{h} + \frac{3}{4}|\dot{h}|_h^2 - tr_h \ddot{h} - \frac{1}{4}tr_h \dot{h}^2\right). \quad (1.92)$$

In particular, when $v = 1$ and $h_t = f(t)^2 g_*$, we have

$$R = nf^{-2}\left[(n-1) - (n-1)\dot{f}^2 - 2f\ddot{f}\right]. \quad (1.93)$$

Proof. Since $g = v^2 dt^2 + h_t$ and $g_b = dt^2 + h_t$, we have

$$|K_b|^2 = \frac{1}{4}h^{ik}h^{lj}\dot{h}_{ij}\dot{h}_{kl} = \frac{1}{4}|\dot{h}|^2, \quad (1.94)$$

$$H_b = \frac{1}{2}\mathrm{tr}_h \dot{h}, \quad (1.95)$$

$$2\partial_t H_b = \dot{h}^{ij}\dot{h}_{ij} + h^{ij}\ddot{h}_{ij} = -h^{ij}\dot{h}_{ij} + \mathrm{tr}_h \ddot{h} = -|\dot{h}|^2 + \mathrm{tr}_h \ddot{h}. \quad (1.96)$$

From (1.90), we obtain

$$R = 2v^{-1}\left[-\Delta_{h_t} v + \frac{1}{2}R_h v\right] + v^{-2}\left(2v^{-1}\partial_t v \mathrm{tr}_h \dot{h} + \frac{3}{4}|\dot{h}|^2 - \mathrm{tr}_h \ddot{h} - \frac{1}{4}\mathrm{tr}_h \dot{h}^2\right), \quad (1.97)$$

as desired.

If $v = 1$ and $h_t = f(t)^2 g_*$, then $\dot{h} = 2f\dot{f}g_*$, $\ddot{h} = 2\dot{f}^2 g_* + 2f\ddot{f}g_*$ and

$$\mathrm{tr}_h \dot{h} = 2nf^{-1}\dot{f}, \quad (1.98)$$

$$\mathrm{tr}_h \ddot{h} = 2nf^{-2}\dot{f}^2 + 2nf^{-1}\ddot{f}, \quad (1.99)$$

$$|\dot{h}|^2 = 4nf^{-2}\dot{f}^2. \quad (1.100)$$

Applying (1.90) again,

$$R = f^{-2}n(n-1) + nf^{-2}\dot{f}^2 - 2nf^{-1}\ddot{f} - n^2f^{-2}\dot{f}^2 \tag{1.101}$$
$$= nf^{-2}\left[(n-1) - (n-1)\dot{f}^2 - 2f\ddot{f}\right]. \tag{1.102}$$

□

1.3 Analysis on manifolds

Some of the arguments that will be used in Chapters 2 and 3 require some terminology and results from PDE theory. For the sake of completeness we include them here and provide references to a more delicate treatment or proofs when needed. We will restrict our discussion to closed Riemannian manifolds.

We define the Banach space $C^k(M)$ as the set of C^k functions on a closed Riemannian manifold (M, g) with the norm

$$|f|_{C^k(M)} = \|f\|_{C^k(M)} = \sum_{m=1}^{k} \sup_M |\nabla^m f|, \tag{1.103}$$

where ∇^m denotes the m-iterated covariant derivative. When $\mathbf{f}$ is a vector valued function on M, we define its norm as the sum of the norms of each component. Similarly, for tensor fields, let $\mathcal{T}^k_p(M)$ denote the set of C^k $(0, p)$-tensor fields with the norm

$$|T|_{C^k} = \|T\|_{C^k(M)} = \sum_{m=1}^{k} \sup_M |\nabla^m T|. \tag{1.104}$$

Similarly, for $\alpha \in (0,1)$, we define the corresponding Hölder spaces with

$$|f|_{C^{k,\alpha}(M)} = \|f\|_{C^{k,\alpha}(M)} = |f|_{C^k(M)} + \sup_{|\beta|=k} \sup_{\substack{x,y\in M \\ x\neq y}} \frac{|\nabla^\beta f(x) - \nabla^\beta f(y)|}{|x-y|^\alpha}, \tag{1.105}$$

where $|x-y|$ denotes the distance between x and y in (M,g) and the difference $|\nabla^\beta f(x) - \nabla^\beta f(y)|$ is made meaningful by parallel transporting $\nabla^\beta f(y)$ to x along a minimal geodesic. Similarly,

$$|T|_{C^{k,\alpha}(M)} = \|T\|_{C^{k,\alpha}(M)} = |T|_{C^k(M)} + \sup_{|\beta|=k} \sup_{\substack{x,y\in M \\ x\neq y}} \frac{|\nabla^\beta T(x) - \nabla^\beta T(y)|}{|x-y|^\alpha}, \tag{1.106}$$

Remark 1.3.1. Since M is closed, it is a well-known fact that we can consider an equivalent norm by expressing the quantities in a fixed local system of coordinates and consider the usual $C^{k,\alpha}$ norm for the coordinate expressions.

We will need the following results from elliptic theory on closed manifolds. Recall that the local expression of the Laplacian on a Riemannian manifold (M,g) (1.25) can be re-written as the elliptic operator

$$\Delta = g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial}{\partial x^k}\right). \tag{1.107}$$

Since on a closed manifold the maximum and minimum of a smooth function are achieved, the following lemma follows from the usual *strong maximum principle* (see [16]).

Lemma 1.3.2. *Let (M,g) be a closed Riemannian manifold. If f is a harmonic ($\Delta_g f = 0$), subharmonic ($\Delta_g f \geq 0$) or superharmonic ($\Delta_g f \leq 0$) function, then f is constant.*

Along the same lines, one can extend the existence results for the Dirichlet problem to closed manifolds (see [21]):

Theorem 1.3.3. *On a closed Riemannian manifold (M, g), if f is a smooth function that integrates to 0 on M, then the Poisson equation $\Delta\psi = f$ has a unique smooth solution up to the addition of a constant.*

Consider a closed Riemannian manifold (M, g) and the elliptic linear operator

$$L = -\Delta_g + \frac{1}{2}R_g, \tag{1.108}$$

where R_g is the scalar curvature of g.

Using standard theory of elliptic operators (see [13, 16]), the first eigenvalue λ_1 of this operator can be obtained via the *Rayleigh quotient*. Using integration by parts we have

$$L(f, f) = \int_M (-\Delta_g f^2 + \frac{1}{2}R_g f^2)\, dA_g = \int_M (|\nabla^g f|_g^2 + \frac{1}{2}R_g f^2)\, dA_g, \tag{1.109}$$

then the first eigenvalue of L is given by

$$\lambda_1 = \inf_f \left\{ \int_M (|\nabla^g f|_g^2 + \frac{1}{2}R_g f^2)\, dA_g\, , \int_M f^2\, dA = 1 \right\}. \tag{1.110}$$

Moreover, the eigenspace corresponding to λ_1 is one-dimensional and has a positive eigenfunction. Note that if $R_g \geq 0$, $\lambda_1 \geq 0$.

In Chapter 2, we will be interested in the case when $M = S^n$ and λ_1 is positive. In particular, when $n = 2$, for a smooth function u in S^2, consider $g = e^{2u}g_*$ and suppose that $\lambda_1 > 0$. We can consider the family of conformal deformations of the standard metric g_* given by

$$g(t) = e^{2tu} g_*, \tag{1.111}$$

to connect g to the standard metric g_*. It turns out that due to the conformal invariance of (1.109) in dimension $n = 2$, the positivity of λ_1 is preserved along the path. Since

$$\int_{S^2} |\nabla^{g_t} f|^2_{g_t} + K_{g_t} f^2 \, dA_{g_t} = \int_{S^2} [e^{-2wt} |\nabla^* f|^2_* + e^{-2wt}(1 - t\Delta_{g_*} w) f^2] \, e^{2wt} dA_* \tag{1.112}$$

$$= \int_{S^2} |\nabla^* f|^2_* + (1 - t\Delta_{g_*} w) f^2 \, dA_*, \tag{1.113}$$

it follows that

$$\begin{aligned} \int_{S^2} |\nabla^{g_t} f|^2_{g_t} + K_{g_t} f^2 \, dA_{g_t} = t \int_{S^2} |\nabla^* f|^2_* + (1 - \Delta_{g_*} w) f^2 \, dA_* \\ + (1-t) \int_{S^2} |\nabla^* f|^2_* + f^2 \, dA_*, \end{aligned} \tag{1.114}$$

$$\geq t\lambda_0 + (1-t)\lambda_1, \tag{1.115}$$

and $t\lambda_0 + (1-t)\lambda_1$ is positive because it is a sum of positive numbers.

Remark 1.3.4. In dimension $n = 2$, by the Uniformization Theorem, any smooth metric on S^2 can be written in the form $g = e^{2u} g_*$ for some smooth function u.

In higher dimensions $n \geq 3$, we do not have the conformal invariance anymore and it is not clear whether a given path of this type will preserve the positivity of λ_1. However, we note that when $R_g > 0$,

$$\int_M (|\nabla^g f|^2_g + \frac{1}{2} R_g f^2) \, dA_g \geq \int_M \frac{1}{2} R_g f^2 \, dA_g > c, \tag{1.116}$$

where $c = c(g) > 0$, i.e., $\lambda_1 > 0$.

1.4 Asymptotically flat spaces

An asymptotically flat space is a Riemannian manifold such that its metric approaches the Euclidean metric at infinity at a certain rate, more precisely (c.f. [1, 2, 5]):

Definition 6. A Riemannian manifold (M^n, g) is said to be *asymptotically flat (AF)* if there is a compact set $K \subset M$ and a diffeomorphism $\Phi : M \setminus K \longrightarrow \mathbb{R}^n \setminus B_1(0)$ such that in the coordinate chart near infinity defined by Φ,

$$g = \sum_{i,j} g_{ij}(x) dx^i dx^j, \tag{1.117}$$

where

$$g_{ij}(x) = \delta_{ij} + O(|x|^{-p}) \tag{1.118}$$

$$|x||g_{ij,k}(x)| + |x|^2 |g_{ij,kl}(x)| = O(|x|^{-p}) \tag{1.119}$$

$$|R_g| = O(|x|^{-q}), \tag{1.120}$$

when $|x| \gg 1$, for some $p > \dfrac{n-2}{2}$ and some $q > n$. Here, commas denote partial derivatives in the coordinate chart.

Asymptotically flat manifolds are of special interest in mathematical relativity since they represent suitable initial data sets for the initial value problem associated to the Einstein equations.

A remarkable property of asymptotically flat manifolds is that they possess a well-defined notion of *total mass*, called ADM-mass [1], denoted by $m_{\text{ADM}}(M)$. This definition is motivated by the Hamiltonian formulation of general relativity and it is obtained through a flux integral at asymptotic infinity.

Definition 7. The ADM mass of an asymptotically flat manifold (M^n, g) is defined by

$$m_{\mathrm{ADM}}(g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{\sigma\to\infty} \int_{S_\sigma} (g_{ij,i} - g_{ii,j})\nu^j \, d\mu, \tag{1.121}$$

where in a coordinate chart near infinity, S_σ is the coordinate sphere of radius σ, ν is the unit normal to S_σ and $d\mu$ is the volume element of S_σ.

A cornerstone result in the development of mathematical relativity is due to Schoen and Yau [36], and to Witten [41]. It establishes the positivity of the ADM-mass and moreover, its rigidity.

Theorem 1.4.1 (Positive Mass Theorem)**.** *Let (M, g) be a complete, asymptotically flat 3-manifold with non-negative scalar curvature and total mass m. Then*

$$m \geq 0, \tag{1.122}$$

with equality if and only if (M, g) is isometric to $\mathbb{R}^3$ with the standard flat metric.

Definition 8. A hypersurface in a Riemannian manifold (M, g) is called minimal if its mean curvature H equals 0. It is outermost minimal if it is minimal and is not entirely contained inside another minimal surface.

Motivated by mathematical relativity, we will call a minimal surface a *horizon*, since they correspond to apparent horizons of black holes in the time-symmetric setting (assuming the dominant energy condition). More generally, a surface Σ in M is called *outer minimizing* if every other surface $\widetilde{\Sigma}$ which encloses it has at least the same area. When Σ is a horizon, the notion of outermost minimal and outer minimizing coincide.

Example 1.4.2. A basic and yet extremely important example of an outer-minimizing horizon in an asymptotically flat 3-dimensional manifold, is the $r = \frac{m}{2}$ surface in the *spatial Schwarzschild* manifold $(\mathbb{R}^3 \setminus B_{\frac{m}{2}}(0), g_m)$, with

$$g_m = \left(1 + \frac{m}{2r}\right)^4 (dr^2 + r^2 g_*), \tag{1.123}$$

where g_* denotes the standard metric on the unit 2-sphere and $m > 0$ is the "mass parameter". It can be easily checked that $m_{\mathrm{ADM}}(g_m) = m$ (see Lemma 1.5.1).

Remark 1.4.3. It is common to define the Schwarzschild manifold of mass m as $(\mathbb{R}^3 \setminus \{0\}, g_m)$, which has two asymptotically flat "ends". Since we only work with asymptotically flat manifolds with one end, we do not adopt this definition.

It is of mathematical and physical interest to know the relation between the total mass m_{ADM} of an initial data set and the area $|\Sigma|$ of its outer-minimizing horizon. This relation is given by the *Riemannian Penrose Inequality*, proved by Huisken and Ilmanen [22] using the Inverse Mean Curvature Flow (IMCF) when the horizon has exactly one connected component, and by Bray [4] in the case when the horizon has of multiple connected components using the so-called conformal flow.

Theorem 1.4.4 (Riemannian Penrose Inequality)**.** *Let (M, g) be a complete, asymptotically flat 3-manifold with non-negative scalar curvature and total mass m whose outer-minimizing horizon has surface area A. Then*

$$m \geq \sqrt{\frac{A}{16\pi}}, \tag{1.124}$$

with equality if and only if (M, g) *is isometric to the spatial Schwarzschild manifold* $(\mathbb{R}^3 \setminus \{0\}, g_m)$*, outside their respective horizons.*

1.5 Schwarzschild manifolds

In Example 1.4.2 we introduced the spatial Schwarzschild space. This asymptotically flat manifold plays an important role in the development of mathematical relativity and, in particular, its special geometric properties are essential to obtain the results described in Chapters 2 and 3. It is worth to review the main features of this class of manifolds, which we do in what follows. We start by stating its definition in higher dimensions (cf. Example 1.4.2) and then we prove some basic facts.

Definition 9. The $(n+1)$-dimensional spatial Schwarzschild space ($n \geq 2$), is defined as the Riemannian manifold $\left(\mathbb{R}^{n+1} \setminus B_{\left(\frac{m}{2}\right)^{\frac{1}{n-1}}}(0), g_m\right)$, where the metric g_m is given by

$$g_m = \left(1 + \frac{m}{2} r^{1-n}\right)^{\frac{4}{n-1}} g_E. \tag{1.125}$$

Here, g_E denotes the Euclidean metric on $\mathbb{R}^{n+1}$ and $r = |x|$.

A Schwarzschild manifold is manifestly an asymptotically flat manifold and hence we can compute its total mass.

Proposition 1.5.1. $m_{ADM}(g_m) = m$

Proof. From Definition 7, to obtain the ADM mass we need to compute the following terms:

$$g_{ij,i} = \frac{4}{n-1}\left(1 + \frac{m}{2|x|^{n-1}}\right)^{\frac{4}{n-1}-1}\left(\frac{m(1-n)}{2}\frac{x^i}{|x|^{n+1}}\right)\delta_{ij}, \tag{1.126}$$

and

$$g_{ii,j} = \frac{4}{n-1}\left(1+\frac{m}{2|x|^{n-1}}\right)^{\frac{4}{n-1}-1}\left(\frac{m(1-n)}{2}\frac{x^j}{|x|^{n+1}}\right)(n+1). \tag{1.127}$$

Since $\nu = \frac{x}{|x|}$, we have $\nu^j = \frac{x^j}{|x|}$, so

$$(g_{ij,i} - g_{ii,j})\nu^j = 2nm\left(1+\frac{m}{2|x|^{n-1}}\right)^{\frac{4}{n-1}-1}\frac{1}{|x|^n}. \tag{1.128}$$

Plugging (1.128) into (1.121) we obtain

$$m_{ADM} = \frac{1}{2n\omega_n}\lim_{\sigma\to\infty}\int_{S_\sigma} 2nm\left(1+\frac{m}{2\sigma^{n-1}}\right)^{\frac{4}{n-1}-1}\frac{1}{\sigma^n}\,dV_\sigma = m. \tag{1.129}$$

□

By setting

$$u = \left(1+\frac{m}{2}r^{1-n}\right), \tag{1.130}$$

we can see g_m as a conformally flat metric and use formula (1.45). According to this formula, to obtain the scalar curvature of g_m we only need to compute Δu. Since

$$\partial_i u = \frac{m}{2}(1-n)\frac{x^i}{|x|^{1+n}}, \tag{1.131}$$

and

$$\partial_{ii} u = \frac{m}{2}(1-n)\frac{1}{|x|^{n+1}} + \frac{m}{2}(1-n)(-1-n)\frac{(x^i)^2}{|x|^{n+3}}, \tag{1.132}$$

it follows that $\Delta u = 0$; as a result, we have obtained the following proposition:

Proposition 1.5.2. $\left(\mathbb{R}^{n+1}\setminus B_{\left(\frac{m}{2}\right)^{\frac{1}{n-1}}}(0), g_m\right)$ *is scalar-flat, that is,* $R_{g_m} = 0$.

Notice that by introducing polar coordinates, we can write the metric of the spatial Schwarzschild space as

$$g_m = \left(1 + \frac{m}{2r^{n-1}}\right)^{\frac{4}{n-1}} (dr^2 + r^2 g_*), \tag{1.133}$$

where g_* denotes the standard metric on S^n. By performing a standard change of variable defined by

$$r\left(1 + \frac{m}{2r^{n-1}}\right)^{\frac{2}{n-1}} = t, \tag{1.134}$$

we may view the Schwarzschild manifold in another familiar form (after replacing t by r), as the Riemannian manifold $([(2m)^{\frac{1}{n-1}}, \infty) \times S^n, g_m)$, where the metric g_m is given by

$$g_m = \left(1 - \frac{2m}{r^{n-1}}\right)^{-1} dr^2 + r^2 g_*. \tag{1.135}$$

Once we have g_m in this form, we can introduce a further change of variable given by

$$s = \int_{(2m)^{\frac{1}{n-1}}}^{r} \left(1 - \frac{2m}{r^{n-1}}\right)^{-1/2} dr, \tag{1.136}$$

to view it as the warped product manifold $([0, \infty) \times S^n, g_m)$, with the metric

$$g_m = ds^2 + u_m(s)^2 g_*. \tag{1.137}$$

In this form, the parameter s represents the distance function to the horizon. From (1.135) we obtain the condition $u_m(s) = r$, hence we can obtain the first derivative of the function $u_m(s)$:

$$u_m'(s) = \frac{dr}{ds} = \left(1 - \frac{2m}{r^{n-1}}\right)^{1/2} = \left(1 - \frac{2m}{u_m(s)^{n-1}}\right)^{1/2}. \tag{1.138}$$

We can go further and compute the second derivative of u_m:

$$u_m''(s) = \frac{1}{2}\left(1 - \frac{2m}{u_m^{n-1}}\right)^{-1/2} \frac{-2m(1-n)}{u_m^n} u_m' = (n-1)\frac{m}{u_m^n}, \tag{1.139}$$

This form of the spatial Schwarzschild manifold is suitable to be modified in a small region to have positive scalar curvature in it. This will be a key step in the construction in Chapter 2. The next proposition summarizes the properties of the function u_m.

Proposition 1.5.3. *A spatial Schwarzschild manifold of mass m can be viewed as the warped product manifold $([0,\infty) \times S^n, g_m)$, with*

$$g_m = ds^2 + u_m(s)^2 g_*, \tag{1.140}$$

where u_m is a function defined on $[0,\infty)$ such that

(a) $u_m(0) = (2m)^{\frac{1}{n-1}}$,

(b) $u_m'(0) = 0$,

(c) $u_m'(s) = \left(1 - \frac{2m}{u_m(s)^{n-1}}\right)^{1/2}$ *for $s > 0$, and*

(d) $u_m''(s) = (n-1)\frac{m}{u_m^n}$.

Using the expression (1.91) we see that the mean curvature of the surface $s = 0$ (i.e., the surface corresponding to $r = \frac{m}{2}$) is given by

$$H = \frac{n u_m'(0)}{u_m(0)} = 0. \tag{1.141}$$

Moreover, by the same formula, it follows that a Schwarzschild manifold of mass m is foliated by mean convex spheres. By a standard argument using the maximum principle, the surface $r = \frac{m}{2}$ is the unique horizon in a Schwarzschild manifold of mass m. To have a picture in mind, in Figure 1.2 we present a diagram of a Schwarzschild manifold that is useful for our later discussion.

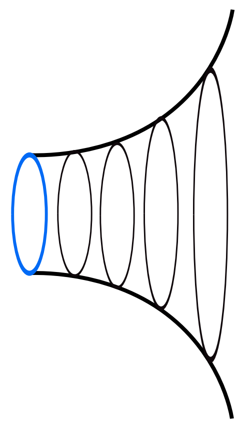

Figure 1.2: This diagram represents a Schwarzschild manifold of mass m. The blue circle represents the horizon $r = m/2$.

Chapter 2

Higher dimensional black hole initial data with prescribed boundary metric

2.1 Motivation

In Section 1.4, we introduced the Riemannian Penrose inequality, which bounds from below the total mass of an asymptotically flat 3-dimensional manifold with non-negative scalar curvature, (M, g), by a quantity depending on the area of its outer minimizing horizon. Specifically, if m denotes the total mass of such manifold, the Riemannian Penrose Inequality states that

$$m \geq \sqrt{\frac{A}{16\pi}}, \tag{2.1}$$

where A is the area of the outer minimizing horizon.

If we further assume that M has a non-empty boundary ∂M which is minimal and outer-minimizing (i.e., it is its horizon), then the Riemannian Penrose inequality reads as

$$m \geq \sqrt{\frac{\text{area}\,(g|_{\partial M})}{16\pi}}. \tag{2.2}$$

One important feature of the Riemannian Penrose inequality is its *rigidity*, that is, equality is achieved if and only if the manifold (M, g) is isometric to a 3-dimensional spatial Schwarzschild space of mass m; in particular, the horizon is a round sphere. It is therefore natural to ask whether the Riemannian Penrose inequality is *stable*, i.e., whether the proximity of the mass of an asymptotically flat manifold (with non-negative scalar curvature) to the optimal value determined by (2.1) implies that the geometry of its horizon is close to being rotationally symmetric (as in a Schwarzschild manifold).

Mantoulidis and Schoen in [25] developed a technique to obtain asymptotically flat extensions with non-negative scalar curvature of a given metric g on S^2, satisfying a special condition motivated by the stability of the horizon of a time-symmetric initial data set. Each of these extensions has (S^2, g) as its horizon and the remarkable property that its total mass can be arranged to be arbitrarily close to the optimal value in (2.1). This result shows the instability of the Riemannian Penrose inequality in dimension 3 in the sense described above.

In recent years, there has been a growing interest in black hole geometry in higher dimensions (see [11]). Galloway and Schoen in [14] obtained a generalization of Hawking's black hole topology theorem [20] to higher dimensions. Their result shows that in initial data sets satisfying the dominant energy condition, the outer-apparent horizons are of positive Yamabe type, i.e., they admit metrics of positive scalar curvature.

Bray and Lee in [5] proved the Riemannian Penrose inequality for dimensions less than eight, which we conveniently to our purpose state as follows:

Theorem 2.1.1 (Riemannian Penrose inequality in dimensions less then eight)**.** *Let* (M^{n+1}, g) *be a complete, asymptotically flat manifold with non-negative scalar curvature, where* $n < 7$*. Suppose that the boundary of* M*,* ∂M*, consists of closed outer-minimizing minimal hypersurfaces, then*

$$m_{ADM}(g) \geq \frac{1}{2}\left(\frac{\mathrm{vol}\,(g|_{\partial M})}{\omega_n}\right)^{(n-1)/n}. \tag{2.3}$$

Furthermore, if M *is spin, then equality occurs if and only if* (M, g) *is isometric to a Schwarzschild manifold outside is unique outer-minimizing horizon.*

Motivated by the above results, in this chapter we extend the techniques developed by Mantoulidis and Schoen to higher dimensions; that is, given an initial metric g of positive scalar curvature on S^n, we construct the corresponding asymptotically flat extensions of non-negative scalar curvature. After obtaining the construction in higher dimensions, we will focus on the geometric conditions that we need to impose on the initial metric g to be able to accomplish it.

The main ingredient of the construction in higher dimensions is the replacement of the use of the Uniformization Theorem in [25] with a suitable smooth path of metrics of positive scalar curvature connecting the given metric to a round metric, specifically, for $n = 3$ we use a fundamental result of Marques [26] regarding deformations of metrics of positive scalar curvature in S^3, and for $n \geq 3$, we apply the results of a special type of inverse curvature flow [15, 37]. The results of this chapter can be found in [7].

2.2 Mantoulidis-Schoen construction in higher dimensions

Let $\mathscr{M}^+ = \{g \text{ smooth metric on } S^2 \,|\, \lambda_1(-\Delta_g + K_g) > 0\}$, where λ_1 denotes the first eigenvalue of the operator $-\Delta_g + K_g$. Here Δ_g denotes the Laplacian on (S^2, g) and K_g the Gaussian curvature of (S^2, g). The asymptotically flat extensions in [25] described above, are obtained through an elegant geometric construction that can be outlined as the following four-step process:

1. Consider an initial metric g in $\mathscr{M}^+$. This metric g will be the metric on the horizon of the asymptotically flat extension. Deform g to g_* (the standard metric on S^2) by a smooth path of metrics obtained by applying the Uniformization Theorem in a way that the path $g_t = g(t)$ is area preserving. It can be shown that such a path remains inside $\mathscr{M}^+$ (c.f. the discussion after Theorem 1.3.3).

2. Construct a "collar extension" of g, which is the manifold $[0, 1] \times S^2$ endowed with a metric of positive scalar curvature, such that the bottom boundary $\{0\} \times S^2$ (having g as the induced metric) is minimal and outer minimizing, while the top boundary $\{1\} \times S^2$ is a round sphere.

3. Pick any $m > 0$ arbitrarily close to $(\text{area}\,(g)/16\pi)^{1/2}$ (see (2.2)). Consider a 3-dimensional spatial Schwarzschild manifold of mass m (which is scalar flat by Proposition 1.5.2) and deform it to have positive scalar curvature in a small region near the horizon.

4. Glue the collar extension to an exterior region of the deformed Schwarzschild manifold by a "positive scalar bridge".

These steps give rise to the desired asymptotically flat extensions, whose properties are summarized in the following theorem [25]:

Theorem 2.2.1. *Given a metric* $g \in \mathscr{M}^+$*, pick any* $m > 0$ *such that* $m > \sqrt{\frac{area(g)}{16\pi}}$*. Then, there exists an asymptotically flat 3-dimensional manifold* M^3 *with non-negative scalar curvature such that*

(i) ∂M^3 *is isometric to* (S^2, g) *and minimal,*

(ii) M^3 *coincides with a Schwarzschild manifold outside a compact set, and*

(iii) M^3 *is foliated by mean convex spheres that eventually coincide with the coordinate spheres in the spatial Schwarzschild manifold.*

Figure 2.1 shows a schematic picture of the construction.

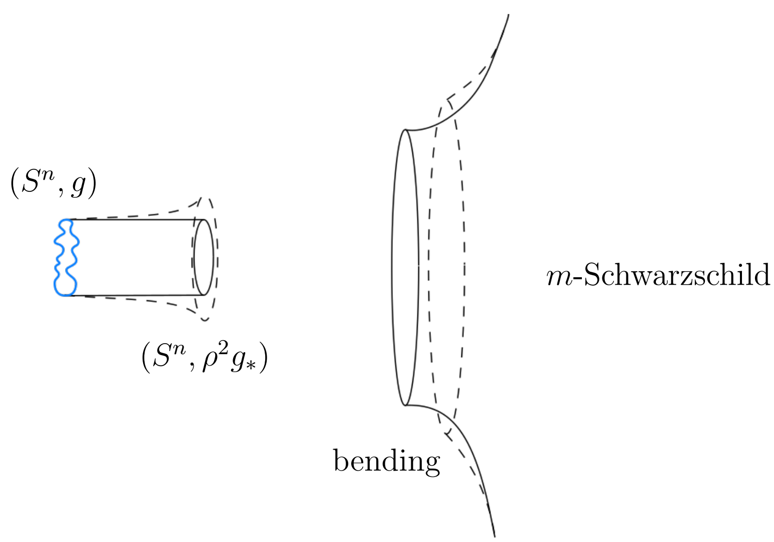

Figure 2.1: Diagram of the construction developed by Mantoulidis and Schoen in [25] (when $n = 2$). The dashed lines represent the collar extension and the bending of the Schwarzschild space.

Though stated for dimension $n = 2$, many of the arguments used in the construction of asymptotically flat extensions in [25] extend in a natural way to higher dimensions $n \geq 3$. In this section, we adapt the arguments to show that Mantoulidis and Schoen's construction can indeed be carried out in higher dimensions. We will restrict ourselves to the set of smooth metrics on S^n with positive scalar curvature, $\mathrm{Scal}^+(S^n)$.

1. Connecting the initial metric with the standard round metric on S^n

For the time being, let us assume that given a metric g on S^n ($n \geq 2$) with positive scalar curvature, there exists a family of metrics on S^n, $\{h(t)\}_{0 \leq t \leq 1}$, such that

- $h(t)$ has positive scalar curvature for each t,
- $h(0) = g$ and $h(1)$ is a round metric,
- $\mathrm{vol}\,(h(t))$ is a constant independent of t, and
- $H = dt^2 + h(t)$ is a smooth metric on $I \times S^n$, where $I = [0, 1]$.

Recall that in [25], this family of metrics is obtained through an application of the Uniformization Theorem. In our case, we will apply the results of Marques [26] and a suitable geometric flow on star-shaped surfaces [37, 15] (see Sections 2.4 and 2.5, cf. [25]).

2. Collar extensions

By a collar extension we mean the manifold $I \times S^n$ endowed with a metric of positive scalar curvature, such that the bottom boundary $\{0\} \times S^n$ is minimal and isometric to (S^n, g), and the top boundary $\{1\} \times S^n$ is round. Moreover, the resulting manifold is foliated by mean convex spheres.

The first step to obtain a collar extension of a given metric g in $\mathrm{Scal}^+(S^n)$ is to construct a manifold which is topologically $S^n \times I$, where $\{0\} \times S^n$ is isometric to (S^n, g), and $\{1\} \times S^n$ is round. To achieve this, we use the family of metrics of positive scalar curvature connecting the given metric g with a round metric described in step 1.

Lemma 2.2.2. *Suppose $\{h(t)\}_{0\le t\le 1}$ is a family of metrics of positive scalar curvature on S^n such that*

- *$h(1)$ is a round metric,*
- *$H = dt^2 + h(t)$ is a smooth metric on $I \times S^n$, and*
- *$\mathrm{vol}\,(h(t))$ is a constant independent of t.*

Then, there exists a smooth metric $G = dt^2 + g(t)$ on $I \times S^n$ satisfying

(i) $g(0)$ is isometric to $h(0)$ on S^n,

(ii) $g(t)$ has positive scalar curvature $\forall\, t \in I$,

(iii) $g(1)$ is round, $g(t) = g(1)\ \forall\, t \in [1/2, 1]$, and

(iv) $\frac{d}{dt} dV_{g(t)} = 0$ for all $t \in [0,1]$.

Proof. Choose a smooth monotone function ζ on $[0,1]$ such that $\zeta(0) = 0$ and $\zeta(t) = 1$, $t \in [1/2, 1]$. Consider $h(\zeta(t))$, $t \in [0,1]$. This new path $\{h(\zeta(t))\}_{0\le t\le 1}$ satisfies the first three conditions and is volume preserving. Relabel $h(\zeta(t))$ as $h(t)$. To achieve condition (iv), we make use of diffeomorphisms on S^n. Let $\{\phi_t\}_{0\le t\le 1}$ be a 1-parameter family of diffeomorphisms on S^n generated by a t-dependent, smooth vector field $X_t = X(\cdot, t)$ to be chosen later. Define $g(t) = \phi_t^*(h(t))$. Let

$\dot{g} = \frac{d}{dt}g$, then by (1.57) and (1.59),

$$\frac{d}{dt}dV_{g(t)} = \frac{1}{2}\mathrm{tr}_g\dot{g}\,dV_{g(t)} \tag{2.4}$$

and

$$\dot{g} = \frac{d}{dt}\phi_t^*(h(t)) = \phi_t^*\left(\frac{d}{dt}h(t)\right) + \phi_t^*(\mathcal{L}_{X_t}h(t)). \tag{2.5}$$

Hence, from Lemmas 1.2.6 and 1.2.7,

$$\mathrm{tr}_g\dot{g} = \mathrm{tr}_{\phi_t^*(h(t))}\left(\phi_t^*\left(\frac{d}{dt}h\right) + \phi_t^*(\mathcal{L}_{X_t}h(t))\right) \tag{2.6}$$

$$= \phi_t^*\left(\mathrm{tr}_h\dot{h} + \mathrm{tr}_h\mathcal{L}_{X_t}h(t)\right) \tag{2.7}$$

$$= \phi_t^*\left(\mathrm{tr}_h\dot{h} + 2\mathrm{div}_hX_t\right). \tag{2.8}$$

Now, let $\psi(t,x)$ be a smooth function on $I \times S^n$ obtained by solving the elliptic equation on S^n,

$$\Delta_h\psi(t,\cdot) = -\frac{1}{2}\mathrm{tr}_h\dot{h}, \tag{2.9}$$

for each t. (2.9) is solvable since $\displaystyle\int_{S^n}\frac{1}{2}\mathrm{tr}_h\dot{h}\,dV_{h(t)} = \frac{d}{dt}\int_{S^n}dV_{h(t)} = 0$ (see Theorem 1.3.3). Furthermore, the solution $\psi(t,\cdot)$ depends smoothly on t by elliptic regularity. Let $X_t = \nabla^{h(t)}\psi$, where $\nabla^{h(t)}$ is the gradient on $(S^n, h(t))$. Clearly, $\mathrm{tr}_g\dot{g} = 0$ by (2.8) and (2.9). Condition (iv) is thus satisfied. □

Remark 2.2.3. Condition (iv) in Lemma 2.2.2 together with (1.91) imply that each t-slice of $I \times S^n$ has zero mean curvature.

Next, given a fixed choice of $\{h(t)\}$, we continue to denote the path provided in Lemma 2.2.2 by $\{g(t)\}$. In the following lemma, we endow the manifold $S^n \times I$

obtained in Lemma 2.2.2 with a metric of positive scalar curvature and such that the boundary $S^n \times \{0\}$ is its horizon.

Lemma 2.2.4. *There exist $A_0 > 0$ such that for all $\varepsilon \in [0,1]$ and $A \geq A_0$, the metric on $[0,1] \times S^n$ given by*

$$\gamma_\varepsilon = A^2 dt^2 + (1 + \varepsilon t^2) g(t), \tag{2.10}$$

has positive scalar curvature on $I \times S^n$, $\{0\} \times S^n$ is minimal, and the spheres $\{t\} \times S^n$ for $t \in (0,1]$ are mean convex with respect to the normal direction ∂_t.

Proof. Consider a metric of the form

$$\gamma = v(t,x)^2 dt^2 + h(t), \tag{2.11}$$

where $v(t,x) = A$ and $h(t) = (1 + \varepsilon t^2) g(t)$ for $A > 0$ and $\varepsilon > 0$ to be determined later (here we are abusing notation by using $h(t)$ again). Applying Lemma 1.2.9, we have

$$R_\gamma = R_h + A^{-2} \left[-\mathrm{tr}_h \ddot{h} - \frac{1}{4} (\mathrm{tr}_h \dot{h})^2 + \frac{3}{4} |\dot{h}|_h^2 \right]. \tag{2.12}$$

From the definition of $h(t)$ and recalling that $\mathrm{tr}_g \dot{g} = 0$, we have the following:

$$\dot{h} = 2\varepsilon t g(t) + (1 + \varepsilon t^2) \dot{g}(t), \tag{2.13}$$

$$|\dot{h}|_h^2 = (1 + \varepsilon t^2)^{-2} (4\varepsilon^2 t^2 n) + |\dot{g}|_g^2 \tag{2.14}$$

$$\mathrm{tr}_h \dot{h} = 2n\varepsilon t (1 + \varepsilon t^2)^{-1} + \mathrm{tr}_g \dot{g} = 2n\varepsilon t (1 + \varepsilon t^2)^{-1}, \tag{2.15}$$

$$\ddot{h} = 2\varepsilon g(t) + 4\varepsilon t\dot{g}(t) + (1+\varepsilon t^2)\ddot{g}(t), \tag{2.16}$$

and

$$\mathrm{tr}_h\ddot{h} = 2n\varepsilon(1+\varepsilon t^2)^{-1} + \mathrm{tr}_g\ddot{g}. \tag{2.17}$$

Using this information, we can bound R_γ from below:

$$R_\gamma \geq R_h + A^{-2}\left[-\frac{2n\varepsilon}{(1+\varepsilon t^2)^{-1}} - \mathrm{tr}_g\ddot{g} - \frac{n^2\varepsilon^2 t^2}{(1+\varepsilon t^2)^2}\right] \tag{2.18}$$

$$\geq R_h + A^{-2}\left[-\frac{2n\varepsilon}{(1+\varepsilon t^2)^{-1}} - \sup_{t,x}|\mathrm{tr}_g\ddot{g}| - \frac{n^2\varepsilon^2 t^2}{(1+\varepsilon t^2)^2}\right], \tag{2.19}$$

By picking $A_0 \gg 1$ sufficiently large, the metric

$$\gamma_\varepsilon = A^2 dt^2 + (1+\varepsilon t^2)g(t) \tag{2.20}$$

has positive scalar curvature for $A \geq A_0$ and $\varepsilon \in [0,1]$; here we show a explicit lower bound of R_γ since it might be useful for future applications. Using (1.91), the mean curvature of any slice $\{t\} \times S^n$, is given by

$$H_t = \frac{n\varepsilon t}{A(1+\varepsilon t^2)}\ . \tag{2.21}$$

Therefore, $H = 0$ when $t = 0$ and $H > 0$ when $t > 0$.

□

Remark 2.2.5. In [25] the previous construction is achieved by making use of the first (positive) eigenfunction $u(t,x)$ of the operator $-\Delta_{g_t} + \frac{1}{2}R_{g_t}$ as a warping factor in (2.10). Since we are imposing the stronger condition $R_{g_t} > 0$, we do not need to make use of the corresponding eigenfunction; however, we point out that Lemma 2.2.4 still holds in higher dimensions using a metric of the form $\gamma_\varepsilon = A^2u(t,x)^2dt^2 + (1+\varepsilon t^2)g(t)$ instead of (2.10).

Once we have constructed the collar extensions, our next goal is to to glue them to an exterior region of a Schwarzschild manifold of mass m, with

$$m > \frac{1}{2}\left(\frac{\mathrm{vol}\,(g)}{\omega_n}\right)^{(n-1)/n}. \tag{2.22}$$

To do so, we bend the Schwarzschild manifold in a small region near the horizon to make its metric positive scalar.

3. Bending the Schwarzschild metric

Recall that the $(n+1)$-dimensional spatial Schwarzschild manifold is given by

$$(M^{n+1}, g_m) = \left([r_0,\infty)\times S^n, \left(1-\frac{2m}{r^{n-1}}\right)^{-1} dr^2 + r^2 g_*\right), \tag{2.23}$$

where g_* denotes the standard metric on S^n and $r_0 = (2m)^{\frac{1}{n-1}}$. Replacing r by s, which is the distance function to the horizon $\{r_0\}\times S^n$, we re-write g_m as

$$g_m = ds^2 + u_m(s)^2 g_*, \tag{2.24}$$

defined on $[0,\infty)\times S^n$. Here the horizon $\{r = r_0\}$ corresponds to $s = 0$ (see Proposition 1.5.3). The function $u_m(s)$ satisfies

(a) $u_m(0) = (2m)^{\frac{1}{n-1}}$,

(b) $u_m'(0) = 0$,

(c) $u_m'(s) = \left(1 - \dfrac{2m}{u_m(s)^{n-1}}\right)^{1/2}$ for $s > 0$, and

(d) $u_m''(s) = (n-1)\dfrac{m}{u_m^n}$ for $s > 0$.

In particular, when $n = 3$,

$$u_m(0) = \sqrt{2m}, \quad u_m'(0) = 0, \tag{2.25}$$

and

$$u_m'(s) = \left(1 - \frac{2m}{u_m(s)^2}\right)^{1/2}, \quad u_m''(s) = \frac{2m}{u_m^3(s)}, \qquad \text{for } s > 0. \tag{2.26}$$

The next lemma bends the metric g_m so that the resulting metric has strictly positive scalar curvature in a small region.

Lemma 2.2.6. *Let $s_0 > 0$. There exist a small $\delta > 0$ and a smooth function $\sigma : [s_0 - \delta, \infty) \to (0, \infty)$ satisfying*

(a) $\sigma(s) = s$ for all $s \geq s_0$,

(b) σ is monotonically increasing, and

(c) the metric $ds^2 + u_m(\sigma(s))^2 g_*$ has positive scalar curvature for $s_0 - \delta \leq s < s_0$ and vanishing scalar curvature for $s \geq s_0$.

Proof. From Lemma 1.2.9, given a metric of the form $\tilde{g} = ds^2 + f(s)^2 g_*$, its scalar curvature is given by

$$\tilde{R} = nf^{-2}\left[(n-1) - (n-1)\dot{f}^2 - 2f\ddot{f}\right]. \tag{2.27}$$

To achieve the condition of $ds^2 + u_m(\sigma(s))^2 g_*$ having positive scalar curvature on

$[s_0 - \delta, s_0)$, we need to have

$$\tilde{R} = nu_m^{-2}\left[(n-1) - (n-1)\left(\frac{d}{ds}u_m\right)^2 - 2u_m\frac{d^2}{ds^2}u_m\right] > 0, \tag{2.28}$$

on $[s_0 - \delta, s_0)$. Thus, it is sufficient to require

$$(n-1) - (n-1)\left(\frac{d}{ds}u_m(\sigma)\right)^2 - 2u_m\frac{d^2}{ds^2}u_m((\sigma)) > 0, \tag{2.29}$$

on $[s_0 - \delta, s_0)$. Since

$$\frac{d}{ds}u_m(\sigma(s)) = u_m'(\sigma(s))\sigma'(s) \tag{2.30}$$
$$\frac{d^2}{ds^2}u_m(\sigma(s)) = u_m''(\sigma(s))\sigma'(s)^2 + u_m'(\sigma(s))\sigma''(s), \tag{2.31}$$

condition (2.29) can be rewritten as

$$\begin{aligned} &(n-1) - (n-1)\sigma'(s)^2 \\ &\qquad + \sigma'(s)^2[(n-1) - (n-1)u_m'(\sigma(s))^2 - 2u_m(\sigma(s))u_m''(\sigma(s))] \\ &\qquad\qquad - 2u_m(\sigma(s))u_m'(\sigma(s))\sigma''(s) > 0. \end{aligned} \tag{2.32}$$

Recall that by Proposition 1.5.2, g_m is scalar flat, i.e.,

$$(n-1) - (n-1)u_m'(\sigma(s))^2 - 2u_m(\sigma(s))u_m''(\sigma(s)) = 0. \tag{2.33}$$

Thus, combining (2.32) and (2.33), to have $\tilde{R} > 0$ we need

$$(n-1) - (n-1)(\sigma')^2 - 2u_m(\sigma)u_m'(\sigma)\sigma'' > 0, \tag{2.34}$$

on $[s_0-\delta, s_0)$. Define $\theta(s) = 1 + e^{-\frac{1}{(s-s_0)^2}}$ and $\theta(s_0) = 1$. For sufficiently small δ, let

$$\sigma(s) = \int_{s_0-\delta}^{s} \theta(s)\, ds + K_\delta, \tag{2.35}$$

where K_δ is chosen so that $\sigma(s_0) = s_0$, and thus can be extended to be equal to s for $s \geq s_0$ (note that $\sigma'(s_0) = 1$). With such a choice of $\sigma(s)$, (2.34) becomes

$$\begin{aligned} &(n-1) - (n-1)[1 + 2e^{-\frac{1}{(s-s_0)^2}} + e^{-\frac{2}{(s-s_0)^2}}] - 4u_m(\sigma)u_m'(\sigma)\frac{e^{-\frac{1}{(s-s_0)^2}}}{(s-s_0)^3} \\ &= e^{-\frac{1}{(s-s_0)^2}}\left(-2(n-1) - (n-1)e^{-\frac{1}{(s-s_0)^2}} - 4u_m(\sigma)u_m'(\sigma)\frac{1}{(s-s_0)^3}\right). \end{aligned} \tag{2.36}$$

By taking δ sufficiently small, this last quantity is positive. □

4. Gluing lemma

Our next goal is to glue a collar extension $(I \times S^n, \gamma_\epsilon)$ from Lemma 2.2.4 to the "bending" of the Schwarzschild metric in Lemma 2.2.6. The following lemma provides a way to construct a "positive scalar bridge" between two manifolds of positive scalar curvature with metrics of the form

$$dt^2 + f(t)^2 g_*. \tag{2.37}$$

By Lemma 1.2.9, the positivity of the scalar curvature of a metric of the form (2.37) translates to the following condition on f:

$$(n-1) - (n-1)\dot{f}(t)^2 - 2f(t)\ddot{f}(t) > 0, \tag{2.38}$$

which is equivalent to

$$\ddot{f}(t) < \frac{n-1}{2f(t)}(1 - \dot{f}(t)^2). \tag{2.39}$$

The assumptions of the gluing lemma (see Lemma 2.2.7) will become transparent when we prove Theorem 2.2.9. At this point, it is useful to point out that the metric in Lemma 2.2.6 is of the form (2.37). In Lemma 2.2.4, the fact that $g(t)$ is constant and round on $[1/2, 1]$ will allow us to make a change of variable to take the metric of the collar extension to the form (2.37) on $[1/2, 1]$.

Lemma 2.2.7. *Let $f_i : [a_i, b_i] \to \mathbb{R}$, $i = 1, 2$, be smooth functions. Suppose that*

(I) $f_i > 0$, $f_i' > 0$ and $f_i'' > 0$ on $[a_i, b_i]$,

(II) the metric $dt^2 + f_i(t)^2 g_*$ has positive scalar curvature, and

(III) $f_1(b_1) < f_2(a_2)$ and $f_1'(b_1) = f_2'(a_2)$.

Then, after translating the intervals so that $a_2 - b_1 = (f_2(a_2) - f_1(b_1))/f_1'(b_1)$, we can construct $f : [a_1, b_2] \to \mathbb{R}$ so that:

(i) $f > 0$ and $f' > 0$ on $[a_1, b_2]$,

(ii) $f = f_1$ on $[a_1, \frac{a_1+b_1}{2}]$,

(iii) $f = f_2$ on $[\frac{a_2+b_2}{2}, b_2]$, and

(iv) The metric $dt^2 + f(t)^2 g_*$ has positive scalar curvature on $[a_1, b_2] \times S^n$.

Proof. Let us translate f_2 by k to be determined, i.e., $\tilde{f}_2(t) = f_2(t - k)$. Then by letting $\tilde{a}_2 = a_2 + k$, we also have $\tilde{f}_2'(\tilde{a}_2) = f_2'(a_2) = f_1'(b_1)$. We want to translate f_2 enough so we can join the graph of f_1 and f_2 by a straight line of slope $f'(b_1)$. Consider the line

$$\ell(x) = f_1'(b_1)x + b.$$

Plug-in $\tilde{a}_2$ and set $\ell(\tilde{a}_2) = \tilde{f}_2(\tilde{a}_2) = f_2(a_2)$ to obtain $b = f_2(a_2) - f_1'(b_1)\tilde{a}_2$. Then, by letting $x = b_1$, we get

$$\frac{f_2(a_2) - f_1(b_1)}{f_1'(b_1)} = \tilde{a}_2 - b_1.$$

We translate enough until this condition is met and relabel $\tilde{f}_2$ as f_2. Now, we can join the graphs of f_1 and f_2 by a straight line of slope $f'(b_1)$. Call this new function $\tilde{f}$. Notice that by assumption, this function is differentiable and its derivative is continuous, hence $f \in C^{1,1}([a_1, b_2])$. Also, $\tilde{f}$ is C^2 away from b_1 and a_2.

Define $m_i = (a_i + b_i)/2$, $i = 1, 2$. Let $\delta > 0$ be such that $m_1 < b_1 - \delta$ and $a_2 + \delta < m_2$. Let η_δ be a smooth cut-off function such that $\eta_\delta(t) = 1$ on $[b_1 - \delta, a_2 + \delta]$ and $\eta_\delta(t) = 0$ on $[a_1, m_1] \cup [m_2, b_2]$. Define the following mollification of $\tilde{f}$:

$$f_\nu(t) = \int_{\mathbb{R}} \tilde{f}(t - \nu\eta_\delta(t)s)\phi(s)\, ds. \tag{2.40}$$

This mollification fixes $\tilde{f}$ on $[a_1, m_1] \cup [m_2, b_2]$ and coincides with the standard mollification on an interval properly containing $[b_1, a_2]$; on the remaining part, its value is given by a standard mollification of f with radius $\nu\eta_\delta(x) \leq \nu$. It can be checked that both $f_\nu \to f$ and $f_\nu' \to f'$ in $C^0([a_1, b_2])$, as $\nu \to 0$ (see Lemma C.2.1 for a proof).

From (2.39), the metric $\tilde{g} = dt^2 + \tilde{f}(t)^2 g_*$ has positive scalar curvature if

$$\tilde{f}''(t) < \frac{(n-1)}{2\tilde{f}(t)} \left(1 - \tilde{f}'(t)^2\right). \tag{2.41}$$

By assumption (II),

$$f_i''(t) < \frac{(n-1)}{2f_i(t)}\left(1 - f_i'(t)^2\right), \text{ on } [a_i, b_i]. \tag{2.42}$$

The condition $f_i'' > 0$ on $[a_i, b_i]$, ensures that the graph of

$$\Omega[\tilde{f}](x) = \frac{(n-1)}{2\tilde{f}(t)}\left(1 - \tilde{f}'(t)^2\right)$$

lies strictly above of the graph of $\tilde{f}''$ (when defined) and the graphs of f_1'' and f_2''. It follows that $\Omega[f_\nu] \to \Omega[\tilde{f}]$ uniformly on $[a_1, b_2]$ as $\nu \to 0$. Let $3d$ be the smallest verical distance from the graph of $\Omega[\tilde{f}]$ to the graph of f_1'' and f_2''. The uniform convergence implies that we can take a small ν so that the graph of $\Omega[f_\nu]$ lies exactly within a distance d from the graph of $\Omega[\tilde{f}]$. Since $\Omega[\tilde{f}]$ is uniformly continuous, there exists a number $\nu > 0$ such that $\Omega[\tilde{f}](s) \leq \Omega[\tilde{f}](t) + d$ on $[t-\nu, t+\nu]$. For simplicity, abusing the notation, set $\tilde{f}''(b_1) = f_1''(b_1)$ and $\tilde{f}''(a_2) = f_2''(a_2)$. Then it follows that for a sufficiently small ν,

$$f_\nu''(t) \leq \sup_{[t-\nu, t+\nu]} \tilde{f}''(s) + d \leq \sup_{[t-\nu, t+\nu]} \Omega[\tilde{f}](s) - 3d + d \leq \Omega[\tilde{f}](t) - d,$$

hence $f_\nu''(t) < \Omega[f_\nu](t)$ on $[a_1, b_2]$. It follows that the metric $dt^2 + f_\nu(t)^2 g_*$ has positive scalar curvature. □

Remark 2.2.8. By Remark 1.2.8, the mean curvature of the cross-sections of $([a_1, b_2] \times S^n, dt^2 + f(t)^2 g_*)$ is positive. Hence, the positive scalar bridge is foliated by strictly mean convex round spheres, with respect to the outward normal direction ∂_t.

5. Gluing of a collar extension to the exterior region of a Schwarzschild manifold

The following theorem allows us to glue a collar extension of the given (S^n, g) to an exterior region of a Schwarzschild manifold with mass m. As a result, we obtain an asymptotically flat extension with non-negative scalar curvature, such that its boundary is minimal and isometric to (S^n, g). Moreover, it is foliated by mean convex spheres that eventually coincide with the coordinate spheres in the Schwarzschild manifold. In terms of mathematical relativity, we obtain a black hole initial data set with prescribed boundary metric, whose mass is arbitrarily close to to the optimal value in the Riemannian Penrose inequality.

Theorem 2.2.9. *Suppose $\{h(t)\}_{0\leq t\leq 1}$ is a family of metrics of positive scalar curvature on S^n such that*

- *$h(1)$ is a round metric,*
- *$H = dt^2 + h(t)$ is a smooth metric on $I \times S^n$, and*
- *$\mathrm{vol}\,(h(t))$ is a constant independent of t.*

Given any $m > 0$ such that $m > \frac{1}{2}\left(\mathrm{vol}\,(g)/\omega_n\right)^{\frac{n-1}{n}}$, there exists an asymptotically flat $(n+1)$-dimensional manifold M^{n+1} with non-negative curvature such that

(i) ∂M^{n+1} is isometric to (S^n, g) and is minimal,

(ii) M^{n+1}, outside a compact set, is isometric to a spatial Schwarzschild manifold of mass m, and

(iii) M^{n+1} is foliated by mean convex n-spheres which eventually coincide with the rotationally symmetric n-spheres in the spatial Schwarzschild manifold.

Proof. Apply Lemma 2.2.2 to the family $\{h(t)\}_{0\leq t\leq 1}$ to obtain $\{g(t)\}_{0\leq t\leq 1}$. Let $m > 0$ be a constant such that

$$m > \frac{1}{2}\left(\frac{\text{vol}\,(g)}{\omega_n}\right)^{(n-1)/n}. \tag{2.43}$$

Consider the collar extensions obtained in Lemma 2.2.4 for $A \geq A_0$, i.e., the family of metrics

$$\gamma_\varepsilon = A^2dt^2 + (1+\varepsilon t^2)g(t), \tag{2.44}$$

with positive scalar curvature on $[0,1]\times S^n$ for $\varepsilon \in [0,1]$. Let g_* denote the round metric on S^n. Then $g(t) = \rho^2 g_*$ for some $\rho > 0$ on $[1/2,1]$ (recall that $g(1)$ is round and $g(t) = g(1)$ for $t \in [1/2,1]$). Let $T = A > 0$ on $[1/2,1]$ and make the change of variables $s = At$ on $[1/2,1]$, obtaining

$$\gamma_\varepsilon = ds^2 + (1+\varepsilon A^{-2}s^2)\rho^2 g_*, \tag{2.45}$$

for $s \in [A/2, A]$.

Define $f_\varepsilon(s) = (1+\varepsilon A^{-2}s^2)^{1/2}\rho$. Then,

$$f'_\varepsilon(s) = \frac{\rho\varepsilon s}{A^2(1+\varepsilon A^{-2}s^2)^{1/2}} > 0, \tag{2.46}$$

$$f''_\varepsilon(s) = \frac{\rho\varepsilon}{A^2(1+\varepsilon A^{-2}s)^{3/2}} > 0. \tag{2.47}$$

This function will play the role of f_1 in Lemma 2.2.7. The role of f_2 will be played by the function $u_m(\sigma(s))$ in the Schwarzschild bending $ds^2 + u_m(\sigma(s))^2 g_*$ from Lemma 2.2.6. To be able to apply Lemma 2.2.7 we need $f_\varepsilon(A) < u_m(\sigma(s_0-\delta))$ and

$f'_\varepsilon(A) = u'_m(\sigma(s_0 - \delta))$; to achieve this condition we will choose ε and δ accordingly. Consider the curves $\Gamma(\varepsilon) = (f_\varepsilon(A), f'_\varepsilon(A))$ and $\Delta(s) = (u_m(s), u'_m(s))$.

Notice that as $\varepsilon \to 0$

$$\Gamma(\varepsilon) \to (\rho, 0) = \left(\left(\frac{\operatorname{vol}(g)}{\omega_n} \right)^{\frac{1}{n}}, 0 \right), \tag{2.48}$$

since the volume of $g(t)$ remains constant. Moreover, the slope $f'_\varepsilon(A)/f_\varepsilon(A)$ is strictly decreasing as

$$\frac{f'_\varepsilon(A)}{f_\varepsilon(A)} = \frac{1}{A}\left(1 - \frac{1}{1+\varepsilon}\right). \tag{2.49}$$

When $s \to 0$, $\Delta(s) \to ((2m)^{1/(n-1)}, 0)$. Since m is chosen so that

$$m > \frac{1}{2}\left(\frac{\operatorname{vol}(g)}{\omega_n}\right)^{(n-1)/n}, \tag{2.50}$$

$\Delta(0)$ lies to the right of $\Gamma(0)$. Using continuity, pick s_0 so the segment of the curve $\Delta(s)$ from 0 to s_0 lies strictly to the right and below the curve $\Gamma(\varepsilon)$. Apply Lemma 2.2.6 with δ sufficiently small so that the curve $u_m(\sigma(s)) : [s_0 - \delta, s_0] \to \mathbb{R}$ has positive second derivative and the curve $\tilde{\Delta}(s) = (u_m(\sigma(s)), (u_m(\sigma(s))')$ still lies to the right and below $\Gamma(\varepsilon)$. Now pick $\varepsilon < 1$ so that $f'_\varepsilon(A)$ and $u'_m(\sigma(s_0 - \delta))$ coincide, and apply Lemma 2.2.7 to construct a positive scalar bridge between the collar extensions and the bending of Schwarzschild. The result follows. □

Remark 2.2.10. In Theorem 2.2.9, (ii) ensures that the mass of the asymptotically flat extension M is m, while (i) and (iii) make ∂M an outer-minimizing horizon by a standard application of the maximum principle, since away from the boundary, M is foliated by strictly mean convex spheres.

In view of the proof of Theorem 2.2.9, to perform the desired asymptotically flat extension of a given metric g on $\mathrm{Scal}^+(S^n)$, it is enough to have a volume-preserving family of metrics of positive scalar curvature $\{h(t)\}_{0\leq t\leq 1}$ such that

- $h(0) = g$ and $h(1)$ is round, and
- $H = dt^2 + h(t)$ is a smooth metric on $I \times S^n$.

As discussed earlier, in [25], for $n = 2$ this is possible by an application of the Uniformization Theorem; moreover, the construction of the collar extensions there is more delicate since their path needs to stay inside $\mathscr{M}^+$. However, for higher dimensions, it is not straightforward how to connect a given initial metric of positive scalar curvature with a round metric, preserving the positivity of the scalar curvature. In the following sections, we will investigate some suitable deformations of the initial metric to obtain the desired family of metrics of positive scalar curvature.

2.3 Conformal class of g_*

Probably the most natural scenario to attempt to construct the desired asymptotically flat extensions is by considering a direct analogous to Theorem 2.2.1. Namely, we can consider the set $\mathscr{M}^+ = \{g \text{ smooth metric on } S^n \,|\, \lambda_1(-\Delta_g + \frac{1}{2}R_g) > 0 \text{ and } g = e^{2w}g_*\}$, where λ_1 denotes the first eigenvalue of the operator $-\Delta_g + \dfrac{1}{2}R_g$, w is a positive smooth function defined on S^n and g_* is the standard metric on S^n. Clearly, such g is in the conformal class of g_*. However, due to the lack of conformal invariance of the quadratic form associated to the operator $-\Delta_g + \dfrac{1}{2}R_g$ in dimensions greater than $n = 2$ (see the discussion at the end of Section 1.3), it is not clear whether the positivity of λ_1 is preserved. This is the main motivation to restrict the initial metric to $\mathrm{Scal}^+(S^n)$.

Then we are naturally led to consider an element in the conformal class of g_* of the form

$$g = u^{\frac{4}{n-2}} g_*, \tag{2.51}$$

where $u > 0$ is a smooth function on S^n. By (1.53),

$$\frac{n-2}{4(n-1)} u^{\frac{n+2}{n-2}} R_g = \frac{n(n-2)}{4} u - \Delta_* u, \tag{2.52}$$

where Δ_* denotes the Laplacian with respect to the standard metric g_*.

It follows that $R_g > 0$ if and only if

$$\Delta_* u - cu < 0, \tag{2.53}$$

where $c = n(n-2)/4$. The simplest path to consider is a linear perturbation of u, that is,

$$g_t = ((1-t)u + t)^{\frac{4}{n-2}} g_*, \text{ for } t \in [0,1]. \tag{2.54}$$

This is a path connecting g to g_*. To check that indeed this path preserves the positivity of the scalar curvature, we use (2.53):

$$\Delta_*((1-t)u + t) - c((1-t)u + t) = (1-t)(\Delta_* u - cu) - ct < 0. \tag{2.55}$$

After an appropriate rescaling to make the path volume-preserving, by a direct application of Theorem 2.2.9, we obtain

Theorem 2.3.1. *Given any $n \geq 3$, let g be a metric on S^n with positive scalar curvature of the form $g = u^{\frac{4}{n-2}} g_*$, where $u > 0$ is a smooth function on S^n. Given any $m > 0$ such that $m > \frac{1}{2}\left(\mathrm{vol}\,(g)/\omega_n\right)^{\frac{n-1}{n}}$, there exists an asymptotically flat $(n+1)$-dimensional manifold M^{n+1} with nonnegative curvature such that*

(i) ∂M^{n+1} is isometric to (S^n, g) and is minimal,

(ii) M^{n+1}, outside a compact set, is isometric to a spatial Schwarzschild manifold of mass m, and

(iii) M^{n+1} is foliated by mean convex n-spheres which eventually coincide with the rotationally symmetric n-spheres in the spatial Schwarzschild manifold.

Since a metric of the form $g = u^{\frac{4}{n-2}} g_*$ is somewhat restrictive, we investigate other scenarios in which the extensions are possible, with more general types of initial metrics.

2.4 Smooth paths in $\mathrm{Scal}^+(S^3)$

Given $g \in \mathrm{Scal}^+(S^3)$, the first step to perform a collar extension of g, is to connect g to a round metric on S^3 via a smooth path in $\mathrm{Scal}^+(S^3)$. We will achieve this by first applying a result of Marques [26] to obtain a continuous path, and then by mollifying this continuous path to obtain a smooth path. Marques' result concerns deformations of metrics of positive scalar curvature on closed 3-manifolds, it was obtained by making use of Ricci flow with surgery, which was proposed by Hamilton [18] for four-dimensional closed manifolds and developed for three manifolds by Perelman in his fundamental work on the Poincaré conjecture [29, 30, 31].

We begin with a general path-smoothing procedure, which may be of independent interest, suggested to us by Marques [27]. Let M be an n-dimensional, $n \geq 2$, smooth closed manifold. Let $\mathcal{S}^k(M)$ denote the space of C^k symmetric $(0,2)$-tensors on M endowed with the C^k topology. Here $k \geq 0$ is either an integer or $k = \infty$. Let $\mathcal{M}^k(M)$ be the open set in $\mathcal{S}^k(M)$ consisting of Riemannian metrics. To simplify the notation, given any $g \in \mathcal{M}^k(M)$ with $k \geq 2$, let $R(g)$ denote the scalar curvature of g.

Lemma 2.4.1. *Let $\{g(t)\}_{0 \leq t \leq 1}$ be a continuous path in $\mathcal{M}^k(M)$, $k \geq 2$. Suppose $R(g(t)) > 0$ for each t. Then there exists a constant $\epsilon > 0$ such that, for any $g \in \mathcal{M}^k(M)$, if $||g - g(t)||_{C^2} < \epsilon$ for some $t \in [0,1]$, then $R(g) > 0$.*

Proof. Suppose the claim is not true. Then for any integer $j > 0$, there exists a metric $g_j \in \mathcal{M}^k(M)$ and some $t_j \in [0,1]$ such that $||g_j - g(t_j)||_{C^2} < \frac{1}{j}$ while $R(g_j) \leq 0$ somewhere on M. Passing to a subsequence, we may assume $\lim_{j \to \infty} t_j = t_*$ for some point $t_* \in [0,1]$. Since $R(g(t_*)) > 0$, there exists $\epsilon_0 > 0$ such that if $g \in \mathcal{M}^k(M)$ and $||g - g(t_*)||_{C^2} < \epsilon_0$, then $R(g) > 0$. For large j, by the continuity of $\{g(t)\}$ in $\mathcal{M}^k(M)$, we now have $||g_j - g(t_*)||_{C^2} < \epsilon_0$, hence $R(g_j) > 0$ which is a contradiction. □

Proposition 2.4.2. *Let $\{g(t)\}_{0 \leq t \leq 1}$ be a continuous path in $\mathcal{M}^k(M)$, $k \geq 2$. Suppose $R(g(t)) > 0$ for each t. Then there exists a smooth path $\{h(t)\}_{0 \leq t \leq 1}$ in $\mathcal{M}^k(M)$ satisfying $h(0) = g(0)$, $h(1) = g(1)$ and $R(h(t)) > 0$ for all t.*

Proof. Let $\epsilon > 0$ be the constant given by Lemma 2.4.1. Since the map $t \mapsto g_t \in \mathcal{M}^k(M)$ is continuous on $[0,1]$, there exists $\delta > 0$ such that, if t, $t' \in [0,1]$ with $|t - t'| < \delta$, then $||g(t) - g(t')||_{C^2} < \epsilon$.

Let $t_0 = 0 < t_1 < \ldots < t_{m-1} < t_m = 1$ be a sequence of points such that

$|t_{i-1} - t_i| < \delta$, $\forall\, i = 1, \ldots, m$. On each $[t_{i-1}, t_i]$, define

$$h^{(i)}(t) = \frac{t - t_{i-1}}{t_i - t_{i-1}} g(t_i) + \frac{t_i - t}{t_i - t_{i-1}} g(t_{i-1}). \tag{2.56}$$

Clearly, $h^{(i)}(t) \in \mathcal{M}^k(M)$ and $\{h^{(i)}(t)\}_{t_{i-1} \le t \le t_i}$ is a smooth path in $\mathcal{M}^k(M)$. Moreover,

$$\begin{aligned} ||h^{(i)}(t) - g(t_{i-1})||_{C^2} &= \frac{t - t_{i-1}}{t_i - t_{i-1}} ||g(t_i) - g(t_{i-1})||_{C^2} < \epsilon, \\ ||h^{(i)}(t) - g(t_i)||_{C^2} &= \frac{t_i - t}{t_i - t_{i-1}} ||g(t_i) - g(t_{i-1})||_{C^2} < \epsilon. \end{aligned} \tag{2.57}$$

In particular, $R(h^{(i)}(t)) > 0$ by Lemma 2.4.1. Let $\{\hat{h}(t)\}_{0 \le t \le 1}$ be the path of metrics obtained by replacing $\{g(t)\}$ by $\{h^{(i)}(t)\}$ on each $[t_{i-1}, t_i]$. Then $\{\hat{h}(t)\}_{0 \le t \le 1}$ satisfy all the properties desired for $\{h(t)\}_{0 \le t \le 1}$ except that it is not smooth at the points $t_1, \ldots, t_{m-1}$.

To complete the proof, we will mollify $\{\hat{h}(t)\}_{0 \le t \le 1}$ near each "corner" t_i, $1 \le i \le m-1$. We demonstrate the construction on $\left(\frac{t_0+t_1}{2}, \frac{t_1+t_2}{2}\right)$ as follows. Let $\phi = \phi(s)$ be a smooth function with compact support in $(-1, 1)$ such that $0 \le \phi \le 1$, $\int_{-\infty}^{\infty} \phi(s) ds = 1$ and

$$\phi(s) = \phi(-s). \tag{2.58}$$

Let $\sigma > 0$ be a fixed constant such that $\sigma < \min\left\{\frac{t_i - t_{i-1}}{4} \mid i = 1, \ldots, m\right\}$. Let $\phi_\sigma(s) = \sigma^{-1}\phi(\frac{s}{\sigma})$. For each $t \in \left(\frac{t_0+t_1}{2}, \frac{t_1+t_2}{2}\right)$, define

$$\begin{aligned} h_\sigma^{(1)}(t) &= \int_{-\sigma}^{\sigma} \hat{h}(t - s)\phi_\sigma(s) ds \\ &= \int_0^1 \hat{h}(u)\phi_\sigma(t - u) du. \end{aligned} \tag{2.59}$$

Evidently, $h_\sigma^{(1)}(t)$ lies in $\mathcal{S}^k(M)$ and is smooth in t. By the convexity of $\mathcal{M}^k(M)$ in $\mathcal{S}^k(M)$, $h_\sigma^{(1)}(t)$ indeed lies in $\mathcal{M}^k(M)$. Moreover,

$$h_\sigma^{(1)}(t) - g(t_1) = \int_{-\sigma}^{\sigma} \left[\hat{h}(t-s) - g(t_1)\right] \phi_\sigma(s)ds,$$

which combined with (2.57) implies

$$||h_\sigma^{(1)}(t) - g(t_1)||_{C^2} < \epsilon. \tag{2.60}$$

Hence, $R(h_\sigma^{(i)}(t)) > 0$ by Lemma 2.4.1. Now suppose $t \in \left(\frac{t_0+t_1}{2}, \frac{t_0+3t_1}{4}\right)$. Then $(t - \sigma, t+\sigma) \subset (t_0, t_1)$. Therefore, by (2.56) and (2.58),

$$\begin{aligned} h_\sigma^{(1)}(t) &= \int_{\mathbb{R}^1} \left[\frac{(t-s)-t_0}{t_1-t_0} g_{t_1} + \frac{t_1-(t-s)}{t_1-t_0} g_{t_0}\right] \phi_\sigma(s)ds \\ &= \int_{\mathbb{R}^1} \left[h^{(1)}(t) + \frac{s}{t_1-t_0}(g_{t_0} - g_{t_1})\right] \phi_\sigma(s)ds \\ &= h^{(1)}(t). \end{aligned} \tag{2.61}$$

Similarly, for $t \in \left(\frac{t_1+3t_2}{4}, \frac{t_1+t_2}{2}\right)$, we have $h_\sigma^{(1)}(t) = h^{(2)}(t)$. In other words, the path $\{h_\sigma^{(1)}(t)\}$ agrees with $\{\hat{h}(t)\}$ near $\frac{t_0+t_1}{2}$ and $\frac{t_1+t_2}{2}$.

Applying the above construction on each $I_i = \left(\frac{t_{i-1}+t_i}{2}, \frac{t_i+t_{i+1}}{2}\right)$ to obtain $h_\sigma^{(i)}(t)$ and then replacing $\hat{h}(t)$ by $h_\sigma^{(i)}(t)$ on I_i, $i = 1, \ldots, m-1$, we obtain a smooth path $\{h(t)\}_{0\le t\le 1}$ meeting all conditions required. This completes the proof. □

Now we state the result of Marques [26, Corollary 1.1], asserting the path connectedness of the space $\mathrm{Scal}^+(S^3) \subset \mathcal{M}^\infty(S^3)$.

Theorem 2.4.3. *Given any $g \in \mathrm{Scal}^+(S^3)$, there exists a continuous path $\{g(t)\}_{0\le t\le 1}$ in $\mathrm{Scal}^+(S^3)$ connecting g to a round metric on S^3.*

The following corollary follows directly from Marques' theorem, Theorem 2.4.3, and Proposition 2.4.2.

Corollary 2.4.4. *Given any* $g \in \mathrm{Scal}^+(S^3)$*, there exists a smooth path* $\{h(t)\}_{0\le t\le 1}$ *in* $\mathrm{Scal}^+(S^3)$ *connecting* g *to a round metric on* S^3*.*

We normalize this path so that it is volume preserving by setting

$$\tilde{h}(t) = \psi(t)h(t), \text{ with } \psi(t) = \left(\frac{\mathrm{vol}\,(g)}{\mathrm{vol}\,(h(t))}\right)^{\frac{2}{n}}, \tag{2.62}$$

which in turn, implies the following lemma:

Lemma 2.4.5. *Let* g *in* $\mathrm{Scal}^+(S^3)$*. There exists a family of metrics* $\{h(t)\}$ *with positive scalar curvature, such that*

- $h(0) = g$ *and* $h(1)$ *is round,*
- $h(t)$ *is volume-preserving, and*
- $H = dt^2 + h(t)$ *is a smooth metric on* $I \times S^n$*.*

Lemma 2.4.5 together with Theorem 2.2.9 provide the proof of the first main result of this section:

Theorem 2.4.6. *Let* g *be a metric with positive scalar curvature on* S^3*. Given any* $m > 0$ *such that* $m > \frac{1}{2}\left(\mathrm{vol}\,(g)/\omega_3\right)^{\frac{2}{3}}$*, there exists an asymptotically flat* 4*-dimensional manifold* M^4 *with non-negative curvature such that*

(i) ∂M^4 *is isometric to* (S^3, g) *and is minimal,*

(ii) M^4*, outside a compact set, is isometric to a spatial Schwarzschild manifold of mass* m*, and*

(iii) M^4 *is foliated by mean convex* 3*-spheres which eventually coincide with the rotationally symmetric* 3*-spheres in the spatial Schwarzschild manifold.*

Remark 2.4.7. It is an interesting fact that a similar result can be obtained by a direct application of the Ricci flow program developed by Hamilton in [17], i.e., without surgery. To do so, though, it is necessary to restrict the initial metric to the set of metrics on S^3 with positive Ricci curvature $\mathrm{Ric}^+(S^3)$. For the interested reader, this is done in Appendix A.

2.5 Paths in $\mathrm{Scal}^+_*(S^n)$

In higher dimensions $n \geq 4$, we do not have results ensuring the path connectedness of $\mathrm{Scal}^+(S^n)$. Moreover, for $n = 8k$ and $n = 8k+1$ (for $k \geq 1$) it is known that $\mathrm{Scal}^+(S^n)$ is disconnected (see [26] and the references therein). Therefore, to be able to carry out our asymptotically flat extensions in higher dimensions, we need to restrict the space of initial metrics.

For $n \geq 2$, let $\mathrm{Scal}^+_*(S^n)$ denote the set of smooth metrics g with positive scalar curvature on S^n such that (S^n, g) isometrically embeds in $\mathbb{R}^{n+1}$ as a star-shaped hypersurface. When $n = 2$, by the results in [28, 33], $\mathrm{Scal}^+_*(S^2)$ agrees with the set of metrics on S^2 with positive Gaussian curvature.

Below, we focus on $n \geq 4$. Given $g \in \mathrm{Scal}^+_*(S^n)$, by applying the work of Gerhardt [15] and Urbas [37], we show that g can be connected to a round metric on S^n via a smooth path in $\mathrm{Scal}^+_*(S^n)$.

2.5.1 Inverse curvature flows on star-shaped surfaces

We are interested in studying the following geometric problem: Let Σ_0 be a smooth, closed hypersurface in $\mathbb{R}^{n+1}$ $(n \geq 2)$, which is given by a smooth isometric embedding $X_0 : S^n \longrightarrow \mathbb{R}^{n+1}$. Consider the initial value problem

$$\begin{aligned} \frac{\partial X}{\partial t}(x,t) &= k(x,t)\nu(x,t), \\ X(\cdot,0) &= X_0, \end{aligned} \tag{2.63}$$

where $k(\cdot,t)$ is a suitable curvature function on the hypersurface Σ_t, which is parametrized by $X_t(\cdot,t) : S^n \longrightarrow \mathbb{R}^{n+1}$ and $\nu(\cdot,t)$ is the outward unit normal vector to Σ_t.

Following [15] and [37], we restrict our attention to a special type of curvature functions $k(x,t)$, namely, we set

$$k(\cdot,t) = \frac{1}{f(\kappa_1, ..., \kappa_n)}, \tag{2.64}$$

where κ_i are the principal curvatures of M_t with respect to the outward unit normal ν, and f is assumed to satisfy the following conditions:

(i) f is homogeneous of degree 1 on a open, convex, symmetric cone $\Gamma \subsetneq \mathbb{R}^n$, with vertex at the origin, which contains the positive cone $\Gamma^+ = \{(\lambda_1, ..., \lambda_n) \,|\, \lambda_i > 0\}$,

(ii) $\dfrac{\partial f}{\partial x_i} > 0$ on Γ,

(iii) f is concave in Γ, and

(iv) $f = 0$ on $\partial\Gamma$.

The most relevant examples of functions f satisfying (i)-(iv), suitable for geometric interpretations, involve the *m-th elementary symmetric function*

$$S_m(\lambda_1, ..., \lambda) = \sum_{1\leq i_1<...<i_m\leq n} \lambda_{i_1}...\lambda_{i_m}. \tag{2.65}$$

Definition 10. Given a closed hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, let $\kappa_1, \ldots, \kappa_n$ denote the principal curvatures of Σ with respect to the outward normal at each point. For $1 \leq k \leq n$, define the normalized k-th mean curvature σ_k of Σ by

$$\sigma_k = \frac{1}{\binom{n}{k}} \sum_{1\leq i_1<i_2...<i_k\leq n} \kappa_{i_1} \cdots \kappa_{i_k}.$$

Remark 2.5.1. Clearly, σ_1 and σ_2 are related to the usual mean curvature H and the scalar curvature R of Σ, respectively, by

$$H = n\sigma_1 \text{ and } R = n(n-1)\sigma_2. \tag{2.66}$$

Gerhardt [15] and Urbas [37] proved the following theorem.

Theorem 2.5.2. *Let Σ_0 be a smooth, closed hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$, given by a smooth embedding $X_0 : S^n \longrightarrow \mathbb{R}^{n+1}$, and suppose that Σ_0 is star-shaped with respect to a point $P_0 \in \mathbb{R}^{n+1}$. Let Γ as above and suppose $f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$ is a positive, symmetric function satisfying (i)-(iv). Suppose that for each point $\xi \in \Sigma_0$ we have*

$$f(\kappa_1(\xi), ..., \kappa_n(\xi)) > 0, \tag{2.67}$$

where $\kappa_1, ..., \kappa_n$ are the principal curvatures of Σ_0 relative to the outward unit normal. Then the initial value problem (2.63), has a unique smooth solution X :

$S^n \times [0,\infty) \longrightarrow \mathbb{R}^{n+1}$. For each $t \in [0,\infty)$, $X(\cdot,t)$ is a parametrization of a smooth, closed hypersurface Σ_t in $\mathbb{R}^{n+1}$, which is star-shaped with respect to P_0. Furthermore, if $\tilde{\Sigma}_t$ is the hypersurface parametrized by $\tilde{X}(\cdot,t) = e^{-\beta t}X(\cdot,t)$, where

$$\beta = f(1,...,1), \tag{2.68}$$

then $\tilde{\Sigma}_t$ converges to a sphere centered at P_0 in the C^∞ topology as $t \to \infty$.

We briefly describe some of the techniques in [37] to prove Theorem 2.5.2 and state the results which are relevant to our goal. We follow the notation used in [37]. Identifying S^n with the unit sphere $\{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ and assuming $X_0 : (S^n, g) \to (\mathbb{R}^{n+1}, g_E)$ is an isometric embedding such that $X_0(S^n)$ is star-shaped with respect to the origin (modulo a diffeomorphism on S^n), we can write X_0 as

$$X_0(x) = \rho_0(x)x, \ x \in S^n. \tag{2.69}$$

Here, $\rho_0 : S^n \to \mathbb{R}^+$ is a smooth positive function on S^n, referred to as the *radial function* representing Σ_0 in [37]. For each $t > 0$, the surface Σ_t in Theorem 2.5.2 is then given by the graph of a function $\rho(\cdot,t)$ over S^n, where

$$\rho(\cdot,\cdot) : S^n \times [0,\infty) \longrightarrow \mathbb{R}^+, \tag{2.70}$$

is a smooth function solving the non-linear parabolic partial differential equation

$$\frac{\partial \rho}{\partial t} = \frac{(\rho^2 + |\nabla \rho|^2)^{\frac{1}{2}}}{\rho F(a_{ij})}, \tag{2.71}$$

with the initial condition $\rho(\cdot,0) = \rho_0$. Here, ∇ denotes the gradient on S^n with respect to the standard metric and $F(a_{ij})$ is the expression of f acting on the principal curvatures of $\rho(\cdot, t)$, normalized so $F(\delta_{ij}) = 1$. For each t, one can rescale ρ to define $\tilde{\rho}(\cdot, t) = e^{-t}\rho(\cdot, t)$. Then $\tilde{\rho}(\cdot, t)$ satisfies

$$\frac{\partial \tilde{\rho}}{\partial t} = \frac{(\tilde{\rho}^2 + |\nabla \tilde{\rho}|^2)^{\frac{1}{2}}}{\tilde{\rho} F(\tilde{a}_{ij})} - \tilde{\rho}, \tag{2.72}$$

where $F(\tilde{a}_{ij})$ is the expression of f acting on the principal curvatures of $\tilde{\rho}(\cdot, t)$. The following estimates on $\tilde{\rho}(\cdot, t)$ and the convergence of $\tilde{\rho}(\cdot, t)$ as $t \to \infty$ are given by (3.38), (3.39) and (3.40) in [37], which we enumerate in the following lemma:

Lemma 2.5.3. *Let $\tilde{\rho}$ be a solution to* (2.72)*. The following holds:*

a) There exist positive constants C and γ such that

$$\max_{S^n} |\tilde{\rho}(\cdot, t) - \rho^*| \le C e^{-\gamma t}. \tag{2.73}$$

Here, $\rho^* > 0$ is some constant.

b) For any positive integer k and any constant $\tilde{\gamma} \in (0, \gamma)$, there exists a positive constant $C_k = C_k(\gamma, \tilde{\gamma})$ such that

$$\int_{S^n} |\nabla^k \tilde{\rho}(\cdot, t)|^2 \le C_k e^{-\tilde{\gamma} t}. \tag{2.74}$$

c) Given any two integers $l \ge 0$ and $k > l + \frac{n}{2}$, there exists a positive $C = C(k, l)$ such that

$$||\tilde{\rho}(\cdot, t) - \rho^*||_{C^l(S^n)} \le C \left[\int_{S^n} |\nabla^k \tilde{\rho}(\cdot, t)|^2 + \int_{S^n} |\tilde{\rho}(\cdot, t) - \rho^*|^2 \right]^{\frac{1}{2}}. \tag{2.75}$$

2.5.2 Metric on $I \times S^n$, $n \geq 3$

Recall that our goal is to obtain a family of metrics $\{h(t)\}_{0 \leq t \leq 1}$ in $\mathrm{Scal}^+(S^n)$ connecting the initial metric g on S^n (which can be embedded as a star-shaped surface in $\mathbb{R}^{n+1}$ with positive scalar curvature) to a round metric. To do so, we apply the results of Urbas and Gerhartd to a suitable choice of curvature function f. We start by recalling a well-know result of geometry of hypersurfaces in Euclidean spaces.

Lemma 2.5.4. *Let Σ be a closed hypersurface in $\mathbb{R}^{n+1}$. Suppose the induced metric on Σ has positive scalar curvature. Then, the mean curvature of Σ with respect to the outward normal is everywhere positive.*

Proof. Let R and H be the scalar curvature and the mean curvature of Σ, respectively. By the Gauss equation (1.36), we have $R = H^2 - |h|^2$, where h is the second fundamental form of Σ in $\mathbb{R}^{n+1}$. Hence, $R > 0$ implies $H^2 > 0$. Since Σ is closed, there exists a point on Σ at which $H > 0$, and hence, we conclude that $H > 0$ everywhere on Σ. □

We say that a smooth map $X : S^n \times [0, \infty) \longrightarrow \mathbb{R}^{n+1}$ is a solution to the σ_1/σ_2 flow (see Remark 2.5.1) if X satisfies

$$\frac{\partial X}{\partial t} = \frac{\sigma_1}{\sigma_2}\nu = \frac{(n-1)H}{R}\nu, \tag{2.76}$$

where ν is the outward unit normal to $\Sigma_t = X(S^n, t)$ and H and R are the mean curvature and the scalar curvature of Σ_t, respectively. By definition, if the $\{\Sigma_t\}$ arise from a smooth solution to (2.76), R does not vanish along Σ_t, hence must be positive. Consequently, by Lemma 2.5.4, H must be positive along Σ_t. Thus, the surfaces Σ_t are moving outward. The σ_1/σ_2 flow is one type of the inverse curvature flows in

$\mathbb{R}^{n+1}$ studied by Gerhardt [15] and by Urbas [37]. For further discussion about the σ_1/σ_2 flow, the reader is referred to [10]. In particular, the following theorem is a special case of Theorem 2.5.2.

Theorem 2.5.5 ([15, 37])**.** *Let $X_0 : S^n \to \mathbb{R}^{n+1}$ be a smooth embedding such that $\Sigma_0 = X_0(\Sigma)$ is star-shaped with respect to a point $P_0 \in \mathbb{R}^{n+1}$. If Σ_0 has positive scalar curvature and positive mean curvature, then the σ_1/σ_2 flow*

$$\frac{\partial X}{\partial t} = \frac{(n-1)H}{R}\nu, \tag{2.77}$$

with the initial condition $X(\cdot, 0) = X_0(\cdot)$ has a unique smooth solution $X : \Sigma \times [0, \infty) \to \mathbb{R}^{n+1}$. In particular, each $\Sigma_t = X(S^n, t)$ has positive scalar curvature R and positive mean curvature H. Moreover, the rescaled surface $e^{-t}X(S^n, t)$ converges to a round sphere centered at P_0 in the C^∞ topology as $t \to \infty$.

In what follows, we argue that the proof of Theorem 2.5.5 in [15, 37] indeed provides a smooth path of metrics, with positive scalar curvature, connecting any $g \in \mathrm{Scal}^+_*(S^n)$ to a round metric on S^n.

It follows directly from a), b) and c) in Lemma 2.5.3 that there exists a constant $\delta > 0$ (say $\delta = \frac{1}{2}\gamma$) such that, for any integer $l \geq 0$,

$$||\tilde{\rho}(\cdot, t) - \rho^*||_{C^l(S^n)} \leq Ce^{-\delta t}. \tag{2.78}$$

This, combined with the PDE (2.72), in turn implies, for any integer $k \geq 1$,

$$\left\| \frac{\partial^k \tilde{\rho}}{\partial t^k} \right\|_{C^l(S^n)} \leq Ce^{-\delta t}, \tag{2.79}$$

for some constants C (see Appendix B).

With these estimates, we can define a path of metrics in $\mathrm{Scal}^+_*(S^n)$ connecting g to a round metric g^* that corresponds to a round sphere in $\mathbb{R}^{n+1}$ of radius ρ^*. Define $\Phi_t : S^n \to \mathbb{R}^{n+1}$ by $\Phi_t(x) = \tilde{\rho}(x,t)x$ and let

$$g(t) = \Phi_t^*(g_{_E}), \tag{2.80}$$

where $g_{_E}$ is the Euclidean metric on $\mathbb{R}^{n+1}$. Theorem 2.5.5 guarantees that $g(t)$ has positive scalar curvature. Moreover, given any integers l and k, it follows from (2.78) and (2.79) that

$$\|g(t) - g^*\|_{C^l(S^n)} \le Ce^{-\delta t} \tag{2.81}$$

and

$$\left\| \frac{\partial^k}{\partial t^k} g(t) \right\|_{C^l(S^n)} \le Ce^{-\delta t}. \tag{2.82}$$

We can now obtain the first step towards the construction of the asymptotically flat extensions.

Lemma 2.5.6. *Let g in $\mathrm{Scal}^+_*(S^n)$. There exists a family of metrics $\{h(t)\}$ with positive scalar curvature, such that*

- *$h(0) = g$ and $h(1)$ is round,*
- *$h(t)$ is volume-preserving, and*
- *$H = dt^2 + h(t)$ is a smooth metric on $I \times S^n$.*

Proof. Let $g(t)$ be the path given by (2.80) and make the change of variable $t = t(s)$ to view the metrics $\{g(t)\}$ as a new family of metrics $\{h(s)\}$ defined on the finite interval $[0,1]$. Specifically, let

$$t(s) = \frac{1}{(s-1)^2} - 1, \text{ (then } s = 1 - \frac{1}{\sqrt{1+t}}) \tag{2.83}$$

and define

$$h(s) = \begin{cases} g(t(s)) & \text{when } s \in [0,1) \\ g^* & \text{when } s = 1 \end{cases}, \tag{2.84}$$

which is continuous by (2.81). Using the exponential decay estimate of the derivatives in (2.82), one concludes that the metric defined by

$$H = ds^2 + h(s), \tag{2.85}$$

is smooth on $I \times S^n$ and it satisfies that $h(0) = g$, $h(s)$ is a metric of positive scalar curvature on S^n for all $s \in [0,1]$, and $h(1)$ is a round metric.

We normalize this path so that it is volume preserving by considering (cf. (A.1) and (A.4))

$$\tilde{h}(t) = \psi(t)h(t), \text{ with } \psi(t) = \left(\frac{\mathrm{vol}\,(g)}{\mathrm{vol}\,(h(t))}\right)^{\frac{2}{n}}. \tag{2.86}$$

The family $\{\tilde{h}(t)\}$ satisfies the desired properties. □

Applying again Theorem 2.2.9, the main theorem of this section follows:

Theorem 2.5.7. *Given any $n \geq 4$, let g be a metric with positive scalar curvature on S^n. Suppose (S^n, g) isometrically embeds into the Euclidean space $\mathbb{R}^{n+1}$ as a star-shaped hypersurface. Given any $m > 0$ such that $m > \frac{1}{2}\left(\mathrm{vol}\,(g)/\omega_n\right)^{\frac{n-1}{n}}$, there exists an asymptotically flat $(n+1)$-dimensional manifold M^{n+1} with non-negative curvature such that* *(i)* ∂M^{n+1} *is isometric to* (S^n, g) *and is minimal,*

(ii) M^{n+1}*, outside a compact set, is isometric to a spatial Schwarzschild manifold of mass* m*, and*

(iii) M^{n+1} *is foliated by mean convex* n*-spheres which eventually coincide with the rotationally symmetric* n*-spheres in the spatial Schwarzschild manifold.*

Chapter 3

Isometric embedding of 2-spheres into Schwarzschild manifolds

In general relativity, it is a challenging problem to give a suitable definition of quasi-local mass, i.e., how much energy (or mass) is present in a given region of a spacetime (see [38, 39] and references therein). Using the Hamiltonian formulation of general relativity, Brown and York in [6] defined the following quasi-local mass, known as the Brown-York mass: let (Ω, g) be a compact Riemannian 3-manifold with boundary $\partial\Omega$ and suppose that the Gauss curvature of $\partial\Omega$ is positive, the Brown-York mass of $\partial\Omega$ is

$$m_{BY}(\partial\Omega) = \frac{1}{8\pi}\int_{\partial\Omega}(H_0 - H)\,d\sigma, \tag{3.1}$$

where H is the mean curvature of $\partial\Omega$ in (Ω, g), H_0 is the mean curvature of $\partial\Omega$ when it is isometrically embedded in the Euclidean space $\mathbb{R}^3$, and $d\sigma$ is the area form on $\partial\Omega$. The fact that $\partial\Omega$ admits such an isometric embedding into $\mathbb{R}^3$ is guaranteed by the embedding theorem of Nirenberg [28] and Pogorelov [33].

In [38, 39], Wang and Yau defined a new quasi-local mass generalizing the Brown-York mass. In particular, they proved an existence theorem (cf. [39]) that provides isometric embeddings of a space-like 2-surface into the Minkowski spacetime $\mathbb{R}^{3,1}$.

The definitions of the Brown-York and the Wang-Yau mass suggest that results concerning isometric embeddings of a 2-surface into a model space often play an important role in quasi-local mass related problems. Recently, Lin and Wang in [24] obtained results regarding isometric embeddings into Anti-de Sitter spacetimes.

3.1 Introduction and statement of results

Motivated by the definitions of Brown-York and Wang-Yau quasi-local mass, we consider isometric embeddings of (S^2, g) into spatial Schwarzschild manifolds. Here S^2 denotes the 2-sphere and g is a Riemannian metric on it. For the remainder of the chapter, we let the Schwarzschild manifold in Definition 9 (with $n = 2$) to have two ends.

Definition 11. A spatial Schwarzschild manifold of mass m is the Riemannian manifold $(\mathbb{R}^3 \setminus \{0\}, g_m)$, where

$$g_m = \left(1 + \frac{m}{2|x|}\right)^4 g_{_E}, \tag{3.2}$$

where $g_{_E}$ is the Euclidean metric on $\mathbb{R}^3$.

When g has positive Gaussian curvature, by the solution of the Weyl problem due to Nirenberg [28] and Pogorelov [33], independently, (S^2, g) can be isometrically embedded in $\mathbb{R}^3$, which is the Schwarzschild manifold with $m = 0$. In this work, following Nirenberg's proof in [28], we prove the following theorem

Theorem 3.1.1. *Given $\alpha \in (0,1)$, there exist $\varepsilon > 0$ and $\delta = \delta(\alpha) > 0$, such that given any metric g' on S^2 with $|g' - \sigma_0|_{2,\alpha} < \delta$ and any Schwarzschild manifold $(\mathbb{R}^3 \setminus \{0\}, g_m)$ with $|m| < \varepsilon$, there exists an isometric embedding of (S^2, g') into $(\mathbb{R}^3 \setminus \{0\}, g_m)$. Here σ_0 denotes the standard metric on S^2.*

We remark that applications of isometric embeddings of nearly round spheres into $\mathbb{R}^3$ can be found in [12, 34].

The Schwarzschild metric g_m is conformally flat. In connection with Theorem 3.1.1, we obtain a sufficient condition that ensures the existence of isometric embeddings of perturbations of a Euclidean convex surface into a conformally flat manifold.

Theorem 3.1.2. *Let Σ be an embedded, convex surface in $\mathbb{R}^3$. Given any positive function u on $\mathbb{R}^3$, let $g_u = u^4 g_{_E}$, where $g_{_E}$ denotes the standard Euclidean metric on $\mathbb{R}^3$. Suppose the following conditions hold*

(i) g_u and $g_{_E}$ induce the same metric h on Σ.

(ii) the mean curvature of Σ in $(\mathbb{R}^3, g_u)$ and $(\mathbb{R}^3, g_{_E})$ agree.

Then, for any Riemannian metric $\tilde{h}$ on Σ that is sufficiently close to h on Σ, there exists an isometric embedding of $(\Sigma, \tilde{h})$ in $(\mathbb{R}^3, g_u)$.

We note that conditions (i) and (ii) in Theorem 3.1.2 coincide with the geometric boundary conditions in the static metric extension conjecture associated to Bartnik's definition of quasi-local mass [3]. In the conformally flat setting, even though such conditions may appear to be restrictive, they are satisfied up to higher-order perturbation on large coordinate spheres in a Schwarzschild manifold.

This chapter is organized as follows. In Section 3.2, we outline part of Nirenberg's argument from [28]. In Section 3.3, we prove Theorem 3.1.2. In Section 3.4, by modifying accordingly the arguments in Section 3.3, we prove Theorem 3.1.1. These results have been published and they can be found in [8].

3.2 Notation and preliminaries

In [40], Weyl considered the following problem: given a metric g on S^2 with positive Gaussian curvature, can (S^2, g) be isometrically embedded into $\mathbb{R}^3$? Positive answers to this question were given by Nirenberg [28] and Pogorelov [33]. Nirenberg in [28] followed the approach of Weyl, the continuity method. This method consists of the following three parts:

(a) The given metric g can be connected to the standard metric on S^2, $g_0 = \sigma_0$, by a family of metrics g_t $(0 \leq t \leq 1)$ with positive Gaussian curvature.

(b) The set of values of t for which the metric g_t can be isometrically embedded in $\mathbb{R}^3$, is open.

(c) The set of values of t for which the metric g_t can be isometrically embedded in $\mathbb{R}^3$, is closed.

We concentrate our attention on part (b). In Nirenberg's paper [28], this statement is proved in the language of functional analysis and its proof relies on the contraction mapping principle. In what follows, we extract certain ingredients in Nirenberg's solution to the Weyl problem that will be used later in our proof of Theorem 3.1.1 and 3.1.2.

We follow the exposition of Nirenberg's approach in [19]. First, we fix some notation: throughout this chapter, indices i, j and k used in differentiation refer to differentiation with respect to coordinates on $\mathbb{R}^3$ and indices α, β and γ refer to local coordinates on S^2. The letter α will also be used to denote a constant in $(0,1)$. For convenience, all metrics on S^2 are assumed to be smooth.

For a map $\mathbf{y} : S^2 \longrightarrow \mathbb{R}^3$ where $\mathbf{y} = (y_1, y_2, y_3)$, let

$$d\mathbf{y} = (dy_1, dy_2, dy_3). \tag{3.3}$$

Given another map $\mathbf{r} : S^2 \longrightarrow \mathbb{R}^3$, define

$$d\mathbf{r} \cdot d\mathbf{y}(u,v) = \sum_{\ell} \left(\frac{dr_\ell \otimes dy_\ell + dy_\ell \otimes dr_\ell}{2} \right)(u,v).$$

With this notation, the fact that a given metric g on S^2 with positive Gaussian curvature can be isometrically embedded in $\mathbb{R}^3$ by $\mathbf{r}$, i.e.,

$$g(u,v) = g_E(d\mathbf{r}(u), d\mathbf{r}(v)), \tag{3.4}$$

for any u and v in T_pM at any p in S^2, is simply expressed as

$$g = d\mathbf{r} \cdot d\mathbf{r}, \tag{3.5}$$

since

$$g_E((d\mathbf{r}(u), d\mathbf{r}(v)) = dr_m dr_m = d\mathbf{r} \cdot d\mathbf{r}(u,v). \tag{3.6}$$

Now suppose that g' is a metric close to g. Consider a perturbation of $\mathbf{r}$ given by $\mathbf{r}' = \mathbf{r} + \mathbf{y}$. Imposing the condition that (S^2, g') can be isometrically embedded into

$\mathbb{R}^3$ by $\mathbf{r}'$ leads to

$$2d\mathbf{r} \cdot d\mathbf{y} = (g' - g) - d\mathbf{y} \cdot d\mathbf{y}. \tag{3.7}$$

To solve (3.7) assuming $(g' - g)$ is small, one applies the contraction mapping method. For this purpose, the existence and estimates of solutions to the corresponding linear equation are needed. The following lemmas were proved in [28] (also see [19]).

Lemma 3.2.1 ([28, 19])**.** *For any smooth quadratic differential form $\overline{q}$ on S^2, there exists a solution $\mathbf{y}$ to*

$$2d\mathbf{r} \cdot d\mathbf{y} = \overline{q}. \tag{3.8}$$

Solutions to (3.8) are not unique. However, after proper normalization, a solution can be found with desired estimates:

Lemma 3.2.2 ([28, 19])**.** *Given any $\alpha \in (0,1)$ and any smooth quadratic differential form $\overline{q}$ on S^2, there exists a smooth solution $\mathbf{y}$ to (3.8) depending linearly on $\overline{q}$ satisfying*

$$|\mathbf{y}|_{2,\alpha} \leq C \left(|\overline{q}|_{1,\alpha} + \left[\frac{1}{\sqrt{|g|}} (\partial_2 c_1 - \partial_1 c_2) \right]_{\alpha} \right), \tag{3.9}$$

where $c_i = \dfrac{1}{\sqrt{|g|}} (\overline{q}_{i2;1} - \overline{q}_{i1;2})$ and $C = C(\alpha, \mathbf{r})$. Here all the norms are computed in a fixed coordinate chart on S^2.

3.3 Isometric embeddings into $(\mathbb{R}^3, g_u)$

Given a smooth function $u > 0$ on $\mathbb{R}^3$, let $g_u = u^4 g_{E}$ where g_{E} is the Euclidean metric. To prove Theorem 3.1.2, we first establish the following proposition.

Proposition 3.3.1. *Let g be a metric on S^2 with positive Gaussian curvature. Suppose that (S^2, g) can be isometrically embedded into $(\mathbb{R}^3, g_u)$ by $\mathbf{r} : S^2 \longrightarrow \mathbb{R}^3$. If $\nabla u = 0$ on $\Sigma = \mathbf{r}(S^2)$, then for any $\alpha \in (0,1)$, there exists $\mu > 0$ such that if g' is another metric on S^2 satisfying $|g - g'|_{2,\alpha} < \mu$, then (S^2, g') can be isometrically embedded into $(\mathbb{R}^3, g_u)$.*

We divide its proof into a few steps.

3.3.1 Derivation of the PDE

Using the notation described in Section 3.2, a map $\mathbf{r} : S^2 \longrightarrow \mathbb{R}^3$ being an isometric embedding of (S^2, g) into $(\mathbb{R}^3, g_u)$ can be expressed as

$$g = u(\mathbf{r})^4 d\mathbf{r} \cdot d\mathbf{r}, \tag{3.10}$$

since in this case, $\mathbf{r}$ being an isometric embedding means

$$g(w, v) = g_u(d\mathbf{r}(w), d\mathbf{r}(v)), \tag{3.11}$$

for all w and v in T_pS^2, and

$$g_u(d\mathbf{r}(w), d\mathbf{r}(v)) = u(\mathbf{r})^4 g_E(d\mathbf{r}(w), d\mathbf{r}(v)) = u(\mathbf{r})^4 d\mathbf{r} \cdot d\mathbf{r}(w, v). \tag{3.12}$$

Consider another metric g' on S^2, g' can be isometrically embedded into $(\mathbb{R}^3, g_u)$ by a perturbation of $\mathbf{r}$, $\mathbf{r}' = \mathbf{r} + \mathbf{y}$, if

$$g' = u(\mathbf{r}')^4 d\mathbf{r}' \cdot d\mathbf{r}' = u(\mathbf{r} + \mathbf{y})^4 (d\mathbf{r} \cdot d\mathbf{r} + 2d\mathbf{r} \cdot d\mathbf{y} + d\mathbf{y} \cdot d\mathbf{y}). \tag{3.13}$$

Since we are interested in $g' - g$, we can add and subtract $g = u(\mathbf{r})^4 d\mathbf{r} \cdot d\mathbf{r}$ to the right hand side of (3.13) to obtain

$$g' - g = (u(\mathbf{r}+\mathbf{y})^4 - u(\mathbf{r})^4) d\mathbf{r} \cdot d\mathbf{r} + u(\mathbf{r}+\mathbf{y})^4 (2d\mathbf{r} \cdot d\mathbf{y} + d\mathbf{y} \cdot d\mathbf{y}). \tag{3.14}$$

Then we see that (3.13) is equivalent to

$$g' - g = \left[\frac{u(\mathbf{r}+\mathbf{y})^4}{u(\mathbf{r})^4} - 1\right] g + u(\mathbf{r}+\mathbf{y})^4 (2d\mathbf{r} \cdot d\mathbf{y} + d\mathbf{y} \cdot d\mathbf{y}). \tag{3.15}$$

To find a solution $\mathbf{y}$ we apply the contraction mapping principle. We set up the iteration problem in the form of Lemma 3.2.1, i.e.,

$$\begin{aligned} 2d\mathbf{r} \cdot d\mathbf{y} &= \left[\frac{1}{u(\mathbf{r}+\mathbf{z})^4}\right](g' - g) - \left[\frac{1}{u(\mathbf{r})^4} - \frac{1}{u(\mathbf{r}+\mathbf{z})^4}\right] g - d\mathbf{z} \cdot d\mathbf{z} \qquad (3.16) \\ &= q(\mathbf{z}) - d\mathbf{z} \cdot d\mathbf{z}, \end{aligned}$$

for any smooth $\mathbf{z} : S^2 \longrightarrow \mathbb{R}^3$.

3.3.2 Estimates

Lemma 3.3.2. *Given $\alpha \in (0,1)$, if $\nabla u(\mathbf{r}) = 0$, then there is a constant $C = C(u, \alpha, \mathbf{r})$ such that for any $t \in [0,1]$ and $\mathbf{z}$ with $|\mathbf{z}|_{2,\alpha} \leq 1$,*

$$|\nabla u(\mathbf{r} + t\mathbf{z})| \leq C |\mathbf{z}|_{2,\alpha}. \tag{3.17}$$

Proof. First, note that there is a constant $\tilde{C} = \tilde{C}_{u,\alpha,\mathbf{r}}$ such that

$$|\partial_{ij}u(\mathbf{r} + \ell\mathbf{z})|_{2,\alpha} \leq \tilde{C}, \tag{3.18}$$

for any $\ell \in [0, 1]$. Then,

$$|\partial_i u(\mathbf{r} + t\mathbf{z})|_{2,\alpha} = \left| \int_0^1 \frac{d}{ds} \partial_i u(\mathbf{r} + st\mathbf{z})\, ds \right|_{2,\alpha} \tag{3.19}$$

$$\leq \tilde{\tilde{C}} \int_0^1 |\nabla \partial_i u(\mathbf{r} + st\mathbf{z})|_{2,\alpha} |\mathbf{z}|_{2,\alpha}\, ds \tag{3.20}$$

$$\leq C|\mathbf{z}|_{2,\alpha}. \tag{3.21}$$

□

Lemma 3.3.3. *Given any $\alpha \in (0, 1)$ and any smooth $\mathbf{z}$ with $|\mathbf{z}|_{2,\alpha} \leq 1$, if $\nabla u(\mathbf{r}) = 0$, then*

$$|u(\mathbf{r} + \mathbf{z}) - u(\mathbf{r})|_{2,\alpha} \leq o(1)|\mathbf{z}|_{2,\alpha}, \tag{3.22}$$

when $|\mathbf{z}|_{2,\alpha} \to 0$.

Proof.

$$|u(\mathbf{r} + \mathbf{z}) - u(\mathbf{r})|_{2,\alpha} \leq \int_0^1 |\nabla u(\mathbf{r} + t\mathbf{z}) \cdot \mathbf{z}|_{2,\alpha}\, dt \tag{3.23}$$

$$\leq \tilde{C} \int_0^1 |\nabla u(\mathbf{r} + t\mathbf{z})|_{2,\alpha} |\mathbf{z}|_{2,\alpha}\, dt \tag{3.24}$$

$$\leq o(1)|\mathbf{z}|_{2,\alpha}, \tag{3.25}$$

when $|\mathbf{z}|_{2,\alpha} \to 0$. □

Remark 3.3.4. Note that using

$$u(\mathbf{r}+\mathbf{z})^4 - u(\mathbf{r})^4 = (u(\mathbf{r}+\mathbf{z})^2 + u(\mathbf{r})^2)(u(\mathbf{r}+\mathbf{z}) + u(\mathbf{r}))(u(\mathbf{r}+\mathbf{z}) - u(\mathbf{z})), \quad (3.26)$$

we obtain similar estimates for $|u(\mathbf{r}+\mathbf{z})^4 - u(\mathbf{r})^4|_{2,\alpha}$. Also, we can find $\mu < 1$ so that $|u(\mathbf{r}+\mathbf{z})^4|_{2,\alpha}$ is bounded away from zero, whenever $|\mathbf{z}|_{2,\alpha} < \mu$.

Lemma 3.3.5. *For $q(\mathbf{z})$ in (3.16), that is*

$$q(\mathbf{z}) = \left[\frac{1}{u(\mathbf{r}+\mathbf{z})^4}\right](g' - g) - \left[\frac{1}{u(\mathbf{r})^4} - \frac{1}{u(\mathbf{r}+\mathbf{z})^4}\right] g, \quad (3.27)$$

there exists $\mu \in (0,1)$ such that for $|\mathbf{z}|_{2,\alpha} \leq \mu$,

$$|q(\mathbf{z})|_{2,\alpha} \leq C_{\mathbf{r},u,\alpha}|g' - g|_{2,\alpha} + o(1)|g|_{2,\alpha}|\mathbf{z}|_{2,\alpha}, \textit{ when } |\mathbf{z}|_{2,\alpha} \to 0. \quad (3.28)$$

Proof. Pick $0 < \mu < 1$ such that $|u(\mathbf{r}+\mathbf{z})^4|_{2,\alpha}$ is bounded away from zero. By Lemma 3.3.3 and Remark 3.3.4, it follows that

$$\left|\frac{1}{u(\mathbf{r})^4} - \frac{1}{u(\mathbf{r}+\mathbf{z})^4}\right|_{2,\alpha} \leq o(1)|\mathbf{z}|_{2,\alpha}, \quad (3.29)$$

when $|\mathbf{z}|_{2,\alpha} \to 0$. □

3.3.3 Contraction mapping

We apply the contraction mapping principle to the map $\mathbf{z} \mapsto \mathbf{y} = \Phi(\mathbf{z})$, that assigns to each $\mathbf{z}$ a solution $\mathbf{y} = \Phi(\mathbf{z})$ of (3.16). To do so, we need to show there exists $\mu > 0$ such that

1. $|\Phi(\mathbf{z})|_{2,\alpha} \leq \mu$ for $|\mathbf{z}|_{2,\alpha} \leq \mu$.

2. $|\Phi(\mathbf{z}) - \Phi(\mathbf{z}')| \leq \delta|\mathbf{z} - \mathbf{z}'|_{2,\alpha}$ for $\delta \in (0,1)$, when $|\mathbf{z}|_{2,\alpha}, |\mathbf{z}'|_{2,\alpha} \leq \mu$.

Lemma 3.3.6. *For any $\alpha \in (0,1)$, there exists a $\mu_* \in (0,1)$ such that for any smooth $\mathbf{z}$ with $|\mathbf{z}|_{2,\alpha} < \mu_*$, there exists a solution $\mathbf{y} = \Phi(\mathbf{z})$, such that for any $\mathbf{z}$ and $\mathbf{z}'$ with $|\mathbf{z}|_{2,\alpha}, |\mathbf{z}'|_{2,\alpha} < \mu_*$,*

$$|\Phi(\mathbf{z})|_{2,\alpha} \leq C_{\mathbf{r},u,\alpha}(|g' - g|_{2,\alpha} + o(1)|g|_{2,\alpha}|\mathbf{z}|_{2,\alpha}), \tag{3.30}$$

when $|\mathbf{z}|_{2,\alpha} \to 0$, and

$$\begin{aligned} |\Phi(\mathbf{z}) - \Phi(\mathbf{z}')|_{2,\alpha} \leq C_{\mathbf{r},u,\alpha}(&o(1)|g' - g|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} + o(1)|g|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} \\ &+ |\mathbf{z} + \mathbf{z}'|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha}), \end{aligned} \tag{3.31}$$

when $|\mathbf{z}|_{2,\alpha}, |\mathbf{z}'|_{2,\alpha} \to 0$.

Proof. By Lemma 3.2.2, for any $\alpha \in (0,1)$ and $\mathbf{z}$, there exists a solution $y = \Phi(\mathbf{z})$ of (3.16) such that

$$|\mathbf{y}|_{2,\alpha} \leq C_{\alpha,\mathbf{r}} \left(|\overline{q}|_{1,\alpha} + \left[\frac{1}{\sqrt{|g|}}(\partial_2 c_1 - \partial_1 c_2) \right]_\alpha \right), \tag{3.32}$$

where $\overline{q} = q(\mathbf{z}) - d\mathbf{z} \cdot d\mathbf{z}$. A direct computation shows that the derivatives of $\mathbf{z}$ appearing in $(|g|)^{-1/2}(\partial_2 c_1 - \partial_1 c_2)$ are up to order two, hence

$$\left[\frac{1}{\sqrt{|g|}}(\partial_2 c_1 - \partial_1 c_2) \right]_\alpha \leq C(|q(\mathbf{z})|_{2,\alpha} + |\mathbf{z}|^2_{2,\alpha}). \tag{3.33}$$

Combining equations (3.32) and (3.33), we obtain

$$|\Phi(\mathbf{z})|_{2,\alpha} \leq C_{\alpha,\mathbf{r}}(|q(\mathbf{z})|_{2,\alpha} + |\mathbf{z}|^2_{2,\alpha}). \tag{3.34}$$

Using Lemma 3.3.5, there exists $0 < \mu_* < 1$ such that for $|\mathbf{z}|_{2,\alpha} < \mu_*$,

$$|q(\mathbf{z})|_{2,\alpha} \leq C_{\mathbf{r},u,\alpha}(|g' - g|_{2,\alpha} + o(1)|g|_{2,\alpha}|\mathbf{z}|_{2,\alpha}), \tag{3.35}$$

when $|\mathbf{z}|_{2,\alpha} \to 0$.

For the second part, note that for any smooth $\mathbf{z}$ and $\mathbf{z}'$ we have

$$\begin{aligned} & q(\mathbf{z}') - q(\mathbf{z}) \\ = & \left[\frac{1}{u(\mathbf{r}+\mathbf{z}')^4} - \frac{1}{u(\mathbf{z}+\mathbf{r})^4}\right](g' - g) + \left[\frac{1}{u(\mathbf{r}+\mathbf{z}')^4} - \frac{1}{u(\mathbf{r}+\mathbf{z})^4}\right] g. \end{aligned} \tag{3.36}$$

If $\mathbf{y}$ and $\mathbf{y}'$ are the solutions corresponding to $\mathbf{z}$ and $\mathbf{z}'$, then

$$2d\mathbf{r} \cdot d\mathbf{y} = q(\mathbf{z}) - d\mathbf{z} \cdot d\mathbf{z} \text{ and } 2d\mathbf{r} \cdot d\mathbf{y}' = q(\mathbf{z}') - d\mathbf{z}' \cdot d\mathbf{z}'. \tag{3.37}$$

Using linearity, it follows that

$$2d\mathbf{r} \cdot d(\mathbf{y}' - \mathbf{y}) = q(\mathbf{z}') - q(\mathbf{z}) + d(\mathbf{z} + \mathbf{z}') \cdot d(\mathbf{z} - \mathbf{z}'). \tag{3.38}$$

By the linear dependence of $\mathbf{y}$ on $\overline{q}$ in Lemma 3.2.2, we have

$$|\mathbf{y}-\mathbf{y}'|_{2,\alpha} \leq C_{\mathbf{r},\alpha}(|q(\mathbf{z}')-q(\mathbf{z})+d\mathbf{z}\cdot d\mathbf{z}-d\mathbf{z}'\cdot d\mathbf{z}'|_{1,\alpha} + \left[\frac{1}{\sqrt{|g|}}(\partial_2 c_1-\partial_1 c_2)\right]_\alpha), \tag{3.39}$$

with c_i as in Lemma 3.2.2 with $\overline{q}=q(\mathbf{z}')-q(\mathbf{z})+d\mathbf{z}\cdot d\mathbf{z}-d\mathbf{z}'\cdot d\mathbf{z}'$.

We can control $|q(\mathbf{z}')-q(\mathbf{z})|_{1,\alpha}$ by $|q(\mathbf{z}')-q(\mathbf{z})|_{2,\alpha}$, for which it is enough to control $|u(\mathbf{r}+\mathbf{z})-u(\mathbf{r}+\mathbf{z}')|_{2,\alpha}$. For this,

$$|u(\mathbf{r}+\mathbf{z})-u(\mathbf{r}+\mathbf{z}')|_{2,\alpha} = \left|\int_0^1 \frac{d}{dt}u(\mathbf{r}+t\mathbf{z}+(1-t)\mathbf{z}')\,dt\right|_{2,\alpha} \tag{3.40}$$

$$\leq C\int_0^1 |\nabla u(\mathbf{r}+t\mathbf{z}+(1-t)\mathbf{z}')|_{2,\alpha}|\mathbf{z}-\mathbf{z}'|_{2,\alpha}\,dt \tag{3.41}$$

$$\leq C_{\mathbf{r},u,\alpha}o(1)|\mathbf{z}-\mathbf{z}'|_{2,\alpha},\ \text{when } |\mathbf{z}|_{2,\alpha}, |\mathbf{z}'|_{2,\alpha}\to 0. \tag{3.42}$$

Note that

$$|d\mathbf{z}\cdot d\mathbf{z}-d\mathbf{z}'\cdot d\mathbf{z}'|_{1,\alpha} = |d(\mathbf{z}+\mathbf{z})\cdot d(\mathbf{z}'-\mathbf{z}')|_{1,\alpha} \leq C|\mathbf{z}+\mathbf{z}'|_{2,\alpha}|\mathbf{z}-\mathbf{z}'|_{2,\alpha}. \tag{3.43}$$

In a similar way as before, a direct computation shows that for $|\mathbf{z}|_{2,\alpha}, |\mathbf{z}'|_{2,\alpha} < \mu_*$

$$\left[\frac{1}{\sqrt{|g|}}(\partial_2 c_1-\partial_1 c_2)\right]_\alpha \leq C_{\mathbf{r},u,\alpha}(o(1)|\mathbf{z}-\mathbf{z}'|_{2,\alpha}|g'-g|_{2,\alpha} + o(1)|\mathbf{z}-\mathbf{z}'|_{2,\alpha}|g|_{2,\alpha}+|\mathbf{z}+\mathbf{z}'|_{2,\alpha}|\mathbf{z}-\mathbf{z}'|_{2,\alpha}), \tag{3.44}$$

when $|\mathbf{z}|_{2,\alpha}, |\mathbf{z}'|_{2,\alpha}\to 0$.

With these estimates and (3.39), we obtain

$$|\Phi(\mathbf{z}) - \Phi(\mathbf{z}')|_{2,\alpha} \leq C_{\mathbf{r},u,\alpha}(o(1)|g - g'|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} + o(1)|g|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} + |\mathbf{z} + \mathbf{z}'|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha}), \tag{3.45}$$

when $|\mathbf{z}|_{2,\alpha}, |\mathbf{z}'|_{2,\alpha} \to 0$.

□

Proof of Proposition 3.3.1. Let $C = C_{\mathbf{r},u,\alpha}$ from Lemma 3.3.6 . Pick $\mu < \mu_*$ such that $o(1) < 1$, $Co(1)|g|_{2,\alpha} < \frac{1}{6}$ and $2C\mu < \frac{1}{6}$. Now take g' such that $C|g' - g|_{2,\alpha} < \dfrac{\mu}{6}$. Then, for $|\mathbf{z}|_{2,\alpha} \leq \mu$,

$$|\Phi(\mathbf{z})|_{2,\alpha} \leq C|g' - g|_{2,\alpha} + Co(1)|g|_{2,\alpha}|\mathbf{z}|_{2,\alpha} \leq \frac{\mu}{6} + \frac{\mu}{6} < \mu, \tag{3.46}$$

and for $|\mathbf{z}|_{2,\alpha}, |\mathbf{z}'|_{2,\alpha} \leq \mu$ we have

$$|\Phi(\mathbf{z}) - \Phi(\mathbf{z}')|_{2,\alpha} \leq (Co(1)|g' - g|_{2,\alpha} + Co(1)|g|_{2,\alpha} + C|\mathbf{z} + \mathbf{z}'|_{2,\alpha})|\mathbf{z} - \mathbf{z}'|_{2,\alpha} \tag{3.47}$$

$$\leq (o(1)\frac{\mu}{6} + \frac{1}{6} + 2\mu C)|\mathbf{z} - \mathbf{z}'|_{2,\alpha} \tag{3.48}$$

$$\leq \frac{1}{2}|\mathbf{z} - \mathbf{z}'|_{2,\alpha}. \tag{3.49}$$

□

3.3.4 Proof of Theorem 3.1.2

We translate conditions (i) and (ii) in Theorem 3.1.2 in terms of the conformal factor u. It is evident that (i) implies $u|_\Sigma = 1$. Using Lemma 1.2.5, the mean curvature of

Σ with respect to g_u, H_u, and with respect to $g_{_E}$, $H_{_E}$, are related by:

$$H_u = u^{-2} H_{_E} + 4u^{-3} \nabla u \cdot \mathbf{n}, \tag{3.50}$$

where $\mathbf{n}$ is the unit normal to Σ with respect to $g_{_E}$. Since $u|_\Sigma = 1$, the above reduces to

$$H_u = H_{_E} + 4 \nabla u \cdot \mathbf{n}. \tag{3.51}$$

Therefore, by (3.51), (i) and (ii) are equivalent to

$$\begin{cases} u|_\Sigma = 1 \\ \nabla u|_\Sigma = 0 \end{cases}. \tag{3.52}$$

Theorem 3.1.2 now follows directly from Proposition 3.3.1.

Remark 3.3.7. It is worth to note that replacing $\mathbb{R}^3$ by $\mathbb{R}^3 \setminus \{0\}$, and letting $g_u = g_m$ (the Schwarzschild metric), we have conditions (3.52) satisfied up to higher-order perturbation on large coordinate spheres. Precisely, if $\Sigma_R = \{|x| = R\}$ denotes a coordinate sphere in $(\mathbb{R}^3 \setminus \{0\}, g_m)$ of radius R (here $|x|$ denote the Euclidean norm), then

$$\begin{cases} u|_{\Sigma_R} = 1 + \dfrac{m}{2R} \\ \partial_i u|_{\Sigma_R} = \ -\dfrac{m}{2} \dfrac{x_i}{R^3} \end{cases}. \tag{3.53}$$

This suggests that a similar result can be established for Schwarzschild manifolds.

3.4 Isometric Embeddings into Schwarzschild Manifolds

In this section, we prove Theorem 3.1.1 . Let σ_0 be the standard metric on S^2. Set

$$u(x) = \left(1 + \frac{m}{2|x|}\right), \ |x| > 0. \tag{3.54}$$

Let $\mathbf{r}_0 : S^2 \to \mathbb{R}^3 \setminus \{0\}$ be the isometric embedding of (S^2, σ_0) into $(\mathbb{R}^3 \setminus \{0\}, g_m)$ given by $\mathbf{r}_0(\omega) = r_0\omega$, where r_0 is the solution to

$$r_0 \left(1 + \frac{m}{2r_0}\right)^2 = 1. \tag{3.55}$$

As m approaches to 0, r_0 approaches to 1 (here we restrict to $m < 1/2$ to ensure the existence of r_0).

Let g' be another metric on S^2 that is close to σ_0. (S^2, g') can be isometrically embedded in $(\mathbb{R}^3 \setminus \{0\}, g_m)$ by perturbation of $\mathbf{r}_0$, $\mathbf{r}'_0 = \mathbf{r}_0 + \mathbf{y}$, if and only if (3.15) holds, i.e.

$$g' - \sigma_0 = \left[\frac{u(\mathbf{r}_0 + \mathbf{y})^4}{u(\mathbf{r}_0)^4} - 1\right] \sigma_0 + u(\mathbf{r}_0 + \mathbf{y})^4 (2d\mathbf{r}_0 \cdot d\mathbf{y} + d\mathbf{y} \cdot d\mathbf{y}). \tag{3.56}$$

As before, we set up the iteration. Given $\mathbf{z} : S^2 \longrightarrow \mathbb{R}^3$, let $\mathbf{y} = \Phi(\mathbf{z})$ be the solution to

$$\begin{aligned} 2d\mathbf{r}_0 \cdot d\mathbf{y} = & \left[\frac{1}{u(\mathbf{r}_0 + \mathbf{z})^4}\right] (g' - \sigma_0) \\ & - \left[\frac{1}{u(\mathbf{r}_0)^4} - \frac{1}{u(\mathbf{r}_0 + \mathbf{z})^4}\right] \sigma_0 - d\mathbf{z} \cdot d\mathbf{z} \end{aligned} \tag{3.57}$$

$$= q(\mathbf{z}) - d\mathbf{z} \cdot d\mathbf{z}, \tag{3.58}$$

whose existence is guaranteed by Lemma 3.2.1.

We proceed to obtain estimates to apply the contraction mapping principle to the map Φ. By the triangle inequality, for any $\mathbf{z} : S^2 \longrightarrow \mathbb{R}^3$ satisfying $|\mathbf{z}|_0 \leq \mu = \dfrac{r_0}{2}$, we have

$$\frac{r_0}{2} \leq |\mathbf{r}_0(\omega) + \mathbf{z}(\omega)| \leq \frac{3}{2} r_0, \tag{3.59}$$

for all $\omega \in S^2$. A direct calculation shows that for $\mathbf{z}$ with $|\mathbf{z}|_0 < \mu$ and $|\mathbf{z}|_{2,\alpha} \leq |\mathbf{r}_0|_{2,\alpha}$, we have

$$|u(\mathbf{r}_0 + \mathbf{z}) - u(\mathbf{r}_0)|_{2,\alpha} \leq |m| \frac{1}{r_0^2} |\mathbf{z}|_{2,\alpha}, \tag{3.60}$$

for some C independent of $\mathbf{z}$. The estimate (3.60) implies the following

Lemma 3.4.1. *For any* $\mathbf{z}$ *satisfying* $|\mathbf{z}|_{2,\alpha} \leq \mu = \dfrac{r_0}{2}$, *we have*

$$|u(\mathbf{r}_0 + \mathbf{z})^4 - u(\mathbf{r}_0)^4|_{2,\alpha} \leq C|m| \frac{1}{r_0^2} |\mathbf{z}|_{2,\alpha} = o(1)|\mathbf{z}|_{2,\alpha} \tag{3.61}$$

as $m \to 0$.

We now fix $\mu = r_0/2$. Repeating the arguments in the proof of Lemma 3.3.6, we can prove the following lemma.

Lemma 3.4.2. *For any* $\mathbf{z}$ *with* $|\mathbf{z}|_{2,\alpha} \leq \mu$, *there is a solution* $\mathbf{y} = \Phi(\mathbf{z})$ *of (3.57) such that*

$$|\Phi(\mathbf{z})|_{2,\alpha} \leq C_{\alpha,\mathbf{r}_0}(|g' - \sigma_0|_{2,\alpha} + o(1)|\sigma_0|_{2,\alpha}|\mathbf{z}|_{2,\alpha} + |\mathbf{z}|^2_{2,\alpha}), \text{ as } m \to 0, \tag{3.62}$$

and given another z' with $|\mathbf{z}'|_{2,\alpha} < \mu$,

$$|\Phi(\mathbf{z}) - \Phi(\mathbf{z}')|_{2,\alpha} \leq C_{\alpha,\mathbf{r}_0} \left(o(1)|g' - \sigma_0|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} + o(1)|\sigma_0|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} \right. \\ \left. +|\mathbf{z} + \mathbf{z}'|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} \right), \text{ as } m \to 0. \tag{3.63}$$

We are in a position to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Instead of working with the isometric embedding $\mathbf{r}_0$ defined above, we consider $\tilde{\mathbf{r}}_0 = r_0^{-1}\mathbf{r}_0$. Divide (3.57) by r_0 to obtain

$$2d\tilde{\mathbf{r}}_0 \cdot d\mathbf{y} = \frac{1}{r_0}(q(\mathbf{z}) - d\mathbf{z} \cdot d\mathbf{z}). \tag{3.64}$$

By the linear nature of the differential equation, all the previous estimates can be carried over. By Lemma 3.4.2, for $|\mathbf{z}|_{2,\alpha} \leq \mu$ we have

$$|\widetilde{\Phi}(\mathbf{z})|_{2,\alpha} \leq \frac{C}{r_0}(|g' - \sigma_0|_{2,\alpha} + o(1)|\sigma_0|_{2,\alpha}|\mathbf{z}|_{2,\alpha} + |\mathbf{z}|^2_{2,\alpha}), \tag{3.65}$$

and

$$|\widetilde{\Phi}(\mathbf{z}) - \widetilde{\Phi}(\mathbf{z}')|_{2,\alpha} \leq \frac{C}{r_0}(o(1)|g' - \sigma_0|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} + o(1)|\sigma_0|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} \\ + |\mathbf{z} + \mathbf{z}'|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha}), \tag{3.66}$$

when $|m| \to 0$.

Let $|m| < \varepsilon_1$ with ε_1 sufficiently small so that $|r_0 - 1| < \frac{1}{2}$. Then $\frac{1}{r_0}$ is bounded, so we have

$$|\widetilde{\Phi}(\mathbf{z})|_{2,\alpha} \leq C(|g' - \sigma_0|_{2,\alpha} + o(1)|\sigma_0|_{2,\alpha}|\mathbf{z}|_{2,\alpha} + |\mathbf{z}|^2_{2,\alpha}), \tag{3.67}$$

and

$$|\widetilde{\Phi}(\mathbf{z}) - \widetilde{\Phi}(\mathbf{z}')|_{2,\alpha} \leq C(o(1)|g' - \sigma_0|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} + o(1)|\sigma_0|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} + |\mathbf{z} + \mathbf{z}'|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha}), \tag{3.68}$$

when $|m| \to 0$. Let $\mu' < \mu$ so that $2C\mu' < \frac{1}{6}$. Take $|m| < \varepsilon < \varepsilon_1$ such that $o(1) < 1$ and $Co(1)|\sigma_0|_{2,\alpha} < \frac{1}{6}$. Take g' so that $|g' - \sigma_0|_{2,\alpha} < \frac{\mu'}{3C}$. Hence,

$$|\widetilde{\Phi}(\mathbf{z})|_{2,\alpha} \leq C|g' - \sigma_0|_{2,\alpha} + Co(1)|\sigma_0|_{2,\alpha}|\mathbf{z}|_{2,\alpha} + C|\mathbf{z}|^2_{2,\alpha} < \mu' \tag{3.69}$$

and

$$|\widetilde{\Phi}(\mathbf{z}) - \widetilde{\Phi}(\mathbf{z}')|_{2,\alpha} \leq Co(1)|g' - \sigma_0|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} + Co(1)|\sigma_0|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha} + C|\mathbf{z} + \mathbf{z}'|_{2,\alpha}|\mathbf{z} - \mathbf{z}'|_{2,\alpha}) \tag{3.70}$$

$$< \nu|\mathbf{z} - \mathbf{z}'|_{2,\alpha}, \tag{3.71}$$

with $\nu < 1$. By the contraction mapping principle, there exists a solution $\mathbf{y}$ to

$$2d\tilde{\mathbf{r}}_0 \cdot d\mathbf{y} = \frac{1}{r_0}(q(\mathbf{y}) - d\mathbf{y} \cdot d\mathbf{y}).$$

Hence, $r_0\mathbf{y}$ is the desired isometric embedding. □

Appendix A

Paths in $\mathrm{Ric}^+(S^3)$

In this appendix we show how to obtain a result similar to those in Chapter 2 for $n = 3$, by a more elementary application of Ricci flow.

A.1 Normalized Ricci flow

The *Ricci curvature* at a unit vector U in T_pM is defined as

$$\mathrm{Ric}(U, U). \tag{A.1}$$

Definition 12. We say that a Riemannian manifold (M, g) has positive Ricci curvature if for all p in M, $\mathrm{Ric}(U, U) > 0$ for all unit vectors U in T_pM.

Let $\mathrm{Ric}^+(S^3)$ denote the set of smooth metrics on S^3 with positive Ricci curvature. Given $g \in \mathrm{Ric}^+(S^3)$, we apply results on the smooth solution to the Ricci flow with initial condition g, developed by Hamilton in [17].

We recall some basic facts about Ricci flow on closed 3-manifolds (readers are referred to [17, 9] for more comprehensive expositions). On a closed manifold M, the Ricci flow equation is given by

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2\mathrm{Ric}(g(t)) \\ g(0) &= g, \end{aligned} \tag{A.2}$$

where $\mathrm{Ric}(\cdot)$ denotes the Ricci curvature of the metric.

To prevent the flow from shrinking to a point in a finite time, we consider the *normalized Ricci flow*, which is obtained by a normalization to keep the volume of the manifold constant along the flow. Let us rescale the metric by a factor $\psi(t)$, i.e., let us define the new metric

$$\tilde{g}(t) = \psi(t) g(t).$$

We seek conditions on ψ to keep the volume constant. Specifically, we impose $\psi(0) = 1$ and $\mathrm{vol}\,(\tilde{g}(t)) = \mathrm{vol}\,(g)$. According to (1.32), the volume transforms as

$$\mathrm{vol}\,(\tilde{g}(t)) = \int_M dV_{\tilde{g}(t)} = \int_M \psi(t)^{3/2} dV_{g(t)} = \psi(t)^{3/2} \mathrm{vol}\,(g(t)), \tag{A.3}$$

hence, since $\mathrm{vol}\,(\tilde{g}(t)) = \mathrm{vol}\,(g)$,

$$\psi(t) = \left(\frac{\mathrm{vol}\,(g)}{\mathrm{vol}\,(g(t))} \right)^{2/3} = \left(\frac{\mathrm{vol}\,(g(t))}{\mathrm{vol}\,(g)} \right)^{-2/3}. \tag{A.4}$$

It follows that the equation satisfied by $\tilde{g}$ is

$$\frac{\partial \tilde{g}}{\partial t} = -2\psi(t)\mathrm{Ric}(g(t)) + \psi'(t) g(t). \tag{A.5}$$

From the definition of $\psi(t)$ and the formula for the evolution of the volume,

$$\frac{\partial}{\partial t}\mathrm{vol}\,(g(t)) = \int_M \frac{1}{2} g^{ij}\dot{g}_{ij}\, d\mu_t, \tag{A.6}$$

we can obtain $\psi'(t)$:

$$\psi'(t) = -\frac{2}{3}\left(\frac{\displaystyle\int_M dV_{g(t)}}{\displaystyle\int_M dV_g}\right)^{-2/3-1}\left(\frac{\displaystyle -\int_M R_{g(t)}\, dV_{g(t)}}{\displaystyle\int_M dV_g}\right) = \frac{2}{3}\psi(t)r(t), \tag{A.7}$$

where $r(t) = \dfrac{\displaystyle\int_M R_{g(t)}\, dV_{g(t)}}{\displaystyle\int_M dV_{g(t)}}$. Using (1.32) and (1.31), we see that

$$\tilde{r}(t) = \frac{\displaystyle\int_M R_{\tilde{g}(t)}\, dV_{\tilde{g}(t)}}{\displaystyle\int_M dV_{\tilde{g}(t)}} = \psi(t)^{-1}r(t), \tag{A.8}$$

so we conclude that

$$\psi'(t) = \frac{2}{3}\psi(t)^2\tilde{r}(t). \tag{A.9}$$

Using (1.30) and (A.9), (A.5) becomes

$$\frac{\partial \tilde{g}}{\partial t} = \psi(t)(-2\mathrm{Ric}(\tilde{g}(t)) + \frac{2}{3}\tilde{g}(t)\tilde{r}(t)). \tag{A.10}$$

Now, let $\tau = \displaystyle\int_0^t \psi(u)\, du$, then $\dfrac{\partial \tilde{g}}{\partial \tau} = \dfrac{\partial \tilde{g}}{\partial t}\dfrac{\partial t}{\partial \tau}$. Since $\dfrac{\partial \tau}{\partial t} = \psi(t)$, hence by the formula for the derivative of the inverse, $\dfrac{\partial t}{\partial \tau} = \dfrac{1}{\psi(t)}$. Therefore, the normalized Ricci flow is given by

$$\frac{\partial \tilde{g}}{\partial \tau} = -2\mathrm{Ric}(\tilde{g}(\tau)) + \frac{2}{3}\tilde{r}(\tau)\tilde{g}(\tau).$$

For simplicity, we will abuse notation by renaming $\tilde{g}$ as g and then, we write the normalized Ricc flow as

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g(t)) + \frac{2}{3}r(t)g(t), \tag{A.11}$$

where $r(t) = \left(\int_M dV_{g(t)}\right)^{-1}\left(\int_M R_{g(t)}\, dV_{g(t)}\right)$. Along solutions $\{g(t)\}$ to (A.11), the volume of $g(t)$ remains a constant. When the initial metric $g = g(0)$ has positive Ricci curvature, the following holds [17] (also cf. [9]).

Theorem A.1.1. *Suppose g is a metric of positive Ricci curvature on a closed manifold M^3. Then* (A.11) *with the initial condition $g(0) = g$ has a unique, smooth solution $\{g(t)\}$ on $[0, \infty)$; moreover, $\{g(t)\}$ converges exponentially fast to a metric g_∞ of positive constant sectional curvature in every C^k-norm. That is, there exists a constant $\delta > 0$, such that for any integer $k \geq 0$,*

$$\|g(t) - g_\infty\|_{C^k(S^3)} \leq Ce^{-\delta t} \tag{A.12}$$

for some positive constants C. Furthermore,

$$\|\nabla^m Ric(g(t))\|_{C^0(S^3)} \leq Ce^{-\delta t}, \tag{A.13}$$

where ∇^m denotes the m-times iterated covariant derivative on $(M^3, g(t))$.

Remark A.1.2.

$$\left\|\frac{\partial g}{\partial t}\right\|_{C^k(S^3)} \leq Ce^{-\delta t},\ \forall\ k. \tag{A.14}$$

The later follows from (A.11) and (A.13), and noticing that

$$\nabla^m \frac{\partial g}{\partial t} = -2\nabla^m \mathrm{Ric}(g(t)), \tag{A.15}$$

since $r(t)$ only depends on t and $\nabla g = 0$.

A.2 Metric on $I \times S^3$

In what follows we take $M = S^3$ and let $\{g(t)\}$ be the solution to (A.11) with the initial condition $g \in \mathrm{Ric}^+(S^3)$. Notice that by Remark 1.2.1, the metric g_∞ of positive constant sectional curvature is round. We make a change of variable $t = t(s)$ to view the metrics $\{g(t)\}$ as a new family of metrics $\{h(s)\}$ defined on the finite interval $[0, 1)$. Specifically, let

$$t(s) = \frac{1}{(s-1)^2} - 1,\ \text{(then } s = 1 - \frac{1}{\sqrt{1+t}}) \tag{A.16}$$

and define

$$h(s) = \begin{cases} g(t(s)) & \text{when } s \in [0,1) \\ g_\infty & \text{when } s = 1 \end{cases}. \tag{A.17}$$

Endow the product $I \times S^3$ with the metric

$$H = ds^2 + h(s). \tag{A.18}$$

By the smoothness of $\{g(t)\}$ on $[0,\infty)$, H is smooth on $[0,1)\times S^3$. We want to verify that H is indeed smooth up to $\{1\}\times S^3$. Let $\{x^i\}$, $i=1,2,3$ denote a fixed local coordinate chart on S^3.

Claim. H is continuous on $I\times S^3$.

Proof. Given $(1,x_0)$ in $\{1\}\times S^3$, we want to show

$$\lim_{(s,x)\to(1^-,x_0)} h_{ij}(s,x) = h_{ij}(1,x_0). \tag{A.19}$$

This follows from (A.12) and the fact

$$\begin{aligned} &|g_{ij}(t(s),x) - (g_\infty)_{ij}(x_0)| \\ \le\ &|g_{ij}(t(s),x) - g_{ij}((t(s),x_0)| + |g_{ij}(t(s),x_0) - (g_\infty)_{ij}(x_0)|. \end{aligned} \tag{A.20}$$

□

Claim. H is $C^1(I\times S^3)$.

Proof. We focus only on checking the derivative with respect to s (the continuity of spatial derivatives at $s=1$ follows directly from (A.12)). For a fixed x, let $h_s = h_{ij}(s,x)$, $g_t = g_{ij}(t,x)$ and $g_\infty = (g_\infty)_{ij}(x)$. Then

$$\begin{aligned} \lim_{s\to 1^-}\left|\frac{h_s-h_1}{s-1}\right| &= \lim_{t\to\infty}\left|\frac{g_t-g_\infty}{-(t+1)^{-1/2}}\right| \\ &\le \lim_{t\to\infty} Ce^{-\delta t}\sqrt{t+1} \\ &= 0. \end{aligned} \tag{A.21}$$

Hence $\left.\dfrac{\partial h}{\partial s}\right|_{s=1}$ exists and is equal to 0. Therefore,

$$\frac{\partial h}{\partial s}(s,x) = \begin{cases} \dfrac{\partial g}{\partial t}(t(s),x)t'(s) & \text{when } s \in [0,1) \\ 0 & \text{when } s = 1 \end{cases} . \tag{A.22}$$

By (A.14),

$$\left\| \frac{\partial h}{\partial s} \right\| \leq Ce^{-\delta t(s)} |t'(s)| \to 0, \tag{A.23}$$

as $s \to 1^-$. (A.22) and (A.23) imply that $\dfrac{\partial h}{\partial s}$ is continuous at $s = 1$. □

For higher order derivatives, when $s \in [0,1)$ we have

$$\frac{\partial^2 h}{\partial s^2} = \frac{\partial^2 g}{\partial t^2}(t(s))(t'(s))^2 + \frac{\partial g}{\partial t}(t(s))t''(s). \tag{A.24}$$

By (A.14), the second term approaches 0 as $s \to 1^-$. To deal with the first term, by (A.11) we have

$$\frac{\partial^2 g}{\partial t^2} = -2\frac{\partial}{\partial t}\mathrm{Ric}(g(t)) + \frac{3}{2}\frac{\partial}{\partial t}r(t)g(t) + \frac{3}{2}r(t)\frac{\partial}{\partial t}g(t). \tag{A.25}$$

Since $r(t)$ tends to a constant as $t \to \infty$, the last term in (A.25) goes to 0 exponentially fast. Since the volume of $(S^3, g(t))$ is independent of t and using the formula for the evolution of the integral scalar curvature,

$$\frac{\partial}{\partial t}\int_M R \, d\mu_t = \int_M (-\dot{g}^{ij}R_{ij} + \frac{1}{2}Rg^{ij}\dot{g}_{ij}) \, d\mu_t, \tag{A.26}$$

we have

$$\frac{\partial}{\partial t} r(t) = \frac{\frac{\partial}{\partial t}\int_{S^3} R_{g(t)} dV_{g(t)}}{\int_{S^3} dV_{g(t)}} = \frac{\int_{S^3}\left[\frac{1}{2}\mathrm{tr}_{g(t)}\dot{g}(t)R_{g(t)} - \langle \mathrm{Ric}(g(t)), \dot{g}(t)\rangle\right] dV_{g(t)}}{\int_{S^3} dV_{g(t)}}, \tag{A.27}$$

where $\dot{g}(t) = \frac{\partial g}{\partial t}$. By (A.14), $\frac{\partial}{\partial t} r(t) \to 0$ exponentially fast. For the first term in the right hand side of (A.25), the formula of the evolution of the Ricci tensor gives

$$\frac{\partial}{\partial t} R_{ij} = \frac{1}{2} g^{ml}\left[\nabla_l \nabla_i \dot{g}_{jm}(t) - \nabla_l \nabla_j \dot{g}_{im}(t) \quad - \nabla_l \nabla_m \dot{g}_{ij}(t) - \nabla_i \nabla_j \dot{g}_{lm}(t)\right], \tag{A.28}$$

but $\nabla_k \dot{g}(t) = -2\nabla_k \mathrm{Ric}$ by (A.15), hence $\frac{\partial}{\partial t}\mathrm{Ric}(t) \to 0$ exponentially fast by (A.13). Thus, we have verified that $\frac{\partial^2 g}{\partial t^2} \to 0$ exponentially fast as t approaches ∞, which implies $\frac{\partial^2 h}{\partial s^2} \to 0$ as $s \to 1^-$ by (A.24).

Claim. H is $C^2(I \times S^3)$.

Proof. Since $\frac{\partial^2 h}{\partial s^2} \to 0$ as $s \to 1^-$, we only need to show that $\frac{\partial^2 h}{\partial s^2}(1, x)$ exists and equals 0. Again, we focus on checking derivatives involving s alone. Fix x and set $h'_s = \frac{\partial h_{ij}}{\partial s}(s, x)$ and $g'_t = \frac{\partial g_{ij}}{\partial t}(t, x)$, then

$$\lim_{s \to 1^-} \frac{h'_s - h'_1}{s - 1} = \lim_{t \to \infty} -2g'_t(1 + t)^2 = 0, \tag{A.29}$$

which proves the claim. □

Note that one can indeed conclude from (A.25), (A.13) and (A.14) that

$$\left\|\frac{\partial^2 g}{\partial t^2}\right\|_{C^k(S^3)} \leq Ce^{-\delta t}. \tag{A.30}$$

Thus, repeating the arguments above, an inductive argument shows that H is C^m on $I \times S^3$ for any m (see Section A.3). We proved

Lemma A.2.1. *Let g in $Ric^+(S^3)$. There exists a family of metrics $\{h(t)\}$ with positive Ricci curvature (hence positive scalar curvature), such that*

- *$h(0) = g$ and $h(1)$ is round,*
- *$h(t)$ is volume preserving, and*
- *$H = dt^2 + h(t)$ is a smooth metric on $I \times S^n$.*

Lemma A.2.1 together with Theorem 2.2.9 provide the proof of the following theorem:

Theorem A.2.2. *Let g be a metric with positive Ricci curvature on S^3. Given any $m > 0$ such that $\omega_3 (2m)^{\frac{3}{2}} > \mathrm{vol}(g)$, there exists an asymptotically flat 4-dimensional manifold M^4 with nonnegative curvature such that*

(i) ∂M^4 is isometric to (S^3, g) and is minimal,

(ii) M^4, outside a compact set, is isometric to a spatial Schwarzschild manifold of mass m, and

(iii) M^4 is foliated by mean convex 3-spheres which eventually coincide with the rotationally symmetric 3-spheres in the spatial Schwarzschild manifold.

A.3 Time regularity of paths in $\mathrm{Ric}^+(S^3)$

To prove higher regularity of the metric $H = ds^2 + h(s)$ on $I \times S^n$ we take a different approach. In the fixed local coordinate system, recall that

$$R_{ij} = \partial_l \Gamma^l_{ij} - \partial_j \Gamma^l_{il} + \Gamma^l_{ij}\Gamma^m_{\lambda m} - \Gamma^m_{lm}\Gamma^\lambda_{jm} \tag{A.31}$$

$$= \tilde{P}(g^{-1}, \partial_x g^{-1}, \partial_x g, \partial_x^2 g) \tag{A.32}$$

$$= P^{(0)}_{\mathrm{Ric}}(g, g^{-1}, \partial_x g, \partial_x^2 g), \tag{A.33}$$

where $P^{(0)}_{\mathrm{Ric}}$ is a polynomial in several variables with $P^{(0)}_{\mathrm{Ric}}(0) = 0$. Here g^{-1}, g, $\partial_x g$ and $\partial_x^2 g$ denote vectors formed by the components of these quantities, and recall that the time derivative of these quantities approach to 0 exponentially as $t \to 0$. Then,

$$\frac{\partial}{\partial t} R_{ij} = \nabla P^{(0)}_{\mathrm{Ric}}(g, g^{-1}, \partial_x g, \partial_x^2 g) \cdot (\dot{g}, \dot{g}^{-1}, \partial_x \dot{g}, \partial_x^2 \dot{g}) \to 0, \tag{A.34}$$

as $t \to \infty$. Here the dot denotes derivative with respect to time. We can rewrite (A.34) as

$$\frac{\partial}{\partial t} R_{ij} = P^{(1)}_{\mathrm{Ric}}(g, g^{-1}, \partial_x g, \partial_x^2 g, \dot{g}, \dot{g}^{-1}, \partial_x \dot{g}, \partial_x^2 \dot{g}) = P^{(1)}_{\mathrm{Ric}}(g^{-1}, \partial_x^{\alpha \le 2} g^{(\beta \le 1)}), \tag{A.35}$$

where α denotes the order of the spatial derivatives of g appearing in the expression and β the order of time derivatives. With this notation,

$$\frac{\partial^2}{\partial t^2} R_{ij} = \nabla P^{(1)}(g^{-1}, \partial_x^{\alpha \le 2} g^{(\beta \le 1)}) \cdot (\dot{g}^{-1}, \partial_x^{\alpha \le 2} \dot{g}^{(\beta \le 1)}) = P^{(2)}_{\mathrm{Ric}}(g^{-1}, \partial_x^{\alpha \le 2} g^{(\beta \le 2)}), \tag{A.36}$$

and we can rewrite the normalized Ricci flow equation (A.11) as

$$\frac{\partial g}{\partial t} = -2P^{(1)}_{\mathrm{Ric}}(g^{-1}, \partial_x^{\alpha\leq 2} g^{(\beta\leq 0)}) + \frac{2}{3} r(t) g(t). \tag{A.37}$$

We need to analyze $r(t) = \left(\int_M dV_{g(t)}\right)^{-1} \left(\int_M R_{g(t)}\, dV_{g(t)}\right)$.

$$\frac{d}{dt} r(t) = \frac{\frac{\partial}{\partial t}\int_{S^3} P^{(0)}_R(g^{-1}, \partial_x^{\alpha\leq 2} g^{\beta\leq 0})\varphi(t) dV_{S^3}}{\int dV_{g(t)}} \tag{A.38}$$

$$= \frac{\int_{S^3} [\partial_t P^{(0)}_R(g^{-1}, \partial_x^{\alpha\leq 2} g^{\beta\leq 0})\varphi(t) + P^{(0)}_R(g^{1}, \partial_x^{\alpha\leq 2} g^{\beta\leq 0})\varphi'(t)] dV_{S^3}}{\int dV_{g(t)}} \tag{A.39}$$

$$= \frac{\int_{S^3} [P^{(1)}_R(g^{-1}, \partial_x^{\alpha\leq 2} g^{\beta\leq 1})\varphi(t) + P^{(0)}_R(g^{1}, \partial_x^{\alpha\leq 2} g^{\beta\leq 0})\varphi'(t)] dV_{S^3}}{\int dV_{g(t)}} \tag{A.40}$$

where $\varphi(t) dV_{S^3} = dV_{g(t)}$, hence $\varphi'(t) dV_{S^3} = g^{ij}\dot{g}_{ij} dV_{g(t)}$ and

$$\frac{d}{dt} r(t) \to 0, \text{ as } t \to 0. \tag{A.41}$$

With this notation, it is clear that $\frac{d}{dt^k} r(t) \to 0$ exponentially fast as $t \to 0$ for any k.

In a similar fashion, we can show that h_{ij} is $C^k(S^3 \times I)$. We illustrate again the C^2 case.

$$\frac{\partial^2 g}{\partial t^2} = -2P^{(2)}_{\mathrm{Ric}}(g^{-1}, \partial_x^{\alpha\leq 2} g^{\beta\leq 1}) + \frac{2}{3} r'(t) g(t) + \frac{2}{3} r(t)\dot{g}(t) \to 0.$$

For the spatial derivatives we use (A.1.2).

The convergence of the third derivative, note that

$$\frac{\partial^3 g}{\partial t^3} = -2P^{(3)}_{\mathrm{Ric}}(g^{-1}, \partial_x^{\alpha\leq 2} g^{\beta\leq 2}) + \frac{\partial^2}{\partial t^2}(r(t)g(t)) \to 0.$$

$$\frac{\partial^3 g}{\partial x_k \partial t^2} = -2\frac{\partial}{\partial x_k} P^{(1)}_{\mathrm{Ric}}(g^{-1}, \partial_x^{\alpha\leq 2} g^{\beta\leq 1}) + \frac{\partial}{\partial t}(r(t)\frac{\partial}{\partial x_k} g(t)) \to 0.$$

Therefore by the structure of the equations, it is clear that $h_{ij}(x,s)$ is $C^k(S^3 \times I)$ by induction.

Appendix B

Further properties of star-shaped surfaces

B.1 Time regularity of paths in $\mathrm{Scal}_*^+(S^n)$

This part is devoted to argue that

$$\left\| \frac{\partial^k \tilde{\rho}}{\partial t^k} \right\|_{C^l(S^n)} \leq Ce^{-\delta t}, \tag{B.1}$$

holds for any integer k.

For this, we will need to explain some of the notation in [37] used in Section 2.4. Recall that $\tilde{\rho}(\cdot, t) = e^{-t}\rho(\cdot, t)$, where $\rho(\cdot, t)$ is the solution to (2.77). Hence, $\tilde{\rho}$ satisfies the differential equation

$$\frac{\partial \tilde{\rho}}{\partial t} = \frac{(\tilde{\rho}^2 + |\nabla \tilde{\rho}|^2)^{\frac{1}{2}}}{\tilde{\rho} F(\tilde{a}_{ij})} - \tilde{\rho}. \tag{B.2}$$

Here, F is a C^∞ homogeneous function of degree one acting on a convex cone of real symmetric $n \times n$ matrices. Moreover, it satisfies $F(\delta_{ij}) = 1$. For a precise description of the properties that F inherits from f in (2.63), see [37]. Straightforward computations show that with respect an orthonormal frame on S^n,

$$\tilde{g}_{ij} = \tilde{\rho}^2 \delta_{ij} + \nabla_i \tilde{\rho} \nabla_j \tilde{\rho}, \tag{B.3}$$

$$\tilde{g}^{ij} = \tilde{\rho}^{-2} \left(\delta_{ij} - \frac{\nabla_i \tilde{\rho} \nabla_j \tilde{\rho}}{\tilde{\rho}^2 + |\nabla \tilde{\rho}|^2} \right), \tag{B.4}$$

where g is the induced metric on the surface $\tilde{\rho}(\cdot, t)$ and ∇ is the gradient on S^n. The second fundamental form, $\tilde{h}$, with respect to the outward unit normal is given by

$$\tilde{h}_{ij} = \frac{1}{(\tilde{\rho}^2 + |\nabla \tilde{\rho}|^2)^{1/2}} (\tilde{\rho}^2 \delta_{ij} + 2 \nabla_i \tilde{\rho} \nabla_j \tilde{\rho} - \tilde{\rho} \nabla_{ij} \tilde{\rho}), \tag{B.5}$$

where $\nabla_{ij} = \nabla_i \nabla_j$.

The principal curvatures are the eigenvalues of the second fundamental form (with respect to an orthonormal basis). In our setting, they correspond to the eigenvalues of the matrix

$$[\tilde{a}_{ij}] = [\tilde{g}^{ij}]^{1/2} [\tilde{h}_{ij}] [\tilde{g}^{ij}]^{1/2}, \tag{B.6}$$

where

$$[\tilde{g}^{ij}]^{1/2} = \tilde{\rho}^{-1} \left[\delta_{ij} - \frac{\nabla_i \tilde{\rho} \nabla_j \tilde{\rho}}{\sqrt{\tilde{\rho}^2 + |\nabla \tilde{\rho}|^2} (\tilde{\rho} + \sqrt{\tilde{\rho}^2 + |\nabla \tilde{\rho}|^2})} \right] \tag{B.7}$$

Note that from Section 2.4, it follows that

$$g_{ij} \to (\rho^*)^2 \delta_{ij} \tag{B.8}$$

$$g^{ij} \to (\rho^*)^{-2} \delta_{ij} \tag{B.9}$$

$$h_{ij} \to \rho^* \delta_{ij} \tag{B.10}$$

$$\tilde{a}_{ij} \to (\rho^*)^{-1}\delta_{ij} \tag{B.11}$$

$$F(\tilde{a}_{ij}) \to (\rho^*)^{-1} \tag{B.12}$$

Let us prove that

$$\left\|\frac{\partial\tilde{\rho}}{\partial t}\right\|_{C^l(S^n)} \leq Ce^{-\delta t}. \tag{B.13}$$

Recall that by (B.2),

$$\begin{aligned}\frac{\partial\tilde{\rho}}{\partial t} &= \frac{(\tilde{\rho}^2+|\nabla\tilde{\rho}|^2)^{\frac{1}{2}}}{\tilde{\rho}F(\tilde{a}_{ij})} - \tilde{\rho} &\text{(B.14)}\\ &= \frac{\tilde{\rho}^2+|\nabla\tilde{\rho}|^2-\tilde{\rho}^4F(\tilde{a}_{ij})^2}{\tilde{\rho}F(\tilde{a}_{ij})((\tilde{\rho}^2+|\nabla\tilde{\rho}|^2)^{\frac{1}{2}}+\tilde{\rho}^2F(\tilde{a}_{ij}))} &\text{(B.15)}\\ &= \frac{|\nabla\tilde{\rho}|^2-\tilde{\rho}^2(1+\tilde{\rho}F(\tilde{a}_{ij}))(1-\tilde{\rho}F(\tilde{a}_{ij}))}{\tilde{\rho}F(\tilde{a}_{ij})((\tilde{\rho}^2+|\nabla\tilde{\rho}|^2)^{\frac{1}{2}}+\tilde{\rho}^2F(\tilde{a}_{ij}))} &\text{(B.16)}\end{aligned}$$

hence, we obtain

$$\left\|\frac{\partial\tilde{\rho}}{\partial t}\right\|_{C^0(S^n)} \leq C\||\nabla\tilde{\rho}|^2\|_{C^0S^n)} + C\|1-\tilde{\rho}F(\tilde{a}_{ij})\|_{C^0(S^n)}. \tag{B.17}$$

and in consequence,

$$\left\|\frac{\partial\tilde{\rho}}{\partial t}\right\|_{C^0(S^n)} \leq Ce^{-\delta t}, \tag{B.18}$$

which follows from (2.78) and (B.12). Note that if we take m-iterated spatial derivatives in (B.2), ultimately their sup-norm will be controlled by the sup-norm of $\nabla^{m+2}\tilde{\rho}$, therefore (using the corresponding gradient estimates replacing l by $l+2$), we obtain

$$\left\|\frac{\partial\tilde{\rho}}{\partial t}\right\|_{C^l(S^n)} \leq Ce^{-\delta t}. \tag{B.19}$$

For $k = 2$, we take a time derivative in (B.2) to obtain

$$\frac{\partial^2 \tilde{\rho}}{\partial^2 t} = \frac{1}{2} \frac{2\tilde{\rho}\partial_t \tilde{\rho} + \partial_t(|\nabla \tilde{\rho}|^2)}{(\tilde{\rho}^2 + |\nabla \tilde{\rho}|^2)^{1/2} \tilde{\rho} F(\tilde{a}_{ij})} + (\tilde{\rho}^2 + |\nabla \tilde{\rho}|^2)^{1/2} \left(-\frac{\partial_t \tilde{\rho} F(\tilde{a}_{ij}) + \tilde{\rho} \partial_t F(\tilde{a}_{ij})}{\tilde{\rho}^2 F(\tilde{a}_{ij})^2} \right) - \partial_t \tilde{\rho}. \tag{B.20}$$

In a similar way, one can directly check that the norm of spatial derivatives will be again control by spatial derivatives of $\partial_t \tilde{\rho}$, and by (B.19), we obtain

$$\left\| \frac{\partial^2 \tilde{\rho}}{\partial t^2} \right\| \leq C e^{-\delta t}. \tag{B.21}$$

It is now evident that, by the structure of (B.2), every time we take a time derivative of it, the norms of the spatial derivatives will be controlled by the norms of the spatial derivatives of the previous time derivatives. This establishes the induction process and hence (B.1).

Appendix C

Geometric and analytical tools

C.1 Second variation formula

Let Σ be a hypersurface of a Riemannian manifold (M, g) with normal vector ν and let $F : \Sigma \times (-\delta, \delta) \longrightarrow M$ be a normal variation of Σ defined by

$$\frac{\partial F}{\partial t}(x,t) = \eta(x,t)\nu(x,t). \tag{C.1}$$

For simplicity, let ∇ denote the connection in M and ∇_Σ the induced connection on Σ. Also, let $\{\partial_1, ..., \partial_n, \partial_t\}$ be a coordinate basis with $\{\partial_1, ..., \partial_n\}$ being a coordinate basis on the tangent space to Σ around an arbitrary point $p \in \Sigma$. The evolution of the mean curvature is given by

$$\frac{\partial H}{\partial t} = \partial_t \sigma^{ij} K_{ij} + \sigma^{ij} \partial_t K_{ij}, \tag{C.2}$$

where σ is the induced metric on Σ.

Since $\sigma_{ij} = g(\partial_i, \partial_j)$ and $[X, Y] = \nabla_X Y - \nabla_Y X$,

$$\partial_t \sigma_{ij} = g(\nabla_{\partial_t}\partial_i, \partial_j) + g(\partial_i, \nabla_{\partial_t}\partial_j) \tag{C.3}$$

$$= g(\nabla_{\partial_i}\partial_t, \partial_j) + g(\partial_i, \nabla_{\partial_j}\partial_t) \tag{C.4}$$

$$= g(\nabla_{\partial_i}(\eta\nu), \partial_j) + g(\partial_i, \nabla_{\partial_j}(\eta\nu)) \tag{C.5}$$

$$= \eta g(\nabla_{\partial_i}\nu, \partial_j) + \eta g(\partial_i, \nabla_{\partial_j}\nu) \tag{C.6}$$

$$= 2\eta K_{ij}, \tag{C.7}$$

and we have the following expression for the first term in (C.2):

$$\partial_t \sigma^{ij} K_{ij} = -\sigma^{il}\sigma^{jk}\partial_t \sigma_{lk} K_{ij} = -2\eta |K|^2_\sigma. \tag{C.8}$$

Now we compute $\partial_t K_{ij}$. From $g(\nu, \nu) = 1$, we know that $\nabla_{\partial_t}\nu$ is tangential to Σ. Also, recall that $\nabla_\Sigma \eta = \nabla\eta - g(\nabla\eta, \nu)\nu$, which in coordinates gives

$$g(\nabla_\Sigma \eta, e_i) = g(\nabla\eta, e_i) = \partial_i \eta. \tag{C.9}$$

Notice that since $\nabla_{\partial_i}\nu$ is tangential, we only need to compute $g(\nabla_{\partial_t}\nu, \partial_i)$ for all i to determine it.

$$g(\nabla_{\partial_t}\nu, \partial_i) = -g(\nu, \nabla_{\partial_i}\partial_t) \tag{C.10}$$

$$= -g(\nu, \partial_i(\eta)\nu) - \eta g(\nu, \nabla_{\partial_i}\nu) \tag{C.11}$$

$$= -\partial_i(\eta), \tag{C.12}$$

hence, we conclude that

$$\nabla_{\partial_t}\nu = -\nabla_\Sigma \eta. \tag{C.13}$$

Therefore, it follows that the second term in (C.2) can be computed as follows:

$$\sigma^{ij}\partial_t K_{ij} = \sigma^{ij} g(\nabla_{\partial_t}\nabla_{\partial_i}\nu, \partial_j) + \sigma^{ij} g(\nabla_{\partial_i}\nu, \nabla_{\partial_t}\partial_j) \tag{C.14}$$

$$= \sigma^{ij} g(\nabla_{\partial_i}\nabla_{\partial_t}\nu, \partial_j) + \sigma^{ij} g(R(\partial_t, \partial_i)\nu, \partial_j) + \sigma^{ij} g(\nabla_{\partial_i}\nu, \nabla_{\partial_t}\partial_j) \tag{C.15}$$

$$= -\sigma^{ij} g(\nabla_{\partial_i}(\nabla_\Sigma \eta, \partial_j) - \sigma^{ij} g(R(\partial_i, \partial_t)\nu, \partial_j) + \eta\sigma^{ij} g(\nabla_{\partial_i}\nu, \nabla_{\partial_j}\nu) \tag{C.16}$$

$$= -\Delta_\Sigma \eta - \eta \mathrm{Ric}(\nu, \nu) + \eta |K|^2_\sigma. \tag{C.17}$$

Here, we used the identity $\sigma^{ij} g(\nabla_{\partial_i}\nu, \nabla_{\partial_j}\nu) = |K|^2_\sigma$, which follows from a straightforward computation in coordinates. Combining (C.8) and (C.17), we obtain the second variation formula

$$\partial_t H = -\Delta_\Sigma \eta - \eta(\mathrm{Ric}(\nu, \nu) + |K|^2_\sigma). \tag{C.18}$$

C.2 A special mollification

Lemma C.2.1. *Consider $a_1 < b_1 < a_2 < b_2$ and let f be a C^1 function on $[a_1, b_2]$ and C^2 except maybe at b_1 and a_2. Given d such that $a_1 + d < b_1$ and $b_2 - d > a_2$, there exists a family of functions f_ν on $[a_1, b_2]$ such that:*

(i) $f_\nu = f$ on $[a_1, a_1 + d] \cup [b_2 - d, b_2]$,

(ii) f_ν coincides with the standard mollification of f on an interval properly containing $[b_1, a_2]$,

(iii) $f_\nu \to f$ in $C^0([a_1, b_2])$ as $\nu \to 0$, and

(iv) $f'_\nu \to f'$ *in* $C^0([a_1, b_2])$ *as* $\nu \to 0$.

Proof. Let $\gamma > 0$ and $\delta > 0$ be such that $b_1 - \gamma$ is away from $a_1 + d$ and b_1 by at least 2δ. Similarly (by taking γ and δ smaller if necessary), $a_2 + \gamma$ is away from a_2 and $b_2 - d$ by at least 2δ. Consider the auxiliary function

$$\eta(x) = \begin{cases} 0, & [a_1, b_1 - \gamma] \cup [a_2 + \gamma, b_2] \\ 1, & (b_1 - \gamma, a_2 + \gamma) \end{cases} \tag{C.19}$$

Let $\eta_\delta(x) = \eta * \phi_\delta(x)$, where ϕ is the standard mollifier. To get the desired family of functions, we define

$$f_\nu(x) = \int_{\mathbb{R}} f(x - \nu\eta_\delta(x)y)\phi(y)\,dy = \int_{\mathbb{R}} f(y)\frac{1}{\nu\eta_\delta(x)}\phi\left(\frac{x-y}{\nu\eta_\delta(x)}\right)\,dy. \tag{C.20}$$

Here, we extended f outside $[a_1, b_2]$ by setting it equal to 0. By the Dominated Convergence Theorem, $f_\nu(x) \to f(x)$ pointwise on $[a_1, b_2]$ as $\nu \to 0$. Clearly,

$$f_\nu(x) = \begin{cases} f(x), & [a_1, b_1 - \gamma - \delta] \cup [a_2 + \gamma + \delta, b_2] \text{ (RI)} \\ f * \phi_\nu(x) & [b_1 - \gamma + \delta, a_2 + \gamma - \delta] \text{ (RII)} \\ f * \phi_{\nu\eta_\delta(x)}(x) & (b_1 - \gamma - \delta, b_1 - \gamma + \delta) \cup (a_2 + \gamma - \delta, a_2 + \gamma + \delta) \text{ (RIII)} \end{cases} \tag{C.21}$$

Moreover, by properties of mollifications, $f * \phi_\nu \to f$ uniformly on $[a_1, b_2]$. To check that $f_\nu \to f$ uniformly on $[a_1, b_2]$, let $\varepsilon > 0$ and pick ν sufficiently small so that $|f * \phi_{\tilde{\nu}}(x) - f(x)| < \varepsilon$ on $[a_1, b_2]$ for any $\tilde{\nu} \le \nu$. In particular, for x in RII, we have $|f_\nu(x) - f(x)| < \varepsilon$. Notice that $\nu\eta_d(x) \le \nu$, so it follows that on RIII, $|f * \phi_{\nu\eta_d(x)}(x) - f(x)| < \varepsilon$. Hence for all x in $[a_1, b_2]$ we have $|f_\nu(x) - f(x)| < \varepsilon$, so

$f_\nu \to f$ uniformly on $[a_1, b_2]$.

Taking a derivative with respect to x, we have

$$f'_\nu(x) = \int_{\mathbb{R}} f'(x - \nu\eta_\delta(x)y)(1 - \nu\eta'_\delta(x)y)\phi(y)\,dy. \tag{C.22}$$

Since $\eta_\delta(x)$ is fixed and ϕ is compactly supported, by the Dominated Convergence Theorem, we have $f'_\nu(x) \to f'(x)$ pointwise on $[a_1, b_2]$. Pick ν so that $|f' * \phi_{\tilde{\nu}}(x) - f'(x)| < \frac{\varepsilon}{2}$ on $[a_1, b_2]$ for all $\tilde{\nu} \le \nu$. Then, on RII, $|f'_\nu(x) - f'(x)| < \varepsilon$.

On RIII, note that

$$f'_\nu(x) = f' * \phi_{\nu\eta_\delta(x)}(x) + \int_{\mathbb{R}} f'(x - \nu\eta_\delta(x)y)(-\nu\eta'_d(x)y)\phi(y)\,dy,$$

hence

$$|f'_\nu(x) - f'(x)| \le |f' * \phi_{\nu\eta_d(x)}(x) - f'(x)| + b_2 \sup_{[a_1,b_2]} f'(x) \sup_{\mathbb{R}} \eta'_\delta(x) \cdot \nu,$$

by taking ν smaller if necessary, we have

$$|f'_\nu(x) - f(x)| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then $f'_\nu \to f'$ uniformly on $[a_1, b_2]$.

On the union of RI and RIII, $f''(x)$ exists, so on this set

$$f''_\nu(x) = \int_{\mathbb{R}} f''(x - \nu\eta_\delta(x)y)(1 - \nu\eta'_\delta(x)y)^2\phi(y)\,dy + \int_{\mathbb{R}} f'(x - \nu\eta_\delta(x)y)(-\nu\eta''_\delta(x)y)\,dy$$

Moreover, note that when x approaches the common boundary of RII and RIII, $f''_\nu(x)$

approaches $f'' * \phi_\nu(x)$. Hence

$$f''_\nu(x) = \begin{cases} f''(x), & \text{(RI)} \\ \dfrac{d}{dx}(f' * \phi_\nu)(x), & \text{(RII)} \\ f'' * \phi_{\nu\eta_\delta(x)}(x) + O(\nu), & \text{(RIII)} \end{cases}, \tag{C.23}$$

□

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