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Theory of weak scattering of stochastic electromagnetic fields from deterministic and random media

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The theory of scattering of scalar stochastic fields from deterministic and random media is treated in Refs. [1–3] in great detail. Some other fields from deterministic and random continuous static scatterings revealed that on scattering, the statistical properties of polarization [1].

The analysis allows for determining the changes in spectrum, coherence, and polarization of electromagnetic fields produced on their propagation from the source to the scattering volume, interaction with the scatterer, and propagation from the scatterer to the far field. An example of scattering of a field produced by a δ-correlated partially polarized source and scattered from a δ-correlated medium is provided.

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I. INTRODUCTION

In classical optics domain scattering of electromagnetic fields from deterministic and random continuous static scattering media is treated in Refs. [1–3] in great detail. Some other aspects of this research area are addressed in Refs. [4–9]. Theoretical and experimental studies relating to scattering from deterministic and random collections of particles were carried out in Refs. [10–13]. Some applications relating to determination of the structure of the medium from scattering experiments can be found in Refs. [14–17]. These investigations revealed that on scattering, the statistical properties of light are influenced by both the correlation properties of the source and the structure and correlation properties of the medium. All these studies were confined to cases in which the incident field is either deterministic or scalar. Hence, it was not possible to account for changes in those characteristics before and after the scattering event (i.e., we incorporate their nature, that scatter from deterministic or random media. We use the assumption that the scatterer is weak, employing the first-order Born approximation. The analysis allows for determining the propagation of the electromagnetic scattered field such as its spectral density, spectral degree of coherence, and various polarimetric features, such as degree of polarization, ellipsometric properties, degree of cross polarization, etc. Since it is often the case in scattering experiments, we predict evolution of the fields before and after the scattering event (i.e., we incorporate their propagation from the source plane to the scattering medium and, after scattering, from the medium to the far field). We carry out the analysis in terms of the cross-spectral density matrices, from which one can determine all other second-order statistical properties of the field of interest. An example in which spectral density and the states of coherence and polarization of the scattered field, produced on scattering of an uncorrelated partially polarized field from a δ-correlated slab, is provided to illustrate the analytical results.

The paper is organized as follows. Section II is devoted to far-field scattering of a deterministic electromagnetic field. In particular, in Sec. II A, the propagation of the electromagnetic field from the source plane to the scattering volume is reviewed; in Sec. II B, the propagation of the scattered field from the medium to the far field is given; in Sec. II C, we combine the formulas of Secs. II A and II B. In Sec. III, we generalize the results of Sec. II to stochastic electromagnetic fields. Section IV provides an example illustrating the usefulness of the theory.

II. PROPAGATION OF THE ELECTRIC-FIELD VECTOR THROUGH THE SCATTERING MEDIUM

A. Propagation of the field from the source plane to the scattering medium

We begin by considering a monochromatic electric field oscillating at angular frequency \( \omega \) and propagating from the source plane \( z = 0 \) into the half space \( z > 0 \) (see Fig. 1). The transverse component of the electric-field vector at a point specified by the position vector \( \mathbf{r} \) in the source plane has the form

\[
E_\perp^{(0)}(\mathbf{r'}, \omega) = \begin{bmatrix} E_{x}^{(0)}(\mathbf{r'}, \omega) \\ E_{y}^{(0)}(\mathbf{r'}, \omega) \end{bmatrix}.
\]

Following Ref. [18], the three components of the electric field generated by the electric field in the source plane and propagated to a point specified by the position vector \( \mathbf{r}_1 \) in the half space \( z > 0 \) can be expressed as

\[
\begin{align*}
E_x(r_1, \omega) &= -\frac{1}{2\pi} \int E_{x}^{(0)}(\rho', \omega) \partial_\rho G(\rho', \mathbf{r}_1, \omega) d^2\rho', \\
E_y(r_1, \omega) &= -\frac{1}{2\pi} \int E_{y}^{(0)}(\rho', \omega) \partial_\rho G(\rho', \mathbf{r}_1, \omega) d^2\rho', \\
E_z(r_1, \omega) &= \frac{1}{2\pi} \int \left[ E_{x}^{(0)}(\rho', \omega) \partial_\rho G(\rho', \mathbf{r}_1, \omega) + E_{y}^{(0)}(\rho', \omega) \partial_\rho G(\rho', \mathbf{r}_1, \omega) \right] d^2\rho',
\end{align*}
\]

where \( \partial \) denotes a partial derivative, \( x_1, y_1, z_1 \) are the Cartesian components of the position vector \( \mathbf{r}_1 \), and \( \mathbf{G} \) is the outgoing free-space Green’s function of the form

\[
G(\mathbf{r'}, \mathbf{r}, \omega) = \frac{\exp (i k |\mathbf{r} - \mathbf{r'}|)}{|\mathbf{r} - \mathbf{r'}|},
\]

where \( k = \omega/c \) is the wave number and \( c \) is the speed of light in vacuum. From Eqs. (2), a linear transformation of the two-dimensional vector space containing the vector \( E_\perp^{(0)}(\mathbf{r'}, \omega) \) onto the three-dimensional vector space containing the vector...
E(\(\mathbf{r}_1, \omega\)) can be conveniently written as
\[
E(\mathbf{r}_1, \omega) = \int E_\perp^{(0)}(\rho', \omega) \circ K(\rho', \mathbf{r}_1, \omega) \, d^2 \rho',
\] (4)
where the small circle \(\circ\) denotes matrix multiplication, and
\[
K(\rho', \mathbf{r}_1, \omega) = \frac{1}{2\pi} \begin{bmatrix} -\partial_2 G(\rho', \mathbf{r}_1, \omega) & 0 \\ -\partial_1 G(\rho', \mathbf{r}_1, \omega) & \partial_1 G(\rho', \mathbf{r}_1, \omega) \\ \partial_1 - x' & \partial_2 - y' \end{bmatrix}.
\] (5a)
(5b)

Here \(x', y'\) are the Cartesian components of the transverse vector \(\rho'\), and
\[
L(R_1, \omega) = \frac{(ikR_1 - 1) \exp(ikR_1)}{R_1^3}.
\] (6)
with \(R_1 = |\mathbf{r}_1 - \rho'|\). More explicitly, Eq. (4) can be written as
\[
E(\mathbf{r}_1, \omega) = \frac{1}{2\pi} \int L(R_1, \omega) \begin{bmatrix} E_\perp^{(0)}(\rho', \omega) \\ E_\parallel^{(0)}(\rho', \omega) \end{bmatrix} \circ \begin{bmatrix} -z_1 & 0 & x_1 - x' \\ 0 & -z_1 & y_1 - y' \end{bmatrix} d^2 \rho'.
\] (7)

B. Propagation of the electric field from the scattering media

When a monochromatic electromagnetic field is incident on a linear isotropic nonmagnetic medium occupying a finite domain \(V\) (see Fig. 2), the scattered field at a point specified by position vector \(\mathbf{r}_s\) \((s^2 = 1)\) may be expressed in the form [1],
\[
E^{(i)}(\mathbf{r}_s, \omega) = \nabla \times \nabla \times \Pi_s(\mathbf{r}_s, \omega),
\] (8)
where \(\Pi_s\) is the electric Hertz potential defined by the formula,
\[
\Pi_s(\mathbf{r}_s, \omega) = \int_V \mathbf{P}(\mathbf{r}_1, \omega) \frac{e^{ik|\mathbf{r}_s - \mathbf{r}_1|}}{|\mathbf{r}_s - \mathbf{r}_1|} \, d^3 r_1.
\] (9)

Here \(\mathbf{P}(\mathbf{r}_1, \omega)\) is the polarization of the medium, which may be expressed, within the accuracy of the first-order Born approximation, as
\[
\mathbf{P}(\mathbf{r}_1, \omega) = \eta(\mathbf{r}_1) E^{(i)}(\mathbf{r}_1, \omega) = \frac{1}{k^2} F(\mathbf{r}_1) E^{(i)}(\mathbf{r}_1, \omega),
\] (10)
where \(\eta\) is the dielectric susceptibility and the quantity
\[
F(\mathbf{r}_1, \omega) = \begin{cases} \frac{1}{4\pi} k^2 [n^2(\mathbf{r}_1, \omega) - 1], & \mathbf{r}_1 \in V, \\ 0, & \mathbf{r}_1 \notin V, \end{cases}
\] (11)
is called the scattering potential of the medium. In general, both depend not only on \(\mathbf{r}_1\) but also on \(\omega\). On substituting Eqs. (9) and (10) into Eq. (8), we obtain the formula for the scattered field outside of the scattering medium,
\[
E^{(i)}(\mathbf{r}_s, \omega) = \frac{1}{k^2} \nabla \times \nabla \times \int_V F(\mathbf{r}_1) E^{(i)}(\mathbf{r}_1, \omega) \frac{e^{ik|\mathbf{r}_s - \mathbf{r}_1|}}{|\mathbf{r}_s - \mathbf{r}_1|} \, d^3 r_1.
\] (12)

Equation (12) is the general result for the scattered field within the accuracy of the first-order Born approximation, which is too complex to be employed in analytical calculations. It, however, simplifies significantly in the far zone of the scatterer. In this case, Eq. (8) is reduced to the following formula:
\[
E^{(i)}(\mathbf{r}_s, \omega) = -k^2 e^{ikr} \mathbf{P}(|\mathbf{r}_s - \mathbf{r}_1|),
\] (13)
where \(\mathbf{P}\) is the three-dimensional Fourier transform of \(\mathbf{P}(\mathbf{r}_1)\), that is,
\[
\mathbf{P}(\mathbf{r}_s, \omega) = \frac{1}{k^2} \int_V \mathbf{P}(\mathbf{r}_1) e^{-ik|\mathbf{r}_s - \mathbf{r}_1|} \, d^3 r_1
= \frac{1}{k^2} \int_V F(\mathbf{r}_1) E^{(i)}(\mathbf{r}_1) e^{-ik|\mathbf{r}_s - \mathbf{r}_1|} \, d^3 r_1.
\] (14)
On substituting Eq. (14) into Eq. (13), after straightforward
vector multiplication, we rewrite Eq. (13) as
\[
\mathbf{E}(\mathbf{s}, \omega) = \frac{e^{ikr}}{r} \int V F(r_1) \left[ \mathbf{E}(\mathbf{r}_1) - [\mathbf{s} \cdot \mathbf{E}(\mathbf{r}_1)] \mathbf{s} \right] \times e^{-iks r_1} d^3 r_1,
\]
(15)
which is the asymptotic expression of Eq. (12) in the far zone. More explicitly, Eq. (15) can be rewritten in matrix form
\[
\mathbf{E}(\mathbf{s}, \omega) = \frac{e^{ikr}}{r} \int V F(r_1) \begin{bmatrix} E_x(\mathbf{r}_1, \omega) \\ E_y(\mathbf{r}_1, \omega) \\ E_z(\mathbf{r}_1, \omega) \end{bmatrix}^T \times e^{-iks r_1} d^3 r_1,
\]
(16)
where \( \mathbf{s} = (s_x, s_y, s_z) \).

It is evident from Eq. (13) that \( \mathbf{s} \cdot \mathbf{E}(\mathbf{r}, \omega) = 0 \) (i.e., that the scattered field in the far zone is orthogonal to \( \mathbf{s} \), or transverse). Therefore, it will be convenient to represent such a transverse field in terms of the spherical polar coordinate system and the spherical coordinate system can be expressed as

\[
\begin{bmatrix} e_r \\ e_\theta \\ e_\psi \end{bmatrix}^T = \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}^T \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix},
\]
(17a)
or in the reverse form

\[
\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}^T = \begin{bmatrix} e_r \\ e_\theta \\ e_\psi \end{bmatrix}^T \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \phi \\ -\sin \phi & \cos \phi & 0 \end{bmatrix},
\]
(17b)
where \( \epsilon_x, \epsilon_y, \epsilon_z, \epsilon_r, \epsilon_\theta, \epsilon_\phi \) are the unit vectors. Thus, we may rewrite Eq. (16) as

\[
\begin{bmatrix} E_x(\theta, \phi) \\ E_y(\theta, \phi) \\ E_z(\theta, \phi) \end{bmatrix}^T = \frac{e^{ikr}}{r} \int V F(r_1) \begin{bmatrix} E_x(\mathbf{r}_1, \omega) \\ E_y(\mathbf{r}_1, \omega) \\ E_z(\mathbf{r}_1, \omega) \end{bmatrix}^T \times e^{-iks r_1} d^3 r_1.
\]
(18)

On passing to Sec. II C, we note that in order for a wave of wave number \( k \), located at distance \( r \) from the scatterer to the observation point to be in the far field of a scatterer with size \( a \), the following restriction should be met (see Ref. [2], Sec. 3.2):

\[
kr \gg \max \left[ 1, \frac{1}{2}(ka)^2 \right].
\]
(19)

C. Propagation of the electric field from source plane
to the far zone

In Sec. II A, we set the origin in the source plane when we deal with the electric field in the half space \( z > 0 \) generated by the source plane. However, in Sec. II B, we set the origin in the scattering medium in order to derive the explicit form of the scattered field. In order to combine the two representations, we suppose the origin is located within the scattering medium coinciding with the origin considered (as in Sec. II B) and specified by the position vector \( \mathbf{r}_0 = (x_0, y_0, z_0) \) from the perspective of the system considered in Sec. II A. Then, Eq. (7) becomes (after replacing \( \mathbf{r}_1 \) by \( \mathbf{r}_1 + \mathbf{r}_0 \))

\[
\mathbf{E}(\mathbf{r}_1, \omega) = \frac{1}{2\pi} \int L(R_1, \omega) \left[ \begin{bmatrix} E_x(\mathbf{r}', \omega) \\ E_y(\mathbf{r}', \omega) \end{bmatrix} \begin{bmatrix} -\epsilon_1 + \epsilon_0 \\ -\epsilon_1 \end{bmatrix} \right] d^2 \rho',
\]
(20)
where now \( R_1 = |\mathbf{r}_1 + \mathbf{r}_0 - \mathbf{r}'| \). On substituting from Eq. (20) into Eq. (18), we obtain for the scattered field in the spherical polar coordinate system (keeping only \( \theta, \phi \) components) the formula

\[
\begin{bmatrix} E_x(\mathbf{s}, \omega) \\ E_y(\mathbf{s}, \omega) \end{bmatrix}^T = \frac{e^{ikr}}{2\pi r} \int V F(r_1) L(R_1, \omega) \times \left[ \begin{bmatrix} -E_x(\mathbf{r}', \omega) \\ E_y(\mathbf{r}', \omega) \end{bmatrix} \begin{bmatrix} -\epsilon_1 + \epsilon_0 \\ -\epsilon_1 \end{bmatrix} \right] \times e^{-iks r_1} d^2 r_1 d^2 \rho',
\]
(21)
where \( \mathbf{s} \cdot \mathbf{r}_1 = \sin \theta \cos \phi x_1 + \sin \theta \sin \phi y_1 + \cos \theta z_1 \) and
III. PROPAGATION OF THE $2 \times 2$ CROSS-SPECTRAL DENSITY MATRIX OF THE ELECTROMAGNETIC FIELD

The transformation law for the cross-spectral density matrix of a stochastic wide-sense statistically stationary electromagnetic field, which interacts with the scattering system of interest, can be determined from Eq. (21). Let the fluctuations of the electric field at the source plane be characterized by the $2 \times 2$ cross-spectral density matrix [3]

$$W^0(\rho_1',\rho_2',\omega) = \left[\{E^{(0)}_\alpha(\rho_1,\omega)E^{(0)}_{\beta}(\rho_1',\omega)\}\right]$$

$$\left(\alpha = x,y; \beta = x,y\right)$$

Then suppose that the correlation properties of the electric field at a pair of points in spherical polar coordinate system, specified by position vectors $r_1$ and $r_2$ $[s = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)]$ are characterized by the $2 \times 2$ cross-spectral density matrix

$$W(r_1,r_2,\omega) = \left[\{E^{(c)}_\alpha(r_1,\omega)E^{(c)}_{\beta}(r_2,\omega)\}\right]$$

$$(\alpha = \theta,\phi; \beta = \theta,\phi),$$

or, equivalently,

$$W(r_1,r_2,\omega) = (E^{(c)}(r_1,\omega) \cdot E^{(c)}(r_2,\omega)),$$

where $\dagger$ denotes the Hermitian adjoint and $E^{(c)}(r_1,\omega) = [E^{(0)}_\alpha (r_1) E^{(0)}_{\beta}(r_1)]$. Using Eq. (21), the matrix identity $(A \cdot B)^\dagger = B^\dagger \cdot A^\dagger$, and after interchanging the order of averaging and integration, we find that

$$W(r_1,r_2,\omega) = \frac{1}{4\pi^2 r_1^2} \int_V \int_V \int_V \left(F^*(r_1)F(r_2)\right) L^\ast(R_1,\omega)$$

$$\times L(R_2,\omega)e^{ik(s_1-r_1-s_2)}\{M^\dagger(\theta_1,\phi_1,r_1,\rho_1)$$

$$\circ E^{(0)}_{\alpha}(\rho_1,\omega) \cdot E^{(0)}_{\beta}(\rho_2,\omega) \circ M(\theta_2,\phi_2,r_2,\rho_2')$$

$$\times d^2\rho'_1 d^2\rho'_2 d^3r_1 d^3r_2 \right),$$

or, more explicitly,

$$\begin{bmatrix}
W_{x\theta} & W_{y\phi} \\
W_{x\phi} & W_{y\theta}
\end{bmatrix} = \frac{1}{4\pi^2 r_1^2} \int_V \int_V \int_V \left(F^*(r_1)F(r_2)\right) L^\ast(R_1,\omega)$$

$$\times L(R_2,\omega)e^{ik(s_1-r_1-s_2)}\{M^\dagger(\theta_1,\phi_1,r_1,\rho_1)$$

$$\circ \begin{bmatrix}
W^{(0)}_{x\alpha} & W^{(0)}_{y\alpha} \\
W^{(0)}_{x\beta} & W^{(0)}_{y\beta}
\end{bmatrix} \circ M(\theta_2,\phi_2,r_2,\rho_2')$$

$$\times d^2\rho'_1 d^2\rho'_2 d^3r_1 d^3r_2,$$

where $M(\theta_1,\phi_1,r_1,\rho_1)$, with $M(\theta_2,\phi_2,r_2,\rho_2')$ given by Eq. (22) and $R_1 = |r_1 + r_0 - \rho_1|$, $R_2 = |r_2 + r_0 - \rho_2'|$.

We note that in the case when the scattering medium is located close to the $z$ axis [i.e., about position $(0,0,z_0)$], the use of the Taylor expansion $\sqrt{1 + \lambda} \sim 1 + \frac{1}{2}\lambda (\lambda \rightarrow 0)$ helps to reduce Eq. (6) to the expression

$$L(R_1,\omega) = \frac{ik}{(z_1 + z_0)} e^{ik(z_1+z_0)}$$

$$\times \exp \left(\frac{ik}{2(z_1 + z_0)}[(x_1 - x')^2 + (y_1 - y')^2]\right),$$

which considerably simplifies the subsequent analysis.

IV. AN EXAMPLE: PROPAGATION OF A PARTIALLY POLARIZED UNCORRELATED ELECTROMAGNETIC SOURCE THROUGH A $\delta$-CORRELATED SCATTERER

We will now restrict our attention to the scattering of a field generated by an uncorrelated source, which in general is partially polarized, from a $\delta$-correlated slab. Suppose that a field is generated in a two-dimensional domain $D$ confined to the plane $z = 0$. Such a source may be characterized by the $2 \times 2$ cross-spectral density matrix of the form [19,20]

$$W^{(0)}(\rho_1',\rho_2',\omega) = S^{(0)}(\rho_1',\omega)\delta^{(2)}(\rho_2' - \rho_1'),$$

where $\delta^{(2)}(\rho_2' - \rho_1')$ is a two-dimensional Dirac $\delta$ function and

$$S^{(0)}(\rho_1',\omega) = [S^{(0)}_{ij}(\rho_1',\omega)] \quad (i = x,y; \ j = x,y).$$

We also suppose that the fluctuations in the scattering volume are $\delta$ correlated, that is,

$$C_{ij}(r_1,r_2,\omega) = (F^*(r_1)F(r_2)) = A(\omega)\delta^{(3)}(r_2 - r_1),$$

and, for simplicity,

$$\delta^{(3)}(r_2 - r_1)$$

being a three-dimensional Dirac $\delta$ function and $A(\omega)$ a function, which is independent of position. On substituting Eqs. (29)–(31) into Eq. (27), we find that

$$W(r_1,r_2,\omega) = \frac{1}{4\pi^2 r_1^2} \int_V \int_V A(\omega) L(R_1,\omega) e^{ik(s_1-s_2)}$$

$$\times L(R_2,\omega) e^{ik(s_1-r_1-s_2)} M^\dagger(\theta_1,\phi_1,r_1,\rho_1)$$

$$\circ \begin{bmatrix}
W^{(0)}_{x\alpha} & W^{(0)}_{y\alpha} \\
W^{(0)}_{x\beta} & W^{(0)}_{y\beta}
\end{bmatrix} \circ M(\theta_2,\phi_2,r_2,\rho_2')$$

$$\times d^2\rho'_1 d^2\rho'_2 d^3r_1 d^3r_2.$$

We will now illustrate the preceding analysis by restricting ourselves to partially polarized incoherent electromagnetic Gaussian shell-model beams, for which the elements of the correlation matrix are

$$S^{(0)}_{ij}(\rho_1',\omega) = \sqrt{I_1I_2} B_{ij} \exp \left(-\frac{\rho'^2}{2\sigma^2}\right) \quad (i = x,y; \ j = x,y),$$

where $\rho' = |\rho'|$ and, for simplicity, $I_1 = I_2 = 1$ (i.e., the field in the source plane is unpolarized). It also follows from the properties of the cross-spectral density matrix that $B_{xx} = B_{yy} = B$ [3]. On substituting Eqs. (28) and (33) into Eq. (32) after some straightforward calculations and decomposition of the $M$ matrix into two, we find that
For a specific case, we consider a δ-correlated hard-edge slab, with dimensions in Cartesian coordinate system \( x_1 \in (-L_1, L_1), \ y_1 \in (-L_2, L_2), \) and \( z_1 \in (-L_3, L_3), \) which also satisfy that \( L_1, L_2, L_3 \ll z_0. \) We note that the assumption that the scattering medium has hard edges does not contradict the first-order Born approximation as long as the refractive index within the slab differs only slightly from that of surrounding free space. We will calculate matrix \( S(r, \omega) = W(r, \omega) \) on which the statistical properties of interest, such as the spectral density \( S \) and the spectral degree of polarization \( P, \) depend. Later, we will turn to the spectral density matrix \( W(r_1, r_2, \omega) \) at two different points on which the degree of coherence \( \mu \) depends in the paraxial propagation regime (i.e., when inclination angle \( \theta \) is small). Therefore, by assuming \( s_1 = s_2 = s, \) we rewrite Eq. (34) as

\[
W(r, s, \omega) = \frac{A(\omega)\sigma^2 k^2 V_0}{2\pi r^2 z_0^2} \begin{bmatrix}
-\cos \theta \cos \varphi & -\cos \theta \sin \varphi & \sin \theta \\
\sin \varphi & -\cos \varphi & 0
\end{bmatrix}
\begin{bmatrix}
1 & B & 0 \\
B & 1 & 0 \\
0 & 0 & L_1^2 + L_2^2 + L_3^2 + z_0^2
\end{bmatrix}
\begin{bmatrix}
-\cos \theta \cos \varphi & \sin \varphi \\
-\cos \theta \sin \varphi & -\cos \varphi \\
\sin \theta & 0
\end{bmatrix}.
\]

For paraxial propagation (\( \theta \) is small), the terms containing the horizontal axis are negligible. Therefore, Eq. (22) becomes

\[
M(\theta, \varphi, r_1) = \begin{bmatrix}
\cos \theta \cos \varphi (z_1 + z_0) & -\sin \varphi (z_1 + z_0) \\
\cos \theta \sin \varphi (z_1 + z_0) & \cos \varphi (z_1 + z_0)
\end{bmatrix}.
\]

On substituting Eq. (36) into Eq. (35), we have, in a more explicit form, the 2 × 2 cross-spectral density matrix in a spherical polar coordinate system,

\[
W(r_{s1}, r_{s2}, \omega) = \frac{A(\omega)\sigma^2 k^2 V_0}{2\pi r^2 z_0^2} \int \frac{e^{i k (s_1 - s_2) \cdot r_1}}{(z_1 + z_0)^2} d^3 r_1 \times \begin{bmatrix}
\cos \theta_1 \cos \varphi_1 & \cos \theta_1 \sin \varphi_1 \\
\sin \varphi_1 & \cos \varphi_1
\end{bmatrix}
\begin{bmatrix}
1 & B & 0 \\
B & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\cos \theta_2 \cos \varphi_2 & -\sin \varphi_2 \\
\cos \theta_2 \sin \varphi_2 & \cos \varphi_2
\end{bmatrix},
\]

where \( s_1 = (\sin \theta_1 \cos \varphi_1, \sin \theta_1 \sin \varphi_1, \cos \theta_1), s_2 = (\sin \theta_2 \cos \varphi_2, \sin \theta_2 \sin \varphi_2, \cos \theta_2) \) are the unit vectors at two different directions. Under the hard-edge slab assumption, the integral in Eq. (37) becomes

\[
\int \frac{e^{i k (s_1 - s_2) \cdot r_1}}{(z_1 + z_0)^2} d^3 r_1 = \frac{V_0}{z_0^2} \frac{2s_x L_1}{\lambda} \frac{2s_y L_2}{\lambda} \times \frac{2s_z L_3}{\lambda},
\]

where \( V_0 = L_1 L_2 L_3 \) is the volume of the slab, \( \lambda \) is the free-space wavelength associated with free-space wave number \( k, \) and \( s_i = (s_i - s_j), (i = x, y, z) \) (i.e., \( s_x = \sin \theta_1 \cos \varphi_1 - \sin \theta_1 \cos \varphi_2, \) etc.). On substituting Eq. (38) into Eq. (37), the 2 × 2 cross-spectral density matrix of the scattered field in the far zone in paraxial propagation, generated by a partially polarized incoherent electromagnetic source through a δ-correlated thin slab, in polar coordinate system, becomes

\[
W(r_{s1}, r_{s2}, \omega) = \frac{A(\omega)\sigma^2 k^2 V_0}{2\pi r^2 z_0^2} \sin \left(\frac{2s_x L_1}{\lambda}\right) \sin \left(\frac{2s_y L_2}{\lambda}\right)
\]

FIG. 3. Normalized spectral density of the scattered field in the far zone vs the inclination angle, with \( \varphi = \frac{\theta}{2}. \)
Formulas [3] in the far zone vs the inclination angle, with the degree of coherence $S_{\lambda}$ spectral density $S_{\omega}$.

From the components of the cross-spectral density matrix the coherence area $\mu$ is expressed, respectively, by the formulas [3]

$$S(rs, \omega) = \text{Tr} W(rs, rs, \omega), \quad (40)$$

$$P(rs, \omega) = \sqrt{1 - \frac{4 \text{Det} W(rs, rs, \omega)}{(\text{Tr} W(rs, rs, \omega))^2}}, \quad (41)$$

$$\mu(rs_1, rs_2, \omega) = \frac{\text{Tr} W(rs_1, rs_2, \omega)}{\sqrt{\text{Tr} W(rs_1, rs_1, \omega) \text{Tr} W(rs_2, rs_2, \omega)}}, \quad (42)$$

where Det and Tr denote determinant and trace of the matrix.

We will now illustrate the results by a set of figures. The following parameters are used for the plots: $\omega = 10^{15}$ s$^{-1}$, $\lambda = 0.2$, $L_1 = L_2 = 5 \times 10^5 \lambda$, $L_3 = 10^5 \lambda$, $z_0 = 0.5 \lambda$, and $\sigma = 1$ mm.

Figure 3 shows the normalized spectral density defined by Eq. (40) for the field in the far zone generated by a partially polarized incoherent electromagnetic Gaussian shell-model (EMGSM) source and scattered by a $\delta$-correlated hard-edge scatterer for several values of the degree of polarization $P_0$ of the field in the source plane.

Figure 4 shows the degree of polarization, defined in Eq. (41), for the scattered field in the far zone. We see from Fig. 4 that the spectral degree of polarization of the scattered field in the far zone (the inclination angle is about zero) has its original value $P_0$. In fact, a similar result can be found from Ref. [20], where propagation of a partially polarized incoherent electromagnetic source in the paraxial approximation is considered without scattering. This may be verified by substituting a Gaussian shell-model beam into Eqs. (9) and (10) of Ref. [20].

In Fig. 5, we show the spectral degree of coherence, defined in Eq. (42), for the scattered field in the far zone with azimuthal separation angle $\Delta \phi$. At first glance, the value of the degree of coherence seems to be independent of the radial distance $r$, but that is not the case, since at larger distances from the scatterer, two points with a fixed azimuth angle difference have larger separation. The coherence area $\Delta A$ is proportional to $r^2$, which is consistent with the van Cittert–Zernike theorem in the far zone. Figure 5 clearly shows the fact that for small inclination angles, the width of the degree of coherence is larger when the distance between two points is smaller.

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