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Embeddings of Curves into Euclidean Spaces

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EMBEDDINGS OF CURVES INTO EUCLIDEAN SPACES

By

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Embeddings of Curves into Euclidean Spaces (May 2012)

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In this treatise we present an introduction to the concepts of curves and embeddings in Euclidean spaces. We discuss what it means for embeddings to be equivalent, and present the most recent findings in the field. The survey is meant to be appropriate for first year graduate mathematics students to grasp, given the appropriate prerequisite knowledge of calculus and abstract algebra. Exercises for student practice are given throughout, and solutions to selected exercises are offered as an appendix. Glossaries of terms and notation, including entries considered part of the prerequisite knowledge and therefore not defined in the text itself, are also available as appendices.

DEDICATION

This work is dedicated in honor of my dear children Amit Batya, Aviv David, Hadar Mazal, and Golan Ben-Tzion, (ages 6, 4½, 3 and 1½ at the time of this publication) and in memory of their late great-grandparents, for whom they are named: Betty and Harry Farber, Mazal and David Zohar, Dorothy Sand and Charles Friedman, and Ruth and Henry Penchansky. It is both for and despite you kids that I persisted in completing this adventure, though none of you existed when it all began!

ACKNOWLEDGEMENTS

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Deep gratitude goes to my unforgettable mentors from Queens College, CUNY. I thank specifically Professors Alan Sultan, William Emerson, Elliot Mendelson, and Alice Artzt. It was you who first kindled my love for mathematics, gave me the confidence and background to continue in my studies, and prepared me to go out and share the world of math with my students.

Beginning from the time I first mentioned a consideration to major in math, my uncle Dr. Mark Farber was there to encourage me to go the distance. Thank you, Dod, for your support, “nudging”, subbing, and general presence as a second father to me from day one. It has been an honor to be your niece and your colleague in the MAS department.

Without the peace of mind that my children were well cared for while my attentions were focused on this effort, I would not have been able to accomplish this. Therefore, a great many thanks to the wonderful staff and faculty of the Beth Torah and Hochberg Prep daycare and day schools.

Intense thanks goes to my husband Sagee, with whom I have the gratifying good fortune to share this crazy beautiful life with. No amount of good deeds or sacrifice can explain what I did to deserve a mate as extraordinary as you, and yet, here you are! I love you.

Finally, thanks to my parents Rabbi Ed and Laurie Farber, who have supported me in everything I do, and who instilled in me and continually model to me the value of learning and the true purpose of our efforts to achieve.

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CHAPTER 1

INTRODUCTION

The aim of this treatise is a quick introduction to some serious results and problems of modern mathematics intended for first year graduate mathematics students. The prerequisites which are necessary for understanding this paper are the standard introductory courses in abstract algebra, real analysis, topology, and complex analysis. In the University of Miami coding system this currently corresponds to the classes MTH 561-562, MTH 533-534, MTH 531-532, and MTH 512.

The problems that are discussed below belong to a branch of algebraic geometry which is called affine algebraic geometry. The latter studies sets of common zeros of polynomial systems in Euclidean spaces - such sets are called affine algebraic varieties. One characteristic of this area of study that makes it so appealing as a choice for a topic of a paper of this type is the simplicity of the formulations of some of its crucial problems. In this aspect affine algebraic geometry is quite similar to another area of mathematics so intertwined with it - number theory.

Indeed, some examples of the zero sets $\{(x,y)\,|\,P(x,y)=0\}$ of polynomials P in two variables x and y are quite familiar to school students. They know such zero sets as algebraic curves: parabolas $y-cx^2=0$, hyperbolas $xy-c=0$, ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, etc. which are nothing but the simplest affine algebraic varieties. However, even for such objects one can formulate very difficult and interesting problems.

For instance, the question whether for $n \geq 3$ there exist points with rational coordinates on the curve $x^n + y^n - 1 = 0$ (i.e. $x = p/r$ and $y = q/r$ where $p,q,r \in \mathbb{Z}$ and $r \neq 0$) is equivalent to the famous Fermat's last problem formulated in 1637 (which states that an equation $p^n + q^n = r^n$ has no integer solution for $n \geq 3$).

Similarly, one can ask concerning a "general" polynomial P of degree at least 5 (or more precisely, a polynomial such that the corresponding curve, while treated as a Reimann surface, is smooth of genus at least 2), whether the curve $\{(x,y) \mid P(x,y) = 0\}$ may contain only a finite number of points with rational coordinates. This is essentially the famous Mordell conjecture formulated in 1922.

Both conjectures were proved respectively by Wiles (1995) and Faltings (1983) but in spite of the simplicity of the formulation, their proofs are based on the deepest methods and achievements of modern mathematics including algebraic geometry. Accordingly our aim is to attract students to such methods via the simplicity of the formulations of the problems.

The fact central to our narrative is the remarkable Abhyankar-Moh-Suzuki theorem (1974-1975) that states that any smooth algebraic curve homeomorphic to a complex line in a complex plane can be viewed as a coordinate axis in a suitable polynomial coordinate system, i.e. it may be given by an equation $x = 0$ [1], [35]. Another crucial result is the Lin-Zaidenberg theorem (1983) that says that in the absence of smoothness such a curve can be given by an equation $x^k - y^l = 0$ for relatively prime k and l in a suitable polynomial coordinate system [23].

In order to explain these formulations more precisely we introduce the notion of a polynomial coordinate substitution (or, in other words, a polynomial automorphism) in Euclidean spaces, and the notion of equivalence of two embeddings of a curve into Euclidean space (which means that one can be obtained from the other by a polynomial automorphism).

This enables us to discuss the general problem of the classification of curve embeddings up to polynomial automorphisms (also called algebraic automorphisms), and present up-to-date results and problems as well as their interplay with other areas of mathematics. For example, we illustrate how these problems are related unexpectedly to knot theory and exploit their extension to analytic curves and analytic automorphisms of Euclidean spaces. This will lead, in particular, to the formulation of the famous Gromov-Eliashberg theorem about optimal embedding dimensions of Stein manifolds into Euclidean spaces [11], [31].

There is also discussion of some very difficult conjectures of affine algebraic geometry related to classification of curved embeddings, including the Jacobian conjecture. We make some historical remarks about these problems and supply exercises throughout the sections. Some standard facts and definitions are reminded in the appendices in order to make this text more self-contained.

CHAPTER 2

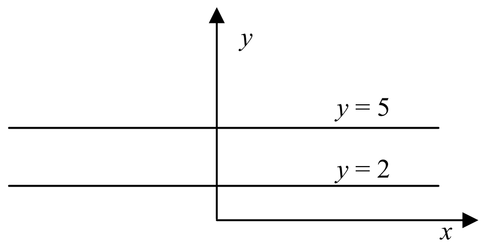
ALGEBRAIC AUTOMORPHISMS

We begin this treatise with some basic definitions and examples to set the stage for the remainder of the discussion. Here we include a refresher on Euclidean spaces and an introduction to algebraic automorphisms. We present some simple and intuitive examples of embeddings and equivalent embeddings before offering more precise definitions, so as to allow the reader to get into the mindset of the discipline before being bogged down with the details.

Definition 2.1. Euclidean space F^n of dimension n over a field F is $F^n = \{(x_1,\dots,x_n) \mid x_i \in F\} =$ the set of all n-tuples with coordinates in the field F.

Mostly we will deal with examples where F is the field of complex numbers, $\mathbb{C}$. However, for illustration we consider also F as the field of real numbers, $\mathbb{R}$. The reader is expected to have familiarity with fields, therefore only a definition for quick reference is included in Appendix D as a refresher. For a more in-depth review of abstract algebra, we recommend E. H. Connell's *Elements of Abstract and Linear Algebra* [40].

Example 2.2. Embeddings of a line $\mathbb{R}$ into $\mathbb{R}^2$



Pictured above are two embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ that differ by translation along the *y*-axis. It is reasonable to consider them as equivalent. How shall we? Consider the translation $(x, y) \mapsto (x, y+3)$. It is an example of an algebraic automorphism of $\mathbb{R}^2$ (defined below), or equivalently, an example of a polynomial coordinate substitution. It is the existence of such a map that will allow us to categorize the two embeddings above as equivalent.

Definition 2.3. A map $\Phi = (\varphi_1, \ldots, \varphi_n) : F^n \to F^n$ is called a **polynomial map** when each coordinate function φ_i is an element of $F[x_1, \ldots x_n]$ (where $F[x_1, \ldots x_n]$ = the ring of polynomials in *n* variables with coefficients in the field *F*). Furthermore, if $\exists$ a polynomial map $\Psi = (\psi_1, \ldots, \psi_n) : F^n \to F^n$ such that $\Phi \circ \Psi = \Psi \circ \Phi = \mathrm{Id}_{F^n}$, then Φ is called an **algebraic (or, polynomial) automorphism** or equivalently, a **polynomial coordinate substitution**.

When Φ is an algebraic automorphism, the functions $\varphi_1, \ldots, \varphi_n$ may be viewed as new coordinates, since $\forall \bar{x} = (x_1, \ldots, x_n) \in F^n$ and $\forall \bar{y} = (y_1, \ldots, y_n) \in F^n$ such that $\bar{x} \neq \bar{y}$, $\exists \varphi_i$ such that $\varphi_i(\bar{x}) \neq \varphi_i(\bar{y})$. (Indeed, otherwise Φ is not injective!) Hence, the term polynomial coordinate substitution is a suitable alternative to the term algebraic automorphism.

Exercise 2.4. Describe all algebraic automorphisms of $\mathbb{C}$. (Hint: Use the Fundamental Theorem of Algebra).

Example 2.5. Let $\Phi = (\varphi_1, \varphi_2) : \mathbb{C}^2_{x,y} \to \mathbb{C}^2_{u,v}$ where (x, y) (and respectively, (u, v)) are coordinates on the corresponding sample of $\mathbb{C}^2$. Then the following are examples where Φ is an algebraic automorphism of $\mathbb{C}^2$.

(1) Linear automorphisms:
$$\begin{aligned} u &= \varphi_1(x,y) = a_{11}x + a_{12}y \\ v &= \varphi_2(x,y) = a_{21}x + a_{22}y \end{aligned}$$

where the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is invertible, i.e. $\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$.

The inverse map Ψ is given by $\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} u \\ v \end{pmatrix}$.

(2) Affine transformations:
$$\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

In this case the inverse map is given by
$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} u - b_1 \\ v - b_2 \end{pmatrix}.$$

(3) Triangular automorphisms:

$$u = \varphi_1(x,y) = x$$

$$v = \varphi_2(x,y) = y + p(x) \qquad \text{(where } p \in \mathbb{C}[x]\text{)}.$$

Here, the inverse Ψ is given by:
$$x = \psi_1(u,v) = u$$

$$y = \psi_2(u,v) = v - p(u)$$

Theorem 2.6. (Jung - Van der Kulk): Every algebraic automorphism of F^2 is a composition of linear and triangular automorphisms.

Exercise 2.7. Show how the affine transformation $\Lambda : F^2 \to F^2$ given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ where A is the invertible matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, can be constructed by the composition of linear and triangular automorphisms.

Exercise 2.8.

(A) Write the inverse of the triangular automorphism of F^3 which by definition is of the form: $(x, y, z) \mapsto (x, y + p(x), z + q(x, y))$ where p and q are polynomials.

(B) Write a composition of the two automorphisms of F^2 given by $\Phi : (x, y) \mapsto (x, y + x^3)$ and $\Psi : (x, y) \mapsto (x + y^3, y)$.

In a more general setting we have the following:

Definition 2.9. Let a Euclidean space F^n be equipped with a coordinate system $x_1, ..., x_n$. An automorphism of the form

$$(x_1, ..., x_n) \mapsto (x_1, x_2 + p_2(x_1), x_3 + p_3(x_1, x_2), ..., x_n + p_n(x_1, ..., x_{n-1}))$$

where each p_i is a polynomial in $i - 1$ variables, is called **triangular**.

Theorem 2.10. There exist algebraic automorphisms of $\mathbb{C}^3$ that are not compositions of linear and triangular automorphisms.

Historical Remark. More precisely, in 1972 Nagata conjectured that the following automorphism of F^3

$$(x, y, z) \mapsto (x + (x^2 - yz)z, y + 2(x^2 - yz)x + (x^2 - yz)^2 z, z)$$

cannot be presented as a composition of linear and triangular automorphisms. Nobody had the slightest idea how to approach this problem until 2004 when Shestakov and Umirbaev proved it using tools from a completely different area [33]. For this solution they were awarded the 2007 E. H. Moore Research Article Prize by the American Mathematical Society (AMS). [41].

CHAPTER 3

ALGEBRAIC DESCRIPTION OF AUTOMORPHISMS

We now employ the power of abstract algebra, namely rings, and establish a connection between algebraic automorphisms and ring isomorphisms. This will allow us to look on the ring level and utilize results from ring theory to make conclusions about the behavior and relationships of the automorphisms themselves. The concept is introduced now, and employed often later.

Let $\Phi = (\varphi_1, ..., \varphi_n) : F^n \to F^n$ be a polynomial map, i.e. $\varphi_i \in F[x_1, ...x_n] \quad \forall i$. Then consider the ring homomorphism $\Phi^* : F[x_1, ..., x_n] \to F[x_1, ..., x_n]$ defined by $\Phi^*(f) = f \circ \Phi$ for every polynomial $f \in F[x_1, ...x_n]$. The best way to get a handle on these ring homomorphisms and their relationships to the polynomial maps is to get one's hands dirty with the details. Hence we present:

Exercise 3.1. Let $\Phi : F^n_{\overline{x}} \to F^n_{\overline{u}}$ be a polynomial map.

(A) Show that $\Phi^* : F[u_1, ..., u_n] \to F[x_1, ..., x_n]$ defined by $\Phi^*(f) = f \circ \Phi$ is a homomorphism of rings, and that $\Phi^*(u_i) = \varphi_i$ where u_i is a coordinate, $u_i : F^n \to F$ given by $(u_1, ..., u_n) \mapsto u_i$.

(B) Show that Φ is an algebraic automorphism of F^n if and only if Φ^* is an isomorphism of the polynomial rings.

It is important to have criteria showing when a given polynomial map $\Phi : F^n \to F^n$ is an automorphism, i.e. when it is invertible. Following is one of such criteria, which is a consequence of a theorem of James Ax described below in Chapter 5 amidst the discussion of morphisms [6].

Proposition 3.2 If the polynomial map $\Phi = (\varphi_1, ...\varphi_n) : F^n \to F^n$ is injective then it is an automorphism.

Remark. It is interesting that, as this Proposition indicates, injectivity is not only a necessary, but also a sufficient condition for a polynomial map $\Phi = (\varphi_1, ..., \varphi_n) : F^n \to F^n$ to be an automorphism. However, if one allows $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ (with $n \geq 2$) to be an analytic map (defined below in Chapter 4), then this proposition is no longer true. In fact there is a famous example of Fatou of an injective analytic map $\mathbb{C}^2 \to \mathbb{C}^2$ which is not surjective [14].

However, the most effective criterion for recognizing when a polynomial map $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ with $n \geq 2$ is an automorphism has not yet been proven. The following is one of the most difficult problems in modern mathematics and is over 70 years old:

Conjecture 3.3 (The Jacobian Conjecture) [2]. Let $(x_1, ..., x_n)$ be a coordinate system on $\mathbb{C}^n$ and $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with coordinate functions $\Phi = (\varphi_1, ..., \varphi_n)$. Suppose that the Jacobian of Φ, $J(\Phi)$ (i.e. the determinant of the Jacobian matrix $\left(\frac{\partial \varphi_i}{\partial x_j}\right)_{i,j=1}^{n}$) is nowhere zero. Then Φ is an automorphism.

Exercise 3.4. Show that if $J(\Phi)$ does not vanish on $\mathbb{C}^n$ then it is a nonzero constant.

Exercise 3.5. Prove the Jacobian conjecture for dimension $n=1$.

Exercise 3.6. Recall the Nagata automorphism given in Section 2 as $(x,y,z)\to(x+(x^2-yz)z, y+2(x^2-yz)x+(x^2-yz)^2z, z)$. Show that its Jacobian is 1. This is of course, no proof of the Jacobian, but a nice illustration of an automorphism whose Jacobian is never zero.

Remark. We know from Calculus that the Jacobian of a differentiable map $\Phi:\mathbb{R}^2\to\mathbb{R}^2$ is nothing but the coefficient of the local change of area. Thus in the real two-dimensional case the Jacobian conjecture can be reformulated as follows: Show that a polynomial map of $\mathbb{R}^2$ into itself locally preserving the area is invertible.

Historical Remark. The Jacobian conjecture was formulated by Keller in 1939. There were several wrong solutions of this problem. The most notorious one is due to W. Engel [12] who announced a solution for the two-dimensional case in 1955 and it was considered correct for around 15 years. However in the 1970's two groups of mathematicians independently found a mistake in Engel's paper. One group was led by Abhyankar in the USA, and the other by Vitushkin in Russia. By now there are hundreds of papers devoted to this problem, but the general case remains resistant even in dimension 2. Clearly, the Jacobian conjecture is not a good choice for a PhD thesis!

CHAPTER 4

HOLOMORPHIC AUTOMORPHISMS AND EQUIVALENCE OF MAPS

Let $I=(i_1,...,i_n)$ be a multi-index with each i_k a non-negative integer. Let $\bar{z}=(z_1,...,z_n)$ be a coordinate system on $\mathbb{C}^n$. Let $a_I \in \mathbb{C}$ and $\bar{z}^I = z_1^{i_1} \cdots z_n^{i_n}$.

Definition 4.1. A power series $\sum_I a_I \bar{z}^I$ mapping $\mathbb{C}^n \to \mathbb{C}$ is an **entire function** (or a function **holomorphic**, equivalently, **analytic** on all of $\mathbb{C}^n$) if this power series is absolutely convergent for every $\bar{z} \in \mathbb{C}^n$. We will use the notation $\mathbb{C}\{z_1,...,z_n\}$ to represent the **set of entire functions** on $\mathbb{C}^n$.

Exercise 4.2. Show that $\mathbb{C}\{z_1,...,z_n\}$ is a ring.

Definition 4.3. A map $\Phi=(\varphi_1,...,\varphi_n):\mathbb{C}^n \to \mathbb{C}^n$ is called **holomorphic** if every coordinate function φ_i is holomorphic on $\mathbb{C}^n$. Furthermore, if a holomorphic map Φ has a holomorphic inverse Ψ such that $\Phi \circ \Psi = \Psi \circ \Phi = Id_{F^n}$, then it is called a **holomorphic automorphism**.

Note that all algebraic automorphisms are holomorphic automorphisms but not all holomorphic automorphsims are algebraic.

Example 4.4. Consider the following function: $\Phi(z_1, z_2) = (z_1, e^{z_1} z_2)$. It is holomorphic, as $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Also, it is an automorphism as its inverse is: $(z_1, z_2) \mapsto (z_1, e^{-z_1} z_2)$. However, it is not algebraic.

Theorem 4.5. A holomorphic map Φ is an automorphism if and only if it is bijective.

Note. As mentioned above, there exists an example of Fatou of an injective holomorphic map $\mathbb{C}^2 \to \mathbb{C}^2$ that is not surjective. Compare this with Proposition 2.11, where the map is a polynomial map, and injectivity is sufficient to conclude the map is bijective.

Exercise 4.6. Show that on $\mathbb{C}$ each holomorphic automorphism is an affine transformation, that is, of the form $f(z) = az + b$, where $a \neq 0$. (Hint: Use the

Note. Consider the map $f(z) = e^z$. It is a holomorphic map of $\mathbb{C}$ to $\mathbb{C}$. However, it is not a counter-example to the conclusion of Exercise 4.6, with inverse $g(z) = \ln(z)$. Indeed, on $\mathbb{C}$, the natural log is a multi-valued function, so it is not a true inverse, and so $f(z) = e^z$ is not a holomorphic automorphism of $\mathbb{C}$ and is not a counter-example to Exercise 4.6. In particular, the Casorati-Weierstrass theorem applies to $f(z)$ at zero, with a change of variable of $w = \frac{1}{z}$, which is another proof that $f(z)$ is not a holomorphic automorphism of $\mathbb{C}$.

Now, we once again employ the power of rings to investigate these holomorphic maps. The idea should be very accessible to those who have mastered the details behind Exercise 3.1.

For every holomorphic map, $\Phi = (\varphi_1, \ldots, \varphi_n) : \mathbb{C}^n \to \mathbb{C}^n$, consider a map $\Phi^* : \mathbb{C}\{z_1, \ldots, z_n\} \to \mathbb{C}\{z_1, \ldots, z_n\}$ given by $\Phi^*(f) = f \circ \Phi$.

Exercise 4.7.

(A) Show that Φ^* as described above is a homomorphism of rings.

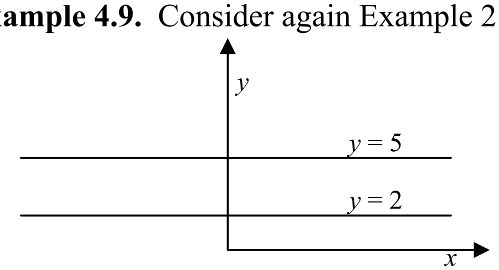
(B) Show that Φ is a holomorphic automorphism of Euclidean space if and only if Φ^* is an automorphism of rings.

Definition 4.8. (Equivalence of maps) Two maps $f : X \to \mathbb{C}^n$ and $g : X \to \mathbb{C}^n$ (of some geometrical object *X*) are called algebraically (respectively, holomorphically) **equivalent** if there exists an algebraic (respectively, holomorphic) automorphism Φ of $\mathbb{C}^n$ such that $g = \Phi \circ f$. That is, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g"'] & \mathbb{C}^n \arrow[r, "\Phi"] & \mathbb{C}^n \end{tikzcd}$$

Remark. In the algebraic case we can use this definition for any field, F (not just the field $\mathbb{C}$).

Example 4.9. Consider again Example 2.2: Embeddings of a line $\mathbb{R}$ into $\mathbb{R}^2$



Let X be the x-axis $=\{x \mid x \in \mathbb{R}\}$. Then the line $y = 2$ can be thought of as the result of the map $f : X \to \mathbb{R}^2$ by $x \mapsto (x,2)$. Similarly, the line $y = 5$ results from $g : X \to \mathbb{R}^2$ by $x \mapsto (x,5)$. Consider the algebraic automorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x, y) \mapsto (x, y+3)$. Then clearly $g = \Phi \circ f$, and so according to the definition above, f and g are algebraically equivalent maps.

CHAPTER 5

AFFINE ALGEBRAIC CURVES AND MORPHISMS

Definition 5.1. A closed **affine algebraic subvariety** X of F^n is the set of common zeros of a system of polynomial equations on F^n. Though an affine algebraic variety is a more general object, it can always be presented as a closed affine algebraic subvariety of F^n. Hence we shall treat the affine algebraic varieties below as such closed subvarieties. **Affine algebraic curves** are one-dimensional affine algebraic varieties. (Dimension of an affine algebraic variety is defined more accurately in Chapter 9.)

Example 5.2.

(1) Euclidean spaces are affine algebraic varieties.

(2) $\{y-x^2=0\}\subset\mathbb{R}^2$ — parabola

(3) $\{xy-1=0\}\subset\mathbb{R}^2$ — hyperbola

(4) $\left\{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\right\}\subset\mathbb{R}^2$ — ellipse

(5) $\{xy-z^2=0\}\subset\mathbb{C}^3_{x,y,z}$ — surface (i.e. two dimensional object)

(6) $\left\{x+x^2y+z^2+t^3=0\right\}\subset\mathbb{C}^4$ — hypersurface in $\mathbb{C}^4$ (that is, it is given by one non-constant polynomial equation)

Remark. For examples (2) through (4) above, one may consider such curves in $\mathbb{C}^2$ where they have the same names.

Definition 5.3. Let X be a closed affine algebraic subvariety of F^n and $F[X]$ be the restriction of the ring of polynomials to X. That is, $F[X] = F[x_1,...x_n]\|_X$. Then $F[X]$ is called the **ring of regular functions** on X.

Exercise 5.4. Show that the ring of regular functions on the hyperbola $xy = 1$ in $\mathbb{C}^2$ is naturally isomorphic to the ring of Laurant polynomials $\mathbb{C}[t, t^{-1}]$.

Exercise 5.5. Prove the following Theorem: Let I be the ideal of all polynomials in $F[x_1,...,x_n]$ that vanish on X, where X is a closed affine algebraic subvariety of F^n. Show that $F[X] \simeq F[x_1,...,x_n]/I$.

Remark. For a hypersurface, this ideal I is principal.

Definition 5.6. Let $X \subset F^n$ and $Y \subset F^m$ be closed affine algebraic subvarieties of the Euclidean spaces. A map $\Phi: X \to Y$ is called a **morphism** if it is a restriction of a polynomial map $\widetilde{\Phi}: F^n \to F^m$. That is, one has the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ \downarrow & & \downarrow \\ F^n & \xrightarrow{\widetilde{\Phi}} & F^m \end{array}$$

where the maps $X \to F^n$ and $Y \to F^m$ are the inclusion maps.

Exercise 5.7. Clearly, the morphism $\Phi: X \to Y$ generates a map, $\Phi^*: F[Y] \to F[X]$. Show that it is a homomorphism of the rings of regular functions.

Definition 5.8. A morphism $\Phi : X \to Y$ is an **isomorphism** if Φ has an inverse morphism $\Phi^{-1} = \Psi : Y \to X$ (i.e. $\Psi \circ \Phi = Id_X$ and $\Phi \circ \Psi = Id_Y$).

Exercise 5.9. Show that $\Phi : X \to Y$ is an isomorphism if and only if $\Phi^* : F[Y] \to F[X]$ is an isomorphism of rings.

Example 5.10. Let $X = \mathbb{C}$ with coordinate *t*, i.e. $\mathbb{C}[X] = \mathbb{C}[t]$ and $Y = \{y - x^2 = 0\} \subset \mathbb{C}^2_{x,y}$. That is, $\mathbb{C}[Y] \simeq \mathbb{C}[x,y]/(y-x^2) \simeq \mathbb{C}[X]$.

Exercise 5.11. Let $X = \mathbb{C}$ and $Y = \{y - x^2 = 0\} \subset \mathbb{C}^2_{x,y}$. Consider $\Phi : \mathbb{C}_t \to \mathbb{C}^2_{x,y}$ given by $t \mapsto (t, t^2)$. Show that $\Phi : X \to Y$ is an isomorphism.

Exercise 5.12. Consider hyperbola $H = \{xy - 1 = 0\} \subset \mathbb{C}^2_{x,y}$ and its projection onto the *x*-axis, $\rho : H \to \mathbb{C}_x$, $\rho(x,y) = x$. Show that ρ is not an isomorphism.

Remark. Note that ρ is not a homeomorphism either.

Fact. If $f : X \to Y$ is an isomorphism of complex (respectively, real) algebraic varieties, then it is automatically a homeomorphism in the standard topology.

Exercise 5.13. Consider a semi-cubic parabola $Y = \{x^3 - y^2 = 0\}$ and $X = \mathbb{C}$ with coordinate *t*. Show that the morphism $\rho : X \to Y \subset \mathbb{C}^2_{x,y}$ given by $\rho(t) = (t^2, t^3)$ is a homeomorphism but not an isomorphism.

Exercise 5.14. Consider

$$X_{1,1} = \{xy - 1 = 0\} \subset \mathbb{C}^2_{x,y} \text{ and}$$

$$X_{k,l} = \{x^k y^l - 1 = 0\} \subset \mathbb{C}^2_{u,v}$$

where k, $l > 0$ are relatively prime. Show that $\Phi : X_{1,1} \to X_{k,l}$ by $(x, y) \mapsto (x^l, y^k)$ is an isomorphism.

Definition 5.15. When an isomorphism $\Phi : X \to Y$ exists between closed affine algebraic subvarieties $X \subset F^n$ and $Y \subset F^m$, we say that ***X* is isomorphic to *Y*** and write $X \simeq Y$. Similarly, when there exists an isomorphism between rings, we say that the rings are isomorphic, and also use the symbol $\simeq$. Thus, in this language and notation, Exercise 5.9 showed that $X \simeq Y \Leftrightarrow F[Y] \simeq F[X]$.

As noted in Chapter 3, it is desirable to have criteria showing when a given endomorphism is an automorphism. We now state the theorem of James Ax, a consequence of which was made use of in the form of Proposition 3.2.

Theorem 5.16 (James Ax) [6]. Let $\Phi : X \to X$ be an injective morphism of an affine algebraic variety into itself. Then Φ is surjective.

Remark. The Ax theorem holds for objects even more general than affine algebraic varieties (algebraic schemes).

CHAPTER 6

EQUIVALENCE

We now consider a relationship between closed affine algebraic subvarieties X and Y that is stronger than their being isomorphic. We consider what it means for X and Y to be *equivalent*.

Definition 6.1. (Equivalence of closed affine algebraic subvarieties) Let X and Y be closed affine algebraic subvarieties of F^n, and $X \simeq Y$ with isomorphism $\Phi : X \to Y$. If there also exists an algebraic automorphism $\widetilde{\Phi} : F^n \to F^n$, such that Φ is the restriction of $\widetilde{\Phi}$, then we say that X and Y are **equivalent**.

Note. Earlier, in Chapter 4, we defined the notion of algebraic equivalence of maps f and g, both mapping some geometrical object X into F^n. It required the existence of an automorphism, Φ of F^n with $\Phi \circ f = g$. This notion of equivalence is essentially the same, but we could not use it in Chapter 4, as we had not yet been introduced to closed affine algebraic subvarieties, nor the notion of isomorphisms of such objects.

Exercise 6.2. Consider $X, Y \subset \mathbb{C}^2_{x,y}$ with $X = \{y = 0\}$ and $Y = \{y - x^2 = 0\}$. In Exercise 5.11 it was shown that $X \simeq Y$. Show that in fact, X and Y are equivalent.

To illustrate the essential “equivalence” of our two notions of equivalence, note that we can think of Exercise 6.2 in the language of equivalence of maps instead of equivalence of closed affine algebraic varieties. Let $X=\{y=0\}$ be represented as the map $f:\mathbb{C}\to\mathbb{C}^2$ by $x\mapsto(x,0)$ and let $Y=\{y-x^2=0\}$ be represented as the map $g:\mathbb{C}\to\mathbb{C}^2$ by $x\mapsto(x,x^2)$. Then *f* and *g* are equivalent maps by the same automorphism as given in the solution to Exercise 6.2.

Example 6.3. Consider $X_{1,1}=\{xy-1=0\}\subset\mathbb{C}^2_{x,y}$ and $X_{k,l}=\{x^k y^l-1=0\}\subset\mathbb{C}^2_{u,v}$ where *k*, *l* > 0 are relatively prime. In Exercise 5.14 it was shown that $X_{1,1}\simeq X_{k,l}$. However, in the next Chapter we will show that although they are isomorphic, they are not equivalent.

Before continuing, those needing to should go through the algebra refresher supplied in Appendix A, as it reviews some results necessary for the following exercise, which will in turn be used in the proof in Chapter 7.

Exercise 6.4. Show that the polynomials $xy-1$ and $x^k y^l-1$ (where *k*, *l* > 0 are relatively prime) are irreducible in the ring $\mathbb{C}[x,y]$.

The following consequence of the Nagata lemma for Unique Factorization Domains will also be useful in the proof in Chapter 7.

Lemma 6.5. Let *p* be an irreducible element of the ring $F^{[n]}$ of polynomials in *n* variables with coefficients in *F* and let *H* be the set of points in F^n where *p* vanishes. Then every polynomial $q\in F^{[n]}$ that vanishes on this hypersurface *H* is divisible by *p*. In particular the ideal *I* of polynomials that vanish on *H* is the principal ideal generated by *p*.

Example 6.6. Consider $X = \left\{xy = z^2\right\} \subset \mathbb{C}^3$. We have that *x*, *y*, and *z* are irreducible on *X*, but *z* is not divisible by *x*.

Exercise 6.7. Consider the ring of continuous real-valued functions on the real line equipped with a coordinate *t*. Show that there is a function in this ring that vanishes at $t = 0$ but is not divisible by *t*. Furthermore, show that the ideal of a function that vanishes at $t = 0$ is not a principal one.

CHAPTER 7

NON-EQUIVALENCE OF $X_{1,1}$ AND $X_{k,l}$

In Exercise 6.4 it is established that the polynomials $xy-1$ and $x^k y^l - 1$ (where $k, l > 0$ are relatively prime) are irreducible in the ring $\mathbb{C}[x, y]$. With the help of this result we are now ready to show that although $X_{1,1}$ and $X_{k,l}$ are isomorphic, still they are not equivalent.

Proof 7.1. Assume they are equivalent. Then by definition, we have an automorphism $\widetilde{\Phi}: \mathbb{C}^2 \to \mathbb{C}^2$ whose restriction, $\Phi: X_{1,1} \to X_{k,l}$ is an isomorphism. These extend to the ring automorphism $\widetilde{\Phi}^*: \mathbb{C}[x, y] \to \mathbb{C}[x, y]$, and the ring isomorphism $\Phi^*: \mathbb{C}[X_{k,l}] \to \mathbb{C}[X_{1,1}]$ both defined in the usual way. Since Φ is an isomorphism, Φ^* must transform the ideal $I_{k,l}$ of functions vanishing on $X_{k,l}$ into the ideal $I_{1,1}$ of functions vanishing on $X_{1,1}$. Both ideals are principal which yields the existence of a nonzero constant c such that:

$$(i) \quad \widetilde{\Phi}^*(x^k y^l - 1) = c(xy - 1)$$

But, since $\widetilde{\Phi}^*$ is a homomorphism, it preserves addition, multiplication and multiplicative identity, so this gives:

$$(ii) \quad \widetilde{\Phi}^*(x^k y^l - 1) = \left(\widetilde{\Phi}^*(x)\right)^k \left(\widetilde{\Phi}^*(y)\right)^l - 1$$

Now, putting together (*i*) and (*ii*), gives

$$(iii)\quad cxy - c = \left(\widetilde{\Phi}^*(x)\right)^k \left(\widetilde{\Phi}^*(y)\right)^l - 1$$

which is not possible, as the degrees do not match up. This gives us a contradiction, which proves that our assumption was false. Thus $X_{1,1}$ and $X_{k,l}$ are not equivalent.

Exercise 7.2. Show that $X_{k,l}$ and $X_{m,n}$ are equivalent if and only if $(k,l) = (m,n)$ or $(k,l) = (n,m)$.

CHAPTER 8

THE THEOREMS

We now list some major theorems regarding this topic, preceded by some relevant definitions.

Definition 8.1.

(1) Let $X \subset \mathbb{C}^m$ be a closed affine algebraic subvariety. Then $\Phi : X \to \mathbb{C}^n$, the restriction of the polynomial map, $\widetilde{\Phi} : \mathbb{C}^m \to \mathbb{C}^n$ is called an algebraic **embedding** when X is isomorphic to its image, $Y = \Phi(X)$ under Φ, and the embedding is called **proper** when additionally its image, $Y = \Phi(X)$ is closed.

(2) To say that an embedding of F^k into F^n (where $n \geq k$) is **rectifiable** is to say that it can be sent to a coordinate line, which is to say that it is equivalent to a linear embedding.

Note. For $(p_1(t), ..., p_n(t)) : \mathbb{C}_t \to \mathbb{C}^n$ to be an embedding, it must be a homeomorphism as well, so we need both of:

(i) $\forall t_1 \neq t_2 \in \mathbb{C}_t \quad \exists\, j$ such that $p_j(t_1) \neq p_j(t_2)$ (injectivity for homeomorphism).

(ii) $\forall t \in \mathbb{C}_t \quad \exists\, j$ such that $p_j'(t) \neq 0$ (no t with derivative everywhere zero; the smoothness of the image, making this an embedding).

Note that in general the image of an affine algebraic variety under a morphism is not necessarily closed. As an example consider $X = \{xy - 1 = 0\} \subset \mathbb{C}^2$ and the forgetting projection $\Phi : X \to \mathbb{C}$ given by $\Phi(x, y) = x$. Then the image $\Phi(X)$ coincides with $\mathbb{C}^*$, the non-zero complex numbers, and in particular it is not closed. However, this map *is* a homeomorphism between X and $\mathbb{C}^*$. This is an example of a non-proper embedding.

Remarks. In the case of $F = \mathbb{C}$ or $F = \mathbb{R}$ every (not necessarily proper) embedding $\Phi : X \to F^n$ yields a homeomorphism (and even diffeomorphism) between X and $\Phi(X)$. If the image of a morphism $\Phi : X \to \mathbb{C}^n$ is not closed, it is not a closed affine subvariety of $\mathbb{C}^n$. However we have the following weak form of Chevalley's theorem.

Theorem 8.2. (Chevalley) Consider the closure $\overline{\Phi(X)}$ of the image of an algebraic variety X under morphism $\Phi : X \to \mathbb{C}^n$. It is always an affine algebraic subvariety of $\mathbb{C}^n$.

As an example, consider again $X = \{xy - 1 = 0\} \subset \mathbb{C}^2$ and $\Phi : X \to \mathbb{C}$, the projection onto the x coordinate. Though the image, $\Phi(X) = \mathbb{C}^*$ is not closed, its closure, $\overline{\Phi(X)} = \overline{\mathbb{C}^*} = \mathbb{C}$ is an affine algebraic subvariety of $\mathbb{C}$.

Theorem 8.3. (Abhyanker-Moh-Suzuki) Let C be a closed affine curve in $\mathbb{C}^2_{x,y}$ such that $C \simeq \mathbb{C}$. Then C is equivalent to $\{y = 0\}$ [1], [35].

Using the language of the above definitions, Theorem 8.3 says: Every polynomial embedding of $\mathbb{C}$ into $\mathbb{C}^2$ is rectifiable.

Historical Remark. This theorem was proven independently around 1975 by Suzuki [35] and by Abhyankar and Moh [1]. The two last mathematicians worked on this theorem while trying to fix Engel's proof of the Jacobian conjecture. It is one of the most beautiful theorems in affine algebraic geometry and now there are at least a dozen different proofs of this fact [5], [22], [25], [30], [36].

Theorem 8.4. (Lin-Zaidenberg) Let L be a closed affine algebraic curve in $\mathbb{C}^2$, homeomorphic to $\mathbb{C}$ but not isomorphic to $\mathbb{C}$. Then there exists relatively prime k and l such that L is equivalent to $\{x^k - y^l = 0\}$ [23].

Note. We saw an example of such an L in Exercise 5.13. It was the semi-cubic parabola $Y = \{x^3 - y^2 = 0\}$.

Historical Remark. This theorem was proven in 1983 by V. Ya. Lin and M. Zaidenberg who were working in Russia [23]. (A special case of this theorem was proven earlier by Lee Rudolph [29] by means of knot theory which will be discussed later.) Before starting to work on this theorem they contacted foreign experts including Pierre Deligne. After a few weeks of reflection Deligne said that in his opinion modern mathematics could not solve this problem. After such inspiring encouragement the problem was quickly solved!

Theorem 8.5. (Craighero-Jelonek) Let C be a closed affine algebraic curve in $\mathbb{C}^n$, $n \geq 4$ and $C \simeq \mathbb{C}$. Then C is equivalent to a coordinate axis. That is, any polynomial embedding of $\mathbb{C} \to \mathbb{C}^n$ for $n \geq 4$, is rectifiable. Note: It is still unknown whether this holds for $n = 3$ [9], [18].

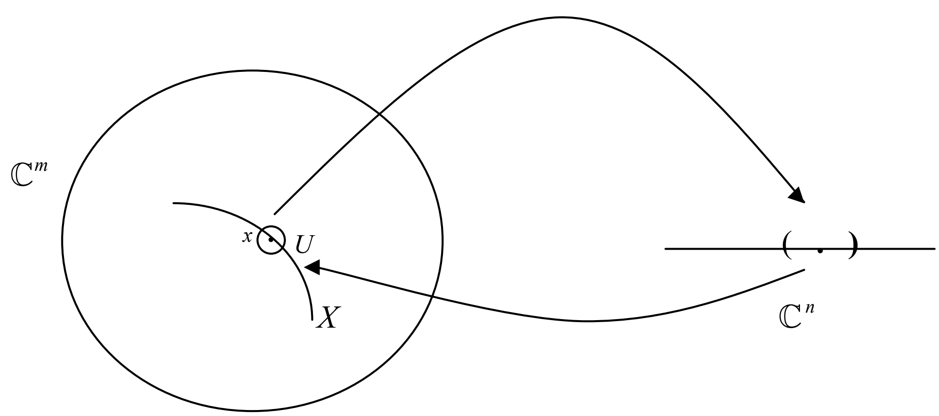
Theorem 8.6. (Kaliman) [20] For all polynomial embeddings, $\Phi:\mathbb{C}\to\mathbb{C}^3$, there is a holomorphic automorphism, $\alpha:\mathbb{C}^3\to\mathbb{C}^3$ such that $\alpha\circ\Phi$ has as its image a coordinate line. Note: It is still unknown however, if there exists some polynomial map which would do the same .

Theorem 8.7. (Rosay, Forstnerič, Globevnik) [15] For all $n\geq 2$, there exists a closed proper holomorphic curve in $\mathbb{C}^n$, biholomorphic to $\mathbb{C}$, which can't be sent to a coordinate line.

CHAPTER 9

Manifolds and Embeddings

Definition 9.1. Let X be a closed affine algebraic or analytic subvariety of $\mathbb{C}^m$. Then we say that X is **smooth of dimension *n*** (or, **an *n* manifold**) if, as illustrated below, $\forall x \in X \quad \exists$ a neighborhood, $U \subset \mathbb{C}^m$ of x, and a holomorphic map $f : U \to \mathbb{C}^n$ such that $f|_{X \cap U} : X \cap U \to B^n \subset \mathbb{C}^n$ is bijective. (Where B^n is a unit ball in $\mathbb{C}^n$.)

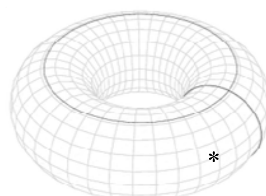


A **Stein *n* manifold** is the set of common zeros of a finite number of analytic functions, $X \subset \mathbb{C}^m$ such that $\forall x \in X \quad \exists$ a neighborhood, $U \subset X$ of x biholomorphic (that is, with a holomorphic inverse) to a ball in $\mathbb{C}^n$. That is, locally, this set of common zeros of analytic functions is organized as a Euclidean space. A one-dimensional manifold is called a curve. A two-dimensional manifold is called a surface.

Remark. Dimension depends on the choice of a field. For example, whereas the complex line $\mathbb{C}$ and the real plane $\mathbb{R}^2$ are essentially the same, $\mathbb{C}$ is one dimensional in the field of complex numbers, while $\mathbb{R}^2$ is two dimensional in the field of reals.

Example 9.2.

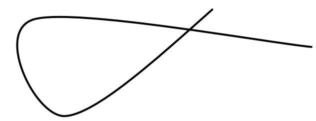
(1) $y^2 = x(x-1)(x-2) \subset \mathbb{C}^2$ (or, any polynomial of degree 3 with three distinct roots) This is a 1-curve in the field of complex numbers; a punctured torus, illustrated below with an asterisk to represent the puncture – as a surface over $\mathbb{R}$.



(2) $p(x,y) = 0 \subset \mathbb{C}^2$ where $p(x,y)$ is an analytic function. This yields a curve *C*. Suppose there is no point of *C* such that $\nabla p = \langle p_x, p_y \rangle = \langle 0,0 \rangle$. Then *C* is smooth because of the implicit function theorem, which in the two-dimensional case states that if $p(x,y) = 0 \subset \mathbb{C}^2$ is continuous and $p_y \neq 0$ at some point *P*, then *y* may be expressed as a smooth function of *x* in some domain about *P*; i.e., there exists a function over that domain such that $y = y(x)$ and locally *C* coincides with the graph of this function.

(3) the cross $xy = 0$ is non-smooth at the origin.

(4) $y^3 = x^2(x-1) \in \mathbb{C}^2$ is not a manifold due to the point of overlap:



(5) $x^3 = y^2$, the semi-cubic parabola, is non-smooth (see Exercise 5.13)

Theorem 9.3. (Kaliman [19], Srinivas [34], Nori) Let X and Y be two isomorphic closed affine algebraic submanifolds of $\mathbb{C}^N$ and of dimension n. If $N \geq 2n+2$, then every isomorphism $\alpha : X \to Y$ can be extended to an automorphism of $\mathbb{C}^N$.

Note. In the case $n=1$, this is simply the Craighero-Jelonek Theorem (8.5).

Fact. Any n-dimensional affine algebraic manifold can be embedded into $\mathbb{C}^{2n+1}$. (It is unknown if this can be improved upon. That is, if it can be embedded into a lower power.)

Theorem 9.4. (Gromov, Eliashberg, Schürmann [11], [31]) Let $n \geq 2$. Then any Stein holomorphic n-manifold can be embedded into $\mathbb{C}^N$ where $N = \lfloor \frac{3n}{2} \rfloor + 1$.

Historical Remark. The conjecture that $\lfloor \frac{3n}{2} \rfloor + 1$ is the optimal dimension N was formulated by Forster in 1970. In 1971 Eliashberg and Gromov announced that they can show that $N \leq \lfloor \frac{3n}{2} \rfloor + 2$. They published the proof in 1992 only, with improvement for even n given by $N = \lfloor \frac{3n}{2} \rfloor + 1$ [11]. During the period (1971 – 1992), many mathematicians tried to recover the proof but failed. As R. Narasimhan wrote with some bitterness in his joint paper [7] with S. Bell, "It has to be confessed that at least one of the authors of this article has been unable to carry out the proof of this theorem." For odd n the final improvement was obtained by Schürmann in 1997, who also used methods from Eliashberg and Gromov.

It is unknown if any non-compact analytic curve can be embedded into $\mathbb{C}^2$. However, for the best analogue of the Gromov-Eliashberg-Schürmann Theorem (9.4) for the case $n=1$, we have:

Theorem 9.5. (Fornæss Wold [37], [38], [39])

(a) A finitely connected domain (that is, a domain with finitely many holes) in $\mathbb{C}$ can be embedded as a closed Stein manifold in $\mathbb{C}^2$.

(b) A torus with a finite number of holes (not punctures) can also be embedded as a closed Stein manifold in $\mathbb{C}^2$.

Below we consider analytic and algebraic curves over $\mathbb{C}$, which from a real point of view, are surfaces (meaning, locally they are like a disc), called Riemann Surfaces.

Example 9.6. Below are examples of compact surfaces; closed Riemann Surfaces:

(a) Perhaps the best example is a sphere.

(b) A disc is *not* an example. With its boundary, a disc is not a manifold because it is not locally like a disc on its boundary. Without its boundary a disc is not closed, as it does not contain all of its limit points – namely its boundary points!

(c) A torus (which is simply a sphere with a handle), as well as a sphere with any non-negative integral number of handles is an example of a closed Riemann surface. The number of handles is called the *genus*.

Exercise 9.7. Show that a compact analytic curve can *NOT* be embedded into the Euclidean space $\mathbb{C}^m$.

Notes.

- Punctured Riemann surfaces (with any finite, positive number of punctures) are Stein. Any algebraic curve is of this type.

- Removing a closed disc from (that is, making a hole in) a closed Riemann surface will give a Stein curve (a one-dimensional manifold). However, it is not biholomorphic to an algebraic curve.

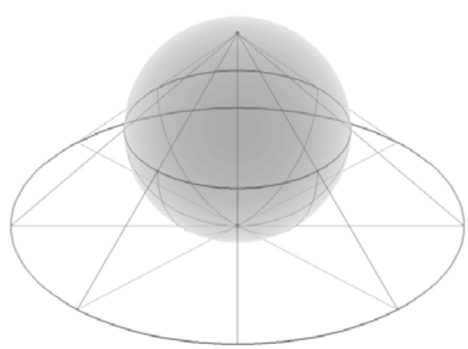
Exercise 9.8. Although a puncture (removal of a point) and a hole (removal of a closed disc) are not the same, show that an annulus, $A_n = \{z \mid \ 0 < r_1 < |z| < r_2\}$ (an open disc with a hole) and a punctured open disc, $\Delta^* = \{z \mid \ 0 < |z| < R\}$ are homeomorphic but not biholomorphic.

CHAPTER 10

FOCUS ON THE REALS

We now focus on real curves. We begin with some discussion of embeddings of $\mathbb{R}$ into $\mathbb{R}^3$.

To set the stage for this topic we first recall a bit about the one-point compactification of *n*-dimensional real space. Recall that by way of the Riemann stereographic projection, we show that the one-point compactification of $\mathbb{R}^2$, that is two-dimensional real space plus one point, gives the two sphere. So, $\mathbb{R}^2 \cup \infty = S^2$. Below is an illustration of the Riemann stereographic projection which represents how S^2 without its north-pole is mapped to the real plane, $\mathbb{R}^2$. Then, the north-pole is mapped to infinity, showing how $S^2 = \mathbb{R}^2 \cup \infty$.



The one-dimensional equivalent is that $\mathbb{R} \cup \infty = S^1$, and although higher dimension cases are difficult to visualize, we have that $\mathbb{R}^3$ plus one point gives S^3 and in general, $\mathbb{R}^n \cup \infty = S^n$.

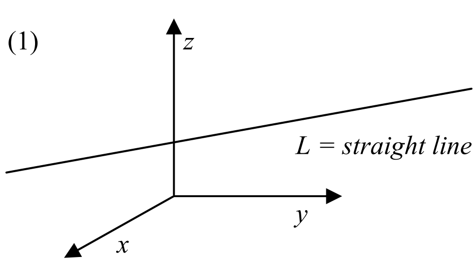
Exercise 10.1. Describe explicitly a relation that shows how $\mathbb{R}$ plus one point gives a circle, S^1.

Definitions 10.2. An embedding $\varphi : \mathbb{R} \to \mathbb{R}^3$ is **proper** if it can be extended to an embedding $S^1 \to S^3$. Two proper embeddings, $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}^3$ are **equivalent** if $\exists$ a homeomorphism $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\alpha \circ \varphi_1 = \varphi_2$. That is, the following diagram commutes:

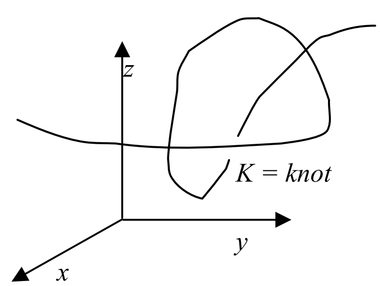
$$\begin{tikzcd} \mathbb{R} \arrow[r, "\varphi_1"] \arrow[rr, bend right, "\varphi_2"'] & \mathbb{R}^3 \arrow[r, "\alpha"] & \mathbb{R}^3 \end{tikzcd}$$

Example 10.3. Below are examples of two embeddings that are not equivalent to each other:

(1)



(2)



Note that the knot does not touch itself because it would not be a manifold if it did.

To show that these embeddings are not equivalent, we make use of the Poincaré fundamental group, that is, the first homotopy group. We know that $\mathbb{R}^3 \setminus L$ is not homeomorphic to $\mathbb{R}^3 \setminus K$ because their associated groups aren't isomorphic, which means of course that *L* and *K* are not equivalent.

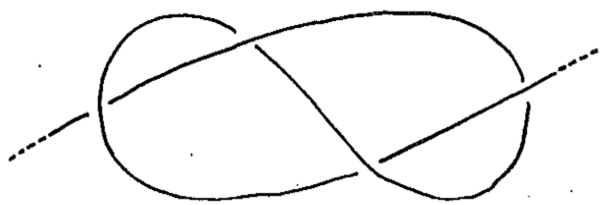
The basic idea due to Poincaré is that we assign to manifolds *X* and *Y*, groups, $\pi_1(X)$ and $\pi_1(Y)$ such that from each continuous map $f : X \to Y$ there arises a natural group homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ which is a group isomorphism whenever *f* is a homeomorphism. As it can be shown that $\pi_1(\mathbb{R}^3 \setminus L)$ is isomorphic to the integers, $\mathbb{Z}$, while $\pi_1(\mathbb{R}^3 \setminus K)$ is not even abelian, it follows clearly that $\pi_1(\mathbb{R}^3 \setminus L)$ is not isomorphic to $\pi_1(\mathbb{R}^3 \setminus K)$. Therefore, there is no homeomorphism $f : (\mathbb{R}^3 \setminus L) \to (\mathbb{R}^3 \setminus K)$, which means that $\mathbb{R}^3 \setminus L$ and $\mathbb{R}^3 \setminus K$ are not equivalent, and so *L* and *K* are not equivalent.

We now give some attention to embeddings of the type $f = (f_1, f_2, f_3) : \mathbb{R} \to \mathbb{R}^3$ where f_1, f_2 and f_3 are polynomials.

Theorem 10.4. (Shastri [32]) Let $\hat{r} = (x(t), y(t), z(t)) : \mathbb{R} \to \mathbb{R}^3$ be a knot *K* such that for any general projection $\rho : \mathbb{R}^3 \to \mathbb{R}^2$, the composition, $\rho \circ \hat{r} : \mathbb{R} \to \mathbb{R}^2$ has a finite number of double points. Then, the knot $K = \hat{r}(t)$ is equivalent to a polynomial knot, that is, a knot represented by a polynomial embedding.

Notes. Here, a "double point" refers to two different points in $\mathbb{R}$ that are mapped to the same point in $\mathbb{R}^2$ under $\rho \circ \hat{r}$. Also, the composition need not have a positive number double points, as we consider the circle (with zero double points) to be the "unknot".

Example 10.5. $\hat{r}:\mathbb{R}_t \to \mathbb{R}^3{}_{x,y,z}$ by $\hat{r}(t)=\left\langle t^3-3t,\ t^5-5t^3+4t,\ t^7-42t\right\rangle$ is non-rectifiable [32].



Remark. Consider $\tilde{\hat{r}}:\mathbb{C}\to\mathbb{C}^3$ defined exactly as $\hat{r}$ is the above example, but in the complex system instead of restricted to the reals. It is unknown whether this embedding of $\mathbb{C}$ into $\mathbb{C}^3$ is rectifiable.

Exercise 10.6. Show that $\mathbb{C}\setminus\tilde{\hat{r}}(\mathbb{C})$ is biholomorphic to $\mathbb{C}^3$\coordinate-line, where $\tilde{\hat{r}}:\mathbb{C}\to\mathbb{C}^3$ is as above. That is, show that $\pi_1\left(\mathbb{C}^3\setminus\tilde{\hat{r}}(\mathbb{C})\right)=\pi_1\left(\mathbb{C}^2\times\mathbb{C}^*\right)$.

Following is some discussion of the Neumann-Rudolph approach to the study of algebraic curves, $\Gamma=\left\{f(x,y)=0\right\}$ in $\mathbb{C}^2$ [28].

Recall that the two-dimensional complex plane is isomorphic to four dimensional real space. Also, a three-sphere of radius R is the same as three-dimensional real space plus one point. That is: $\mathbb{R}^3\cup\infty=S_R^3\subset\mathbb{R}^4\simeq\mathbb{C}^2$.

Consider γ_R defined here by $\gamma_R = \Gamma \cap S_R^3$. Then γ_R is a link in S_R^3. That is, γ_R is a disjoint union of several knots. (So, knot theory is related to algebraic curves in $\mathbb{C}^2$.) For sufficiently large *R*, the equivalence type of $\gamma_R \subset S^3$ is independent of *R*.

Neumann and Rudolph [28] studied these embeddings $\gamma_R \subset S^3$ in order to describe algebraically equivalent embeddings of $\Gamma \subset \mathbb{C}^2$, and using these ideas they reproved the Abhyankar-Moh-Suzuki and Lin-Zaidenberg theorems (8.3 and 8.4). Furthermore, Neumann himself then wrote the following:

Theorem 10.7. (Neumann) [27]

(1) Every once punctured torus (a curve, analytically) in $\mathbb{C}^2$ is equivalent algebraically to a curve in $\mathbb{C}^2_{x,y}$ given by $y^2 = x^3 + ax + b$.

(2) Every embedding of $\mathbb{C}$ into $\mathbb{C}^2$ with one node is equivalent to $y^2 = x^3 + x^2$.

(3) Every once punctured surface of genus 2 is equivalent to a curve of the form $y^2 = x^5 + ax^3 + bx^2 + cx + d$.

Neumann's Theorem describes embeddings of once punctured curves.

Developing these methods further, Nakazawa and Oka [26] classified embeddings of all smooth once-punctured curves of genus up to 16.

Exercise 10.8. The simplest twice punctured surface is $\mathbb{C}^*$. Justify that indeed $\mathbb{C}^*$ is a twice punctured surface.

Russel, Koras, Cassou-Nogues [3], classified almost all equivalent embeddings of $\mathbb{C}^*$ into $\mathbb{C}^2$. The final classification is expected to appear in the coming paper of M. Koras who announced the solution of the problem in 2011. The formulation presented by Russel, Koras and Cassou-Nogues is too complicated to present here. Instead, we consider one special case discovered approximately 3 years prior to Russel, Koras and Cassou-Nogues' paper. This example demonstrates the difficulty of the problem.

Theorem 10.9. (Kaliman [21]) Let Γ be an algebraic $\mathbb{C}^*-$curve, $\Gamma = \{f(x,y)=0\} \subset \mathbb{C}^2_{x,y}$ such that the genus of $\{f(x,y)=c\}$ is zero $\forall c \in \mathbb{C}$. Then Γ is equivalent to the zero fiber of one of the following functions:

$$(1)\ \frac{\Psi^{mn+1} + (\Psi^n + x)^m}{x^m}, \qquad \text{or} \qquad (2)\ \frac{\Psi^{mn-1} + (\Psi^n + x)^m}{x^m}$$

where $m \geq 2$, $n \geq 1$ in (1) and $m = 2$, $n \geq 2$ in (2), and $\Psi(x,y)$ is defined as $\Psi(x,y) = x^m y + a_{m-1}x^{m-1} + \cdots + a_1 x - 1$, where the coefficients $a_{m-1}, \ldots, a_1$ are such that (1) and (2) are polynomials.

CHAPTER 11

EMBEDDINGS OF HIGHER-DIMENSIONAL EUCLIDEAN SPACES

A line $\mathbb{C}$ is nothing but a one-dimensional Euclidean space. Hence the question about embeddings of $\mathbb{C}$ into $\mathbb{C}^n$ admits a natural generalization. That is, to describe equivalent proper embeddings $\mathbb{C}^k \to \mathbb{C}^n$ for $k < n$. It should be emphasized that in the case of $k = 1$ the word proper can be omitted as it is automatic.

Exercise 11.1. Prove that any embedding of $\mathbb{C}$ into $\mathbb{C}^n$ is automatically proper.

Exercise 11.2. Consider the so-called Danielewsky surface *S* given by the equation $xy = z^2 - 1$ in $\mathbb{C}^3$. It contains the line *L* given by $z - 1 = x = 0$. Find a one-to-one polynomial morphism from $\mathbb{C}^2$ onto $S \setminus L$. This is an example of a non-proper embedding of $\mathbb{C}^2$ into $\mathbb{C}^3$.

As a consequence of the Kaliman-Srinivas-Nori theorem, one has the following.

Theorem 11.3. For $n \geq 2k + 2$ every proper algebraic embedding $\Phi : \mathbb{C}^k \to \mathbb{C}^n$ is rectifiable, i.e. the image $\Phi(\mathbb{C}^k)$ can be viewed as a coordinate *k*-dimensional subspace $\mathbb{C}^k$ of $\mathbb{C}^n$ after a suitable coordinate substitution.

Except for the Abhyankar-Moh-Suzuki theorem nothing is known about the case of $k < n \leq 2k + 1$. There is however the following hypothesis.

Conjecture 11.4. (Abhyankar-Sathaye). Every proper algebraic embedding $\mathbb{C}^{n-1} \to \mathbb{C}^n$ is rectifiable.

Example 11.5. There are startling examples of non-rectifiable hypersurfaces *H* in $\mathbb{C}^n$ that are diffeomorphic to Euclidean space (i.e. equivalent to $\mathbb{R}^{2n-2}$ as real manifolds) such that they are either non-isomorphic to $\mathbb{C}^{n-1}$ or it is unknown whether they are isomorphic to $\mathbb{C}^{n-1}$. One of the most beautiful among them is the so-called Russell cubic *R* which is the hypersurface $x + x^2 y + z^2 + t^3 = 0$ in $\mathbb{C}^4$. It is indeed diffeomorphic to $\mathbb{R}^6$ (by the Choudary-Dimca [4] theorem mostly based on a very difficult *h*-cobordism theorem) but not isomorphic to $\mathbb{C}^3$ [24].

Historical Remark. This cubic appeared in connection with another important problem of affine algebraic geometry and it was important to distinguish it from $\mathbb{C}^3$. The old methods did not work in this case until Makar-Limanov [24] introduced a new invariant (a very similar invariant was found later by Derksen) which showed that the Russell cubic *R* is not $\mathbb{C}^3$.

It is unknown whether $R \times \mathbb{C}$ is isomorphic to $\mathbb{C}^4$ or not. In particular, the Russell cubic is a potential counterexample to another famous hypothesis:

Conjecture 11.6. (Zariski-Ramanujam). If a complex affine algebraic variety *X* is such that $X \times \mathbb{C}$ is isomorphic to $\mathbb{C}^{n+1}$ then *X* is isomorphic to $\mathbb{C}^n$.

The answer is known only in the case of $n = 2$ [16].

Theorem 11.7. (Fujita [16]). If X is a smooth two-dimensional affine algebraic variety such that $X \times \mathbb{C}^k$ is isomorphic to $\mathbb{C}^{k+2}$ then X is isomorphic to the plane $\mathbb{C}^2$.

Historical Remarks. Actually the original question of Zariski was whether an isomorphism of $X \times \mathbb{C} \simeq Y \times \mathbb{C}$ implies an isomorphism $X \times Y$. The first counterexample was constructed by Danielewsky who showed, for instance, that the surfaces $X = \left\{xy = z^2 - 1\right\}$ and $Y = \left\{x^2 y = z^2 - 1\right\}$ are not isomorphic, although $X \times \mathbb{C}$ and $Y \times \mathbb{C}$ are. He never published this result.

CHAPTER 12

SUMMARY

The modern mathematical area of study known as affine algebraic geometry is replete with interesting questions, elegant answers, decades-old resistant conjectures, and entertaining historical anecdotes. This survey was meant to introduce the graduate mathematics student to some of the basic definitions, powerful results, modern theorems, and classic open problems in the area, in an effort to whet the appetite and attract the student to further study of this topic.

We began with a definition and some examples of polynomial maps and algebraic automorphisms of *n* dimensional Euclidean space. This tool was later used as a means to establish important notions of equivalence. To gain confidence with algebraic automorphisms, the reader was encouraged through stated exercises to investigate automorphisms of one dimensional complex space, to build an affine transformation of two dimensional space by composition of linear and triangular automorphisms, and to practice building an inverse of a triangular automorphism of Euclidean 3-space. We mentioned the fascinating result that whereas all automorphisms of F^2 are the composition of linear and triangular automorphisms, the same cannot be said for automorphisms of F^3, as illustrated by an automorphism of Nagata. It took over 30 years from the time Nagata suggested it until Shestakov and Umirbaev finally proved conclusively that Nagata's automorphism cannot be expressed as a composition of linear and triangular automorphisms.

Once the definition of an algebraic automorphism was developed, we investigated this tool on the ring level by building for any polynomial map, a corresponding ring homomorphism such that the polynomial map is an algebraic automorphism if and only if the ring homomorphism is a ring automorphism. This gave us a new strategy as to how to view these maps and make conclusions about their behavior and properties – through the power of ring theory. Criteria for identifying when a polynomial map is an automorphism were discussed, including Ax's result that all injective polynomial maps are automorphisms, and the famous Jacobian conjecture – the nearly 80-year old, as-yet unproven claim that if the determinant of the Jacobian matrix of a polynomial map is nowhere zero, then the map is an automorphism.

Time was taken to investigate holomorphic automorphisms. Similarities between these and polynomial automorphisms include that they too have a natural extension to the ring level, (this time to the ring of entire functions), and that still the only automorphisms of the complex line remain the non-constant linear functions. One remarkable difference is that injectivity is no longer a sufficient condition for bijectivity, as it was in the algebraic case.

Next, we saw how automorphisms are used to establish the notion of equivalence of maps, and later, equivalence of closed affine algebraic subvarieties. First, morphisms and isomorphisms of affine algebraic curves were considered. We saw how isomorphisms of closed affine algebraic subvarieties are similarly extended to ring isomorphisms, and used this connection to investigate some classic examples. We saw how the complex line and the parabola are shown to be isomorphic, how the hyperbola is neither isomorphic to nor even homeomorphic to the coordinate axis, and how the semi-cubic parabola, while homeomorphic to the coordinate axis, is *not* isomorphic to it. Finally we saw how $X_{1,1}$ and $X_{k,l}$ are isomorphic, which led us into the discussion of equivalence of affine algebraic subvarieties, which is a stronger relationship than isomorphism.

Through the power of ring theory, we were able to show how the isomorphic $X_{1,1}$ and $X_{k,l}$ fail to be equivalent to each other, while the isomorphic parabola and *x*-axis do share this stricter relationship of equivalence.

We then defined what it means for an embedding to be rectifiable, and reviewed some of the most central theorems of this discipline. Chevalley guaranteed that the closure of the image of any algebraic variety under a morphism will always be an affine algebraic variety. Predominantly, Abhyanker-Moh-Suzuki gave the result that every polynomial embedding of $\mathbb{C}$ into $\mathbb{C}^2$ is rectifiable. Lin-Zaidenberg took this further and found that in the absence of smoothness, a closed affine algebraic curve in $\mathbb{C}^2$ that is homeomorphic to but *not* isomorphic to $\mathbb{C}$, is equivalent to $\{x^k - y^l = 0\}$, with relatively prime k and l. Craighero-Jelonek similarly stood on the shoulders of the giants Abhyankar-Moh-Suzuki and found that every polynomial embedding of $\mathbb{C}$ into $\mathbb{C}^n$ is rectifiable for $n \geq 4$. Though it is still not known if this holds for $n = 3$, we have a theorem of Kaliman that guarantees for all polynomial embeddings of $\mathbb{C}$ into $\mathbb{C}^3$, a holomorphic automorphism of $\mathbb{C}^3$ with a coordinate line as its image.

We then discussed manifolds and embeddings. We gave some classic examples such as the punctured torus, and counter-examples such as the cross and the semi-cubic parabola, which are not manifolds. This led us to some interesting results concerning optimal embedding dimensions of manifolds. Kaliman, Srinivas and Nori took the Craighero-Jelonek Theorem a step further with the result that every isomorphism of two closed affine algebraic submanifolds of dimension N can be extended to an automorphism of $\mathbb{C}^N$ if $N \geq 2n+2$. Finally, we made it to the Gromov-Eliashberg-Schürmann result that for $n \geq 2$, any Stein holomorphic n-manifold can be embedded into $\mathbb{C}^N$ when $N = \lfloor \frac{3n}{2} \rfloor + 1$. We saw also Fornæss Wold's results giving the best current analogue of Gromov-Eliashberg-Schürmann for the case $n = 1$.

We spent some time focusing on the field $\mathbb{R}$, and in specific, embeddings and proper embeddings of $\mathbb{R}$ into $\mathbb{R}^3$. We saw that in 3 dimensional real space, a straight line and a knot are not equivalent, and saw a particular example of a knot of Shastri that is not rectifiable in real space, but is not yet known to be rectifiable or not in complex space. We spoke briefly of the many remarkable results of equivalence classes such as Neumann's theorem about embeddings of once punctured curves, Nakazawa and Oka's classification of embeddings of all smooth once-punctured curves of genus up to 16, and Russel, Koras and Cassou-Nogues' classification of almost all equivalent embeddings of $\mathbb{C}^*$ into $\mathbb{C}^2$, whose completion is expected to be published soon by Koras.

Finally we gave some attention to embeddings of higher dimensional Euclidean spaces. A consequence of the Kaliman-Srinivas-Nori theorem gives that every proper algebraic embedding $\Phi : \mathbb{C}^k \to \mathbb{C}^n$ is rectifiable for $n \geq 2k+2$. The conjecture of

Abhyankar-Sathaye that every proper algebraic embedding of $\mathbb{C}^{n-1} \to \mathbb{C}^n$ is rectifiable is mentioned. The incredible Russel cubic which has been shown to be diffeomorphic to $\mathbb{R}^6$ but *not* isomorphic to $\mathbb{C}^3$ is noted as an example of a non-rectifiable hypersurface in $\mathbb{C}^4$ that is not isomorphic to $\mathbb{C}^3$. Finally, we present the conjecture of Zariski-Ramanujan, that if a complex affine algebraic variety X is such that $X \times \mathbb{C} \simeq \mathbb{C}^{n+1}$, then $X \simeq \mathbb{C}$. This has been proven so far only for the case of $n = 2$ by Fujita.

It is our hope that this paper may serve as an introduction for the graduate mathematics student to this area of the discipline. Perhaps exposure to this topic will have inspired the next Keller to go forth and study and eventually bring another worthy conjecture for contemplation, or even better - the next Wiles to supply finally a proof to a waiting conjecture such as the Jacobian!

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APPENDIX A

ALGEBRA REVIEW TO PRECEDE CHAPTER 7

We now go through a short review of some algebraic concepts that are necessary for results used in Chapter 7 of the text. It is assumed that the reader has some working knowledge of groups, rings and fields, and the definitions below are a refresher, not an introduction. For the following, let R be a commutative ring with identity.

Definition A1. R^* is the multiplicative group formed by the set of all units of R, where a unit of R is an element of R that has a multiplicative inverse in R.

Example A2. $\mathbb{C}\left[x_1, \ldots, x_n\right]^* = \mathbb{C}^* =$ all non-zero complex numbers.

Definition A3. Two elements a and b of a ring are said to be **associates** if there exists a unit u in R such that $a = ub$. In this case we write $a \sim b$.

Exercise A4. Show that ~ is an equivalence relation. Also, show that when a and b are associates, the principal ideal generated by a, denoted (a), is the same as the principal ideal generated by b.

Definition A5. To say that an element a of a ring is **irreducible** means that if $a = bc$ (where b and c are elements of the ring), then either $a \sim b$ or $a \sim c$.

Example A6. In $\mathbb{C}[x]$ only the linear functions are irreducible.

Definition A7.

(1) An element p of a ring is called **prime** if $p|ab \Rightarrow p|a$ or $p|b$, where $p|a$ (p divides a) means $\exists! c \in R$ with $pc = a$. Fact: if an element is prime, then it is irreducible.

(2) A **domain** is a ring with no zero divisors (where a zero divisor is a non-zero element of the ring that can be multiplied by another non-zero element of the ring, giving the product $\underline{0}_R$).

(3) A domain R is called a **Unique Factorization Domain (UFD)** if $\forall a \in R \setminus R^*, a \neq 0$, a can be written in the form $a = up_1^{s_1} \cdots p_t^{s_t}$ where u is a unit in the ring, each p_i is a distinct irreducible in the ring, each s_i is a positive integer, and this representation of a is unique up to permutation or switching to associates. Whereas primes are always irreducible, in a UFD all irreducibles are prime as well. The ring $\mathbb{C}[x_1,\ldots,x_n]$ is a UFD.

Example A8. The integers form a UFD. The rings $\mathbb{C}[x]$ and $\mathbb{C}[x,y]$ are UFDs.

An important result we have involving these concepts is that the following are equivalent in a commutative ring R:

(i) $p \in R$ is a prime element

(ii) the ideal generated by p, denoted (p), is a prime ideal

(iii) the quotient ring $R/(p)$ is a domain

A principal ideal, pR is prime IFF p is prime. Also, two ideals, pR and qR are the same IFF their generating elements p and q are associates.

APPENDIX B

SOLUTIONS TO SELECTED EXERCISES

Exercise 2.4. Use the Fundamental Theorem of Algebra to show that if the polynomial map $\Phi : \mathbb{C} \to \mathbb{C}$ is of degree > 1, then it cannot be one-to-one. Similarly, a zero-degree polynomial (a constant) is not one-to-one. This leaves only one-degree polynomials (non-constant linear functions) which are all algebraic automorphisms of $\mathbb{C}$.

Exercise 2.7. Let H be the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then consider the following automorphisms:

$$\Phi_{HA} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a_{21}x + a_{22}y \\ a_{11}x + a_{12}y \end{pmatrix} \qquad \text{(linear with matrix } HA\text{)}$$

$$\Psi_{b_1} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y + b_1 \end{pmatrix} \qquad \text{(triangular with } p(x) = b_1\text{)}$$

$$\Phi_{H} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix} \qquad \text{(linear with matrix } H\text{)}$$

$$\Psi_{b_2} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y + b_2 \end{pmatrix} \qquad \text{(triangular with } p(x) = b_2\text{)}$$

It is readily checked that $\Psi_{b_2} \circ \Phi_H \circ \Psi_{b_1} \circ \Phi_{HA} = \Lambda$, showing one way that the affine transformation Λ can be constructed by the composition of linear and triangular automorphisms.

Exercise 2.8. (A) The inverse of the triangular automorphism defined as $(x,y,z)\mapsto(x,y+p(x),z+q(x,y))$ is given by $(x,y,z)\mapsto(x,y-p(x),z-q(x,y-p(x)))$.

(B) Given $\Phi:(x,y)\mapsto(x,y+x^3)$ and $\Psi:(x,y)\mapsto(x+y^3,y)$, the compositions are given by $(\Phi\circ\Psi)(x,y)=\left(x+y^3,y+\left(x+y^3\right)^3\right)$ and

$$(\Psi\circ\Phi)(x,y)=\left(x+\left(y+x^3\right)^3,y+x^3\right).$$

Exercise 3.1. (B) $(\Rightarrow)$ If Φ is an algebraic automorphism of F^n, then let Ψ be the inverse of Φ, and define $\Psi^*:F\left[x_1,...,x_n\right]\to F\left[u_1,...,u_n\right]$ according to $\Psi^*(g)=g\circ\Psi$. Then Ψ^* is the inverse of Φ^*, making Φ^* a ring isomorphism whenever Φ is an algebraic automorphism.

$(\Leftarrow)$ If Φ^* is an isomorphism of rings, then define $\Psi=(\psi_1,...,\psi_n):F^n\to F^n$ according to $\psi_i=\Psi^*(x_i)$ where $x_i:F^n\to F$ is given by $(x_1,...,x_n)\mapsto x_i$. Then Ψ is the inverse of Φ so that whenever Φ^* is an isomorphism of rings, then Φ is an algebraic automorphism.

Exercise 3.4. In the one-dimensional case, the Jacobian of Φ is simply its derivative, which is a polynomial in one variable. As a simple consequence of the Fundamental Theorem of Algebra, if this polynomial is never zero, it must be a non-zero constant. As for higher dimensions, the Jacobian is obtained from multiplying and adding of the first partials, which are all polynomials, and is thus itself a polynomial in n variables. The Fundamental Theorem of Algebra still applies and thus if this polynomial is non-constant then it has a zero.

Exercise 3.5. From Exercise 3.4 we have that if the Jacobian is nowhere zero then it is a non-zero constant. In dimension 1 the Jacobian being a non-zero constant gives that Φ is a non-constant linear function. Then from Exercise 2.4 we have that Φ is an automorphism.

Exercise 3.6 The Jacobian matrix of the Nagata is:

$$\begin{bmatrix} 1+2xz & -z^2 & x^2-2yz \\ 6x^2-2yz+4x^3z-4xyz^2 & 1-2xz-2x^2z^2+2yz^3 & -2xy+x^4-4x^2yz+3y^2z^2 \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant is calculated as

$$(1+2xz)(1-2xz-2x^2z^2+2yz^3)(1)+0+0-0-0-(-z^2)(6x^2-2yz+4x^3z-4xyz^2)(1)$$

which simplifies to 1.

Exercise 4.6. If *f* has only finitely many non-zero coefficients, then by the Fundamental Theorem of Algebra *f* must be an affine transformation in order to be a holomorphic automorphism (as shown in Exercise 2.4).

When *f* has infinitely many non-zero coefficients, we can show that *f* is not a holomorphic automorphism by employing a change of variable and calling on the Casorati-Weierstrass Theorem which states that an analytic function comes arbitrarily close to all values in any neighborhood of an essential singularity.

Consider $f(z)=\sum_i a_i z^i$ with finitely many zero coefficients, and f holomorphic on all of $\mathbb{C}$. Using a change of variable, $w=\frac{1}{z}$ we get a function which is holomorphic for all w except $w=0$, has an essential singularity at $w=0$, and has infinitely many non-zero terms with negative exponents. The Casorati-Weierstrass Theorem applies to this function and gives that it is dense in $\mathbb{C}$ on any open disc with radius $R > 0$ and center $w = 0$. Such a disc corresponds to the exterior of a disc of points z centered at $z = 0$. Thus $f(z)$ is dense in $\mathbb{C}$ on the exterior of any open disc centered at $z = 0$. Consider such an open disc, D of radius r, centered at $z = 0$. We know that $f(z)$ maps this disc holomorphically to a disc of radius R, with center $f(0)$. Since f is dense in $\mathbb{C}$ on the exterior of the disc D, we can find a z_0 outside of D, with $f(z_0) = f(0)$, although $z_0 \neq 0$. Therefore, f is not injective, and is not an automorphism.

Exercise 4.7 This essentially a repetition of Exercise 3.1, but in the analytic case.

Exercise 5.4. Let us call the hyperbola $\{(x, y) \in \mathbb{C}^2 \mid xy = 1\} = H$. Then the ring of regular functions on H is denoted $\mathbb{C}[H]$. An obvious map to build would be $\Phi : \mathbb{C}[H] \to \mathbb{C}[t, t^{-1}]$ defined by $\Phi : f(x, y) \mapsto f(t, t^{-1})$, which gives a natural isomorphism between the ring of regular functions on the hyperbola $xy = 1$ and $\mathbb{C}[t, t^{-1}]$.

Exercise 5.5. Construct the surjective homomorphism $\Phi : F[x_1, ..., x_n] \to F[X]$ defined by $\Phi : f \mapsto f|_X$. Its kernel is I. Then use the First Isomorphism Theorem for Rings which states that in a ring homomorphism, the domain mod the kernel is isomorphic to the image, giving us here that $F[x_1, ... x_n]/I \simeq F[X]$.

Exercise 5.9. $(\Rightarrow)$ If Φ is an isomorphism with inverse Ψ, then define $\Psi^*: F[X] \to F[Y]$ by $\Psi^*(g) = g \circ \Psi$. Then it is readily verified that $\Psi^* = (\Phi^*)^{-1}$ so that Φ^* is a ring isomorphism whenever Φ is an isomorphism.

$(\Leftarrow)$ If Φ^* is an isomorphism of rings, then build $\Psi = (\psi_1, ..., \psi_n): Y \to X$ according to $\psi_i = f_i$ where $f_i \in F[Y]$ is the pre-image of the projection map $g_i \in F[X]$ under Φ^*, where $g_i : X \to F$ is defined by $g_i : \bar{x} = (x_1, ..., x_n) \mapsto x_i$. Such an f_i is guaranteed to exist for each $i \in \{1, ..., n\}$ due to the surjectivity of Φ^*. It is easily checked that Ψ is a morphism, $\Phi^*(\psi_i) = \psi_i \circ \Phi = g_i$ for all *i*, and in fact Ψ is the inverse morphism of Φ, so that Φ is an isomorphism whenever Φ^* is a ring isomorphism.

Exercise 5.11. Its inverse is given by $\Psi: Y \to X$ defined by $(x, x^2) \to x$.

Exercise 5.12. ρ is clearly not an isomorphism, as it is not surjective. In particular, 0 is not in the image of ρ.

Exercise 5.13. ρ is a homeomorphism, as it is continuous and bijective, with a continuous inverse given by $\rho^{-1}: Y \to X$ by $(x^2, x^3) \mapsto x$. However, ρ is not an isomorphism, as the associated ring homomorphism, $\rho^*: \mathbb{C}[Y] \to \mathbb{C}[X] = \mathbb{C}[t]$ defined by $\rho^*(f) = f \circ \rho$ is not an isomorphism, because all polynomials in the image have zero derivative at $t = 0$. In particular, ρ^* is not surjective, as the polynomial $t \in \mathbb{C}[t]$ is not in the image of ρ^*.

Exercise 5.14. Show that when *k* and *l* are relatively prime we can find positive integers m', m'', n' and n'' such that $m'k - n'l = 1$ and $m''l - n''k = 1$. Then check that the map $\Psi : X_{k.l} \to X_{1,1}$ by $(u,v) \mapsto (u^{m''}v^{n''}, u^{n'}v^{m'})$ is the inverse of Φ, showing that $X_{1,1}$ and $X_{k,l}$ are isomorphic.

Exercise 6.2. Consider $\widetilde{\Phi} : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $\widetilde{\Phi} : (x,y) \mapsto (x, y + x^2)$. It is an automorphism with inverse $\widetilde{\Psi} : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $\widetilde{\Psi} : (x,y) \mapsto (x, y - x^2)$, the restriction of which is the isomorphism $\Phi : X \to Y$. Thus *X* and *Y* are not only isomorphic, they are equivalent.

Exercise 6.4. To show $xy - 1$ is irreducible check first that no two non-constant polynomials can multiply to give $xy - 1$. Thus whenever two polynomials multiply to give $xy - 1$, one of them is a unit, making the two associates, and $xy - 1$ irreducible. For $x^k y^l - 1$ where *k* and *l* are relatively prime, recall that in a UFD such as $\mathbb{C}[x,y]$, irreducibles and primes are the same. Therefore $xy - 1$ which was shown above to be irreducible, is also prime. Apply the result that $p \in R$ is a prime element $\Leftrightarrow$ the quotient ring $R/(p)$ is a domain, gives that $\mathbb{C}[x,y]/(xy-1)$ is a domain. Now, since $\mathbb{C}[x,y]/(xy-1) \simeq \mathbb{C}[X_{1,1}] \simeq \mathbb{C}[X_{k,l}] \simeq \mathbb{C}[x,y]/(x^k y^l - 1)$, we have that $\mathbb{C}[x,y]/(x^k y^l - 1)$ must also be a domain, and therefore $x^k y^l - 1$ is prime, so $x^k y^l - 1$ is irreducible.

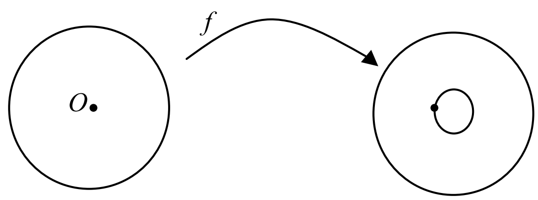
Exercise 6.7. An answer to the first question is $\sqrt{|t|}$. In the second question, for any *f* that vanishes at $t=0$, one can consider $\sqrt{|f|}$.

Exercise 7.2. The proof is very similar to Proof 7.1 of the text.

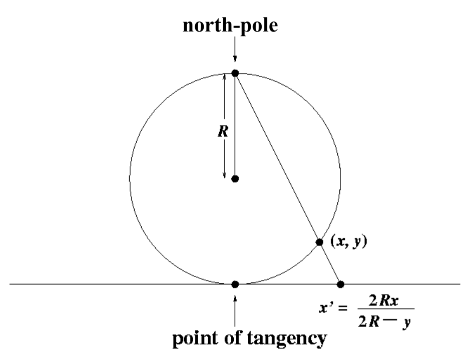
Exercise 9.7. Assume that there is a compact analytic curve that can be embedded into the Euclidean space $\mathbb{C}^n$. Then it would be analytic in any disc, and have no maximum there as it is an embedding with derivative nowhere zero. However, because it is compact, it must attain a maximum on the disc, and so by the maximum principle, it must be constant. Therefore it is a point, not an analytic curve.

Exercise 9.8. Consider these structures in their polar form, each point $z=a+bi$ represented as (r,θ) where $r=\sqrt{a^2+b^2}$, $a=r\cos\theta$, and $b=r\sin\theta$. The homeomorphism $f:\Delta^*\to A_n$ defined by $f(r,\theta)=\left(r_1+r\frac{r_2-r_1}{R},\theta\right)$ and with inverse, $f^{-1}:A_n\to\Delta^*$ given by $f^{-1}(r,\theta)=\left(R\frac{r-r_1}{r_2-r_1},\theta\right)$ shows that the annulus and the punctured disc are homeomorphic.

Proof that they are not biholomorphic: Consider *f*, a bounded analytic map from Δ^* to A_n. By the Riemann theorem about deleting a singularity, *f* extends to the origin. Then the origin would map to a point on the boundary of the annulus. However, the disc and the annulus plus a point on the boundary are not even homeomorphic, thus the punctured disc and the annulus are not biholomorphic.



Exercise 10.1. On the real x, y plane, take S^1 to be the circle of radius R, centered at $(0, R)$ and therefore tangent to the x–axis at the origin. That is, $S^1 = \left\{(x, y) \in \mathbb{R}^2 \middle| \; x^2 + (y-R)^2 = R^2\right\}$. Call the point $(0, 2R)$ on the circle the "north-pole". Now, for each point $P = (x, y)$ on S^1 that is not the north-pole, draw the line segment connecting the north-pole to the x–axis through P. This line segment hits the x–axis at $x' = \dfrac{2Rx}{2R-y}$.



This gives a one-to-one, onto function from $S^1 \setminus (0, 2R)$ to $\mathbb{R}$ (represented here by the x-axis). Now map the north-pole to infinity, and this shows that the circle and the reals union infinity are the same.

Exercise 10.6. Use Kaliman Theorem (8.6).

Exercise 10.8. We showed in Exercise 10.1 that $\mathbb{R}^2 \cup \infty = S^2$. We also know that $\mathbb{R}^2$ and $\mathbb{C}$ are the same. Since $\mathbb{C}^*$ is just $\mathbb{C}$ with zero removed, consider starting with S^2 and removing both 0 and ∞. When you remove ∞ from S^2, you are back to $\mathbb{R}^2$ (which is $\mathbb{C}$), and then when you remove 0, you have left $\mathbb{C}^*$. Therefore $\mathbb{C}^*$ is just a twice punctured S^2. That is, $\mathbb{C}^*$ is S^2 with both 0 and ∞ removed.

APPENDIX C

GLOSSARY OF NOTATION

Entries here are given by order of appearance, and each entry is listed under the chapter or appendix in which the symbol first appears.

CHAPTER 1

$\{(x,y)\,|\,P(x,y)=0\}$; set notation. Here specifically we indicate the set of all points (x,y) such that the polynomial P maps (x,y) to zero.

$\in$; indicative of an element being a member of a set.

$\mathbb{Z}$; the set of integers.

CHAPTER 2

$\mathbb{C}$; the field of the complex numbers. That is, the set of all $a+bi$ where a and b are real numbers and $i^2=-1$.

$\mathbb{R}$; the field of real numbers.

$\mapsto$; indicating where a particular element is sent under the mapping in discussion.

Φ, φ; respectively, the upper and lower case Greek letters, Phi.

$\Phi : F^n \to F^n$; indicating that the map Φ maps the domain F^n to the codomain F^n.

$F[x_1, ..., x_n]$; the ring of polynomials in *n* variables with coefficients in the field *F*.

$\exists$; there exists.

Ψ, ψ; respectively, the upper and lower case Greek letters, Psi.

$\Phi \circ \Psi$; the composition of the two maps, Φ composed with Ψ, here composition is from right to left.

Id_{F^n}; the identity function on F^n. Explicitly, the function which maps each element of F^n to itself. In general, Id_R is the identity function on the set *R*.

$\forall$; for all

$\mathbb{C}^2_{x,y}$, $\mathbb{C}^2_{u,v}$; two-dimensional complex space with coordinates specified as *x* and *y* or, respectively, *u* and *v*.

A^{-1}; the inverse of *A*. Here, *A* is a matrix, so this is multiplicative inverse. However, the -1 superscript is also used as compositional inverse.

$\mathbb{C}[x]$; the field of polynomials in variable *x* with coefficients in the complex numbers.

Λ; the upper case Greek letter, Lambda.

CHAPTER 3

$J(\Phi)$; the Jacobian of a map Φ. That is, the determinant of the Jacobian matrix of the map Φ. If the Jacobian is non-zero, the equations have a non-trivial solution.

$\left(\frac{\partial \varphi_i}{\partial x_j}\right)_{i,j=1}^{n}$; the Jacobian matrix of the map $\Phi = (\varphi_1,...\varphi_n): \mathbb{C}^n \to \mathbb{C}^n$. Specifically the $n \times n$ matrix whose i^{th} row is the vector of partial derivatives of the i^{th} function.

CHAPTER 4

$\sum_I a_I \bar{z}^I$; summation notation. Here, specifically indicating the sum of all terms of the form $a_I z^I$ such that $I = (i_1,...i_n)$ with each i_k a non-negative integer, $a_I \in \mathbb{C}$ and $\bar{z}^I = z_1^{i_1} \cdots z_n^{i_n}$.

$\mathbb{C}\{z_1,...,z_n\}$; the ring of entire functions on $\mathbb{C}^n$.

e; the Euler number. Specifically, $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots$, or $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. The approximate value of e is 2.718281828.

$\sum_{n=0}^{\infty} \frac{z^n}{n!}$; again, summation notation. Here, the sum of all terms of the form $\frac{z^n}{n!}$ as n ranges through the non-negative integers.

CHAPTER 5

$\subset$; a subset of. For example, $\{y - x^2 = 0\} \subset \mathbb{R}^2$ refers to the set of all points (x, y) with $y = x^2$ and $x, y \in \mathbb{R}$ so that the set $\{y - x^2 = 0\}$ is a subset of $\mathbb{R}^2$.

$F[X]$; where X is a closed affine algebraic subvariety of F^n, the notation $F[X]$ denotes the ring of regular functions on *X*. That is, the restriction of the ring $F[x_1, ...x_n]$ to *X*.

$\simeq$; is isomorphic to. Used to indicate when two algebraic structures are isomorphic to each other. That is, when an isomorphism exists between the two.

$F[x_1, ...x_n]/I$; more generally, R/I where *R* is any ring and *I* is an ideal of the ring. This denotes the quotient ring "*R* mod *I*".

$\widetilde{\Phi}$; pronounced "phi tilde". Used here often to represent a map whose restriction to a closed affine algebraic subvariety is also of importance.

Φ^{-1}; the inverse of Φ. Here, Φ is a map, so this refers to the compositional inverse. That is, Φ^{-1} is the map which, when composed with Φ gives the identity map on the domain of Φ.

ρ; the lower case Greek letter, rho.

$X_{1,1}$; the set of all $(x, y) \in \mathbb{C}^2$ such that $xy - 1 = 0$.

$X_{k,l}$; the set of all $(x, y) \in \mathbb{C}^2$ such that $x^k y^l - 1 = 0$, where $k, l > 0$ are relatively prime.

$\Leftrightarrow$; if and only if.

CHAPTER 6

$F^{[n]}$; another notation for $F\left[x_1, ..., x_n\right]$, the ring of polynomials in n variables with coefficients in the field F.

CHAPTER 7

$I_{k,l}$ and $I_{1,1}$; the ideals of functions vanishing on $X_{k,l}$ and respectively, $X_{1,1}$.

CHAPTER 8

$\mathbb{C}^*$; the units of $\mathbb{C}$, namely the non-zero complex numbers. In general, the notation R^* denotes the units of the ring R.

$\overline{\Phi(X)}$; the closure of $\Phi(X)$.

CHAPTER 9

$\cap$; intersection. As in, the set of common elements of two sets.

$f|_{X \cap U}$; the function *f* restricted to the subset $X \cap U$ of its domain.

B^n ; a unit ball in *n*-space. That is, $B^n = \{x| \quad |x| < 1\}$.

$\nabla p = \langle p_x, p_y \rangle$; the gradient of the map *p*.

$\left\lfloor \frac{3n}{2} \right\rfloor$; a specific use of the floor function, where in general $\lfloor x \rfloor =$ the largest integer no larger than *x*.

$A_n = \{z \mid 0 < r_1 < |z| < r_2\}$; an annulus. Here specifically, the open disc centered at $z = 0$ of radius r_2, with a hole. The hole is has been made by removing the closed disc centered at $z = 0$ of radius $r_1 < r_2$.

$\Delta^* = \{z \mid 0 < |z| < R\}$; a punctured open disc. Here specifically, the open disc of radius *R* centered at $z = 0$, but with the point $z = 0$ removed. This removal of a point is a "puncture".

CHAPTER 10

S^2, S^3, S^n and S^1 ; S^2 is the two-sphere – the set of all points $(x, y, z) \in \mathbb{R}^3$ a specific distance from a specific point. In general, S^n is the *n*-sphere – all *n*-tuples in $\mathbb{R}^{n+1}$ that are a specific distance from a specific point. In particular, the one-sphere, S^1 is a circle.

$\mathbb{R}^3 \setminus L$, $\mathbb{R}^3 \setminus K$; the "\" indicates removal. Thus these are representing three-dimensional real space with *L* and *K*, respectively, removed, where *L* is a straight line and *K* is a knot.

$\pi_1(X)$, $\pi_1(Y)$, $\pi_1(\mathbb{R}^3 \setminus L)$, and $\pi_1(\mathbb{R}^3 \setminus K)$; the Poincaré fundamental groups of *X*, *Y*, $\mathbb{R}^3 \setminus L$ and $\mathbb{R}^3 \setminus K$ respectively.

$\hat{r}$; pronounced "*r* hat". Used here in reference to a knot discussed in the Shastri theorem.

$\mathbb{C}^2 \times \mathbb{C}^*$; the Cartesian product of $\mathbb{C}^2$ and $\mathbb{C}^*$. In general, the Cartesian product of two sets *A* and *B* is defined by $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$.

Γ; the capital Greek letter, gamma. Used here to represent an algebraic curve, $\Gamma = \{f(x,y) = 0\} \subset \mathbb{C}^2$.

S_R^3; a three-sphere with radius specified as *R*.

γ_R; pronounced "gamma sub-*R*". Defined here as the link $\gamma_R = \Gamma \cap S_R^3$ where Γ is an algebraic curve $\Gamma = \{f(x,y) = 0\} \subset \mathbb{C}^2$.

APPENDIX A

$a \sim b$; indicating that *a* and *b* are associate elements in their ring.

$p \mid ab$; indicating that *p* divides *ab*.

$\underline{0}_R$; the zero element in the ring *R*. That is, the additive identity in the ring.

(p) or pR; the ideal generated by the element p. That is, $\{pr \mid r \in R\}$.

IFF; also, $\Leftrightarrow$. "if and only if".

APPENDIX B

$(\Rightarrow)$; indicating that a proof of the "if" part of an *if and only if* statement follows.

$(\Leftarrow)$; indicating that a proof of the "only if" part of an *if and only if* statement follows.

APPENDIX D

GLOSSARY OF TERMS

Absolutely convergent, (of a series) such that the series of absolute values of its terms converges. [8]

Algebraic geometry, the study of geometry by algebraic methods. [8]

Associates, two elements a and b of a ring R such that there exists a unit u in R with $a = ub$. In this case we write $a \sim b$.

Automorphism, as isomorphism, the domain and range of which are identical. [8]

Biholomorphic mapping, (or, conformal), a holomorphic function that is one-to-one, onto, and has a holomorphic inverse. [17]

Cartesian product, the set of ordered n-tuples, the elements of which are respectively members of any sequence of given sets. [8]

Casorati-Weierstrass Theorem, states that an analytic function comes arbitrarily close to all values in any neighborhood of an essential singularity, that is, that the image of any ball centered on the singularity is dense in the complex numbers. [8]

Closure, the set of points in a space every neighborhood of which has a non-empty intersection with a given set. [8]

Compact, having the property that every collection of open sets the union of which is the whole space has a finite subcollection with the same property. [8]

Diffeomorphism, a differentiable mapping that has a differentiable inverse. [8]

Disc, an open or closed ball – the set of all points whose distance from a fixed point is less than (or equal to in the closed disc) a fixed number, the radius of the disc.

Endomorphism, a homomorphism of a structure into itself. [8]

Embedding, another word for an injection. That is, a mapping whereby different members of the domain are associated with different members of the range. [8]

Field, a commutative ring whose non-zero elements form a group under multiplication. [40]

First Isomorphism Theorem for Rings, the result that, in a ring homomorphism, the domain mod the kernel is isomorphic to the image.

Genus, (of an algebraic plane curve) the difference between the maximum number of double points (points at which a curve intersects itself) a curve of the given degree may possess, and the actual number of the given curve. [8]

Group, a non-void set that is closed under an associative binary operation with respect to which there exists a unique identity element within the set, and every element has an inverse within the set. [8]

Hole, the removal of a closed disc from a space.

Homeomorphic, related by a homeomorphism. [8]

Homeomorphism, a one-to-one correspondence that is continuous in both directions between the points of two geometric figures or between two topological spaces. [8]

Homomorphism, (of rings) a function between rings that commutes with the group operation, preserve multiplicative identity, and commutes with the multiplicative operation. [8]

Ideal, a subring of a ring that is closed under subtraction and under multiplication by any ring element whatever. [8]

Injective, a function $f : X \to Y$ is *injective* or *1-1* ("*one-to-one*") provided that if x_1 and x_2 are distinct elements of *X*, then $f(x_1)$ and $f(x_2)$ are distinct elements of *Y*. [40]

Isomorphism, (of rings) a bijective homomorphism. [8]

Kernel, (of a homomorphism), the pre-image of the zero element of a ring under the homomorphism.

Neighborhood, a neighborhood of a point *p* in a space *X* is a subset of *X* which includes an open set containing *p*.

Node, a point at which a continuous curve crosses itself.

One-Point Compactification, adding a single point, designated ∞, to a Hausdorff space, in such a manner that the result is compact.

Poincaré fundamental group, denoted as $\pi(X, x)$ consists of all equivalence classes of loops based at *x* and the product operation between them. [42]

Proper holomorphic, holomorphic but not a polynomial.

Prime ideal, in a commutative ring, *R*, an ideal *P* is called a *prime ideal* if $P \neq R$ and whenever the product *ab* of two elements $a, b \in R$ is an element of *P*, then at least one of *a* and *b* is an element of *P*. [10]

Principal ideal, an ideal in a ring that is generated by a single element. [8]

Puncture, the removal of a single point from a neighborhood.

Quotient Ring, of ring *R*, modulo ideal *I*, is the ring of all equivalence classes created by the equivalence relation $a \sim b \Leftrightarrow a - b \in I$.

Relatively prime, a pair of integers not having any common divisors other than unity. [8]

Riemann surface, a topological device for rendering multiple-valued complex functions into single-valued functions. [13]

Ring, a commutative additive group with an additional binary, associative operation (multiplication), such that the distributive law holds, and there is a multiplicative identity [40].

Smooth, (of a function or curve) differentiable at every point. [8]

Stein manifold, a complex submanifold of the vector space of *n* complex dimensions.

Surjective, a function $f : X \to Y$ is *surjective* or *onto* if for all $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$. [40]

Unit, (*Algebra*) a multiplicatively invertible element of a ring. [8]

VITA

Dahlia Zohar was born Dahlia Michal Farber to parents Rabbi Edwin and Laurie Farber in Miami, Florida in 1978. She attended the Goldstein Hebrew Academy in South Dade through 6th grade, and the Samuel Scheck Hillel Community Day School through 12th grade. She then went on to earn a Bachelors of Arts in Mathematics with a minor in Secondary Education (1999) from Queens College, CUNY and a Masters of Science in Secondary Mathematics Education (2000) from the same institution. After four years of freezing and miserable weather in New York, Dahlia moved back to Miami to pursue her doctoral degree at the University of Miami while serving as a teaching assistant in their Mathematics department. In 2003 she began teaching Business Calculus and Statistics in the Management Sciences department of the School of Business Administration at the University of Miami, where she still teaches at the time of this publication. In March 2004 she met Sagee Zohar and in March 2005 married him. Dahlia's most "productive" years were 2006-2010 when she gave birth to daughter Amit Batya (March 2006), son Aviv David (September 2007), daughter Hadar Mazal (June 2009), and son Golan Ben-Tzion (December 2010). After Golan was born, Dahlia stopped having babies long enough to finish her degree, and in May 2012 she was awarded the Doctor of Arts degree in Mathematics with a concentration in Higher Education by the University of Miami. It is an open problem as to whether more babies are to come. Sagee's conjecture is a solid negative, while Dahlia remains hopefully optimistic.

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