High Resolution Simulation of High Reynolds Number Mixing in a 2D Gravity Current Under Variable Forcing

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HIGH RESOLUTION SIMULATION OF HIGH REYNOLDS NUMBER MIXING IN A 2D GRAVITY CURRENT UNDER VARIABLE FORCING

By

Silvia Matt

A DISSERTATION

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HIGH RESOLUTION SIMULATION OF HIGH REYNOLDS NUMBER MIXING IN
A 2D GRAVITY CURRENT UNDER VARIABLE FORCING

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We investigate the impact of high-frequency variability in forcing on the dynamics of a density current using a high-order non-hydrostatic spectral element model, SEOM (Spectral Element Ocean Model) in the streamfunction-vorticity formulation. Turbulent structures and instabilities depend strongly on forcing and boundary conditions. We introduce time-dependent disturbances through forcing at the inlet boundary and through variation in background transport. Steady forcing and forcing at the inlet boundary at very short periods for experiments at Reynolds number, $Re=15,000$, result in a regime where the passage of the gravity current head with a strongly overturning tail gives way to a stable two-layer system with internal waves on the density interface. Time-dependent forcing at intermediate periods results in turbulent flow regimes with a wide range of time and length scales. At longer forcing periods, individual turbulent bore heads are observed propagating through the system. Forcing through variation in background transport rather than at the inlet boundary changes the distribution of density classes across the flow. Experiments at very high Reynolds number, $Re=50,000$, result in a highly non-linear flow regime, where the mixing is less affected by temporal variability in forcing.
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Chapter 1

Introduction

Dense overflows from marginal seas are an important part of the oceanic thermohaline circulation and global climate system. The mixing of these overflows with interior oceanic waters determines the properties and formation rates of most of the intermediate and deep water masses in the world’s oceans. Dense waters from marginal seas flow over shallow sills and enter the open ocean as dense currents. As they descend the continental slope, they mix with the oceanic ambient water, and it is this mixing that ultimately determines the properties of the product waters that enter the global circulation. The problem of accurately implementing gravity current mixing and entrainment in global scale ocean circulation models remains challenging to the present day. These models have difficulties to correctly capture the properties of the product waters due to computational limitations on the explicit resolution and reproduction of the mixing processes. Mixing and entrainment happen on small spatial - both vertically and horizontally - and temporal scales. The dominant mechanism of mixing between density currents and the ambient fluid is via small-scale shear instabilities. In order to numerically represent and resolve the mixing dynamics in overflows, a numerical model requires vertical and horizontal resolution on very small scales. The dynamics relating to small scale mixing in realistic oceanic overflow plumes are still scarcely studied. Unlike laboratory currents, these overflows flow over very gentle
slopes and are subject to many external influences, such as ambient stratification, rotation and the effects of temporal variability in forcing. The impact of such temporal variability on entrainment and mixing in overflows is the subject of this work. In our study, we focus on the impact of high-frequency variability in forcing on the flow evolution and mixing behavior of a density current on a gentle slope.

1.1 Motivation

It has been observed that oceanic overflows are subject to a wide range of temporal variability in forcing. The time scales of this forcing range from tidal in the Mediterranean outflow [Bryden et al., 1994] and the Antarctic slope currents [Padman et al., 2009], to weather-band scale in the Denmark Strait [Girton et al., 2001] and Faroe Bank Channel overflows [Saunders, 1990], and seasonal in the Red Sea [Matt and Johns, 2007]. Investigation of transport and entrainment in the Red Sea outflow plume also shows that the estimated transport of Red Sea water from the Bab-el-Mandeb strait downstream into the Gulf of Aden is subject to the effects of strong sub-mesoscale temporal variability. The impact of this time-dependent forcing on the outflow plume could not be explicitly resolved by the available observational data [Matt and Johns, 2007].

A very recent study of internal waves in the Mediterranean outflow finds that regions of high dissipation within the dense outflow are also regions where large-amplitude oscillations can be observed [Peters et al., 2010]. The same study also finds strong tidal modulation of the gravity current and available potential energy related to super-tidal processes.

Generally, oceanographic models studying gravity currents and overflows neglect tides or other temporal variations. It is not yet understood how the phenomena induced by these time-dependent variations in forcing affect the mixing of the dense overflow currents with the oceanic ambient waters. Attempting to understand this problem provided the initial
motivation for the present study.

### 1.2 Computational challenges

When attempting to model mixing in gravity currents, we have to address a number of significant computational challenges. There are requirements on the model resolution: We need small vertical and horizontal grid spacing, as well as small enough time steps, to resolve the scales at which the mixing and turbulence occur.

Furthermore, in traditional finite difference ocean models, numerical dissipation can contribute a considerable amount to the total dissipation. This purely numerical dissipation can make it difficult to understand the details of the mixing processes and turbulence in the flow. We have eliminated numerical dissipation in our study through the use of a high-order spectral element method. Furthermore, we attempt to model the instabilities and turbulence directly without resorting to a sub-grid scale (SGS) model, which would smooth over the instabilities and parameterize the underlying physics. Examples of SGS models are the Smagorinsky model [Smagorinsky, 1963] used in Large Eddy Simulation (LES), the K-Profile Parameterization (KPP) [Large et al., 1994][Large et al., 1997] and Turner’s parameterization (TP) [Turner, 1986][Hallberg, 2000] of layered models, or the Reynolds Averaged Navier-Stokes (RANS) models used in turbulence closure (see for example Sagaut [1998]). Our numerical approach here can be considered a Direct Numerical Simulation (DNS) without a specific SGS model. This type of modeling requires very high model resolution, e.g. a fine grid and high spectral order.

This use of high-order state-of-the-art numerical methods as well as the very high model resolution necessary in this study come at a considerable computational cost. The computational resources available to us, though considerable - especially since the recent availability of the Center for Computational Science at the University of Miami, allowing for high-performance parallel runs on tens to hundreds of processors - limited our study to a
pure process study, removed from the large-scale oceanographic setting.

The research we present here, while motivated by larger scale oceanographic processes, is therefore in essence an idealized fluid dynamics process study.

1.3 Statement of the problem

We simulate, in very high resolution and detail, a gravity current on a gentle slope, in 2D and without rotation. Disturbances are introduced in the domain by adding time-dependent variations in forcing. These disturbances are high-frequency oscillations, their periods are short compared to the time it takes the dense current to cross the domain.

We work at Reynolds numbers much higher than generally used in numerical models or laboratory experiments, coming to within two orders of magnitude of the oceanic value. Reynolds numbers this high are prohibitive in the laboratory and generally unheard of in numerical simulations of gravity currents resolving turbulent overturning scales and mixing.

Our main interest is in the question whether the disturbances introduced through time-dependent variability in forcing affect the overall mixing and entrainment in the current. We ask, are mixing and flow regime sensitive to conditions imposed on the current, variability in forcing, and different types of forcing (baroclinic versus barotropic)?

In the transition from waves on the interface between two layers of fluids of different density to that of breaking waves and turbulence through shear instability, which can be observed in our simulations, it is of interest to relate our results to linear stability theory.

The findings from this work help to advance our understanding of mixing in density currents at very high Reynolds numbers in an idealized setting and under the influence of time-dependent forcing. The results, while from an idealized process study, are expected to be helpful to our understanding of mechanisms which may be at work in oceanic density currents.
1.4 Outline

We present the work by starting with a review of the theoretical background relevant to our problem in Chapter 2. Gravity currents are briefly introduced, as well as internal waves and waves at a density interface. The Kelvin-Helmholtz instability and related phenomena are discussed and we highlight certain special features of two-dimensional turbulence.

The numerical model is presented in Chapter 3, along with the equations for the budget of kinetic energy and enstrophy. The model setup and parameters are explained. Details of the numerical grid are given, and we explain how the time-dependent variations are introduced in the model. The chapter closes with a matrix of experiments performed as part of this work. Additional information about the model such as the spectral element formulation and details on boundary condition implementation can be found in Appendix A.

The core results of this work are presented in individual chapters based on the different types of forcing applied. We start in Chapter 4 by presenting exploratory experiments of a gravity current on a free-slip bottom at very high Reynolds number. These experiments were done without added time-dependent variations in forcing.

Chapter 5 extends these experiments on a no-slip bottom at moderately high Reynolds number, and includes time-dependent forcing, using both baroclinic and barotropic forms of forcing, for a range of forcing periods. The resulting flow regimes are discussed and the impact of the forcing on mixing is investigated.

The following chapter, Chapter 6, provides a detailed analysis from a select number of experiments performed at very high Reynolds number, on a no-slip bottom and with time-dependent forcing.

Energy and enstrophy in the flow are presented in Chapter 7, and entrainment rates calculated via the entrainment parameter are discussed.

The transition of waves to turbulence and how our results relate to linear stability theory
are considered in Chapter 8.

We conclude by summarizing results and discussing the broader impact and importance of our findings in Chapter 9.
Chapter 2

Background

The gravity current simulations exhibit a range of flow features and flow regimes, including turbulence. These have been the subject of numerous investigations. Here we review some of the most salient results, paying particular attention to those of an analytical/asymptotic nature. The regimes in question include (Figure 2.1):

- a) waves on a density interface
- b) Kelvin-Helmholtz instabilities
- c) Tollmien-Schlichting waves near the channel inlet and the bottom boundary
- d) gravity currents and hydraulic jumps

The last three are intimately involved in mixing while the first is not. However, unstable waves can lead to the generation of Kelvin-Helmholtz rolls and subsequent mixing. We note that by necessity most of the results above are obtained from perturbation analysis in highly idealized contexts. Their applicability to characterize processes in the present simulations have to be considered with care.

In our study, we investigate the mixing dynamics of a gravity current on a gentle slope (Figures 2.1, 2.3). We will thus start by reviewing results that describe the large-scale flow,
Figure 2.1  Picture of gravity current simulation (density field) illustrating the hydrodynamic phenomena we observe in our experiments: (a) waves on a density interface, (b) Kelvin-Helmholtz instabilities, (c) possibly Tollmien-Schlichting instabilities and (d) gravity current head.

the gravity current, before we go on to present background relevant to the smaller-scale phenomena we observe embedded in the large-scale flow.

2.1 Gravity currents

The simplest case of a gravity current is the lock-exchange, which is the intrusion of a front into a fluid of different density [Turner, 1973]. Examples are seabreeze fronts or salt water intrusions into fresh water at the mouth of a river. As the dense (light) fluid flows into the light (dense) fluid a gravity current front forms. Mixing occurs at the density interface behind the gravity current head via shear instabilities [Simpson, 1997].

We can consider a two-dimensional inviscid flow, in a channel of depth $d$. The density of the dense fluid is taken as $\rho_1 = \rho_0 + \rho'$, where $\rho_0$ is the background density and $\rho'$ is a density anomaly (see also Section 2.2.2 below). Here $g' = g \frac{\rho'}{\rho_0}$ is the reduced gravity. Equations for continuity and Bernoulli’s equation can be applied above the interface and will give expressions for the final velocity of the upper layer $U_0$ and the depth of this layer, $h_0$, which is constant (Figure 2.2). We can consider the hydraulic approach, where we neglect mixing and assume the pressure in the wake of the head to be hydrostatic, For the case of no friction, the ‘flow force’ [Turner, 1973], e.g. the total pressure force plus the
momentum flux per unit span, is constant and we can write

$$\frac{1}{2} \rho_1 (U^2 d + g'd^2) = \rho_1 (U_0^2 h_0 + \frac{1}{2} g'h_0^2)$$  \hspace{1cm} (2.1)

The layer velocity can then be written as

$$U_0^2 = 2g'(d - h_0) = \frac{g'(d^2 - h_0^2)}{(2d - h_0)h_0}.$$  \hspace{1cm} (2.2)

whose solution is $h_0 = \frac{1}{2}d$, the solution in which the advancing layer fills half the channel [Benjamin, 1968]. The Froude number for this symmetrical case is then subcritical $\frac{U}{(g/d)^{1/2}} = \frac{1}{2}$ for the approaching flow and supercritical $\frac{U_0}{(g/h_0)^{1/2}} = \sqrt{2}$ for the receding flow. Experiments of high Reynolds number flow show good agreement with the inviscid case presented above [Turner, 1973]; [Barr, 1967]. At low viscosities, experiments show that the nose velocity $U$ is nearly constant over long distances, and the nose speed is close to that of the inviscid case predicted above.

We now look at the case of a gravity current where the advancing nose is shallower than half the total channel depth - which is what we observe in the early stages of our gravity current simulations. In this case, there must be an energy loss near the front and this energy loss is accompanied by ‘breaking’ [Turner, 1973]; [Benjamin, 1968]. We observe this in gravity currents in the laboratory and numerical experiments, where we have a head with a thickness larger than that of the dense fluid behind the head [Baines, 1995]; [ÖZgökmen et al., 2004]. We also observe turbulence in the region just beyond the head, the tail region of the gravity current.
We can now look at this case where energy dissipation is included in the force balance. It is necessary to include energy dissipation in the formulation here, however, no actual mixing of fluid densities is accounted for. The velocity of the advancing nose for this case is given by

\[ U = U_0 \frac{d - h_1}{d} = \sqrt{2g' h_1}, \]  

(2.3)

for \( \frac{h_1}{d} \ll 1 \). Here, \( h_1 \) is the layer depth of dense water behind the gravity current ‘head’.

This nose velocity changes almost linearly with \( \frac{h_1}{d} \), between \( 2^{-\frac{1}{2}} (g' h_1)^{\frac{1}{2}} \) for \( h_1 = \frac{1}{2} d \) with no energy losses, to \( (2g' h_1)^{\frac{1}{2}} \) as \( \frac{h_1}{d} \rightarrow 0 \), when losses are included in the force balance. It may be noted that a common condition in laboratory experiments is \( \frac{h_1}{d} = 0.1 \), a scale similar to what we observe in several of our experiments.

The previous considerations were done for the case of a flat bottom. Observations of gravity current noses on inclined beds show that the velocity of the nose can also be described by the local density difference and the height \( D \) of the head by the relation

\[ U = 0.75 (g' D)^{\frac{1}{2}}. \]  

(2.4)

This relationship appears robust across a range of slopes [Turner, 1973]. This is consistent with the finding that propagation speed is largely independent of slope angle, because increased buoyancy due to higher entrainment compensates for increased gravitational force [Turner, 1986]; [Özgökmen et al., 2003].

Another way to describe a gravity current is as a special limiting case of a two-layer hydraulic jump, where the upstream depth of the lower layer is zero [Baines, 1995]. We can take \( U \) to be the speed of the head and \( u_1 \) and \( h_1 \) the speed and depth of the layer flowing towards the head. If we then assume that \( d \), the total depth, is very large, that \( g' = \frac{g \Delta \rho}{\rho_0} \), and that \( Q \) is the rate of inflow of heavy fluid relative to the ground, scaling will give the
following relationships for the layer depth and velocities

\[ h_1 \sim \left( \frac{Q^2}{g' f} \right)^{\frac{1}{2}} \]  

(2.5)

and

\[ U + u_1 \sim (Q g')^{\frac{1}{3}} \]  

(2.6)

The observed head speed \( U \) is approximately equal to \( 1.2 \left( g' h_1 \right)^{\frac{1}{2}} \) for \( \frac{h_1}{d} \ll 1 \) with a steady decrease with increasing \( \frac{h_1}{d} \). Empirical data show that the dependence of \( \frac{u_1}{U} \approx 15\% \) on \( \frac{h_1}{d} \) is weak, where \( u_1 \) is the observed inflow velocity relative to the head [Baines, 1995].

We reviewed above the case of a gravity current, the large-scale flow feature we simulate in our numerical experiments. Within the framework of this large-scale flow, we observe smaller-scale features, ranging from internal waves at the density interface to Kelvin-Helmholtz overturns that contribute to mixing. We will describe these phenomena next.

### 2.2 Internal waves

Stratification allows the existence of internal waves, and these in turn can contribute to mixing if they become unstable. The dispersion properties of internal waves depend on the ambient stratification. Here, we review two common types: waves on density interfaces and waves in linearly stratified fluids.

#### 2.2.1 Waves at boundary between two homogeneous layers

The simplest case of waves on a density discontinuity between two layers can be treated by looking at the general case of waves on the interface between two homogeneous layers. We will consider only two-dimensional waves, the motion remains confined to the \( x-z \)-plane.
Assume two layers, initially at rest and with densities $\rho_0$ and $\rho_1 = \rho_0 + \rho'$, where $\rho'$ is a small density perturbation. The motion within the layers can be considered irrotational and so, following Turner [1973], we can write the velocity potential $u = -\nabla \phi$.

Continuity dictates that $\nabla^2 \phi = 0$ in each layer and the equations satisfied by the velocity potentials can be obtained by integrating the linearized equations

$$\rho \frac{\partial u}{\partial t} = -\nabla p' + \rho' g$$  \hspace{1cm} (2.7)

and they are

$$\frac{p_0}{\rho_0} = \frac{\partial \phi_0}{\partial t}$$  \hspace{1cm} (2.8)

$$\frac{p_1}{\rho_1} = \frac{\partial \phi_1}{\partial t} - g \frac{\rho'}{\rho_1} z$$  \hspace{1cm} (2.9)

Now consider the case of a sinusoidal wave travelling on the interface with an interfacial displacement $\eta$ of

$$\eta = a \cos k x e^{i\omega t}$$  \hspace{1cm} (2.10)

Since the solution depends on the boundary condition away from the interface, we again follow Turner [1973] and present two important cases below: waves on the interface between two infinitely deep layers and waves between layers of finite thickness.

Let us first consider the case of waves on the interface between two infinitely deep layers, we can assume that $\phi_0 \to 0$ at $z = \infty$ and $\phi_1 \to 0$ at $z = -\infty$ and we obtain the
velocity potentials that satisfy these conditions and which are:

\[ \phi_0 = A_0 e^{-kz} \cos kxe^{\mathrm{i}\omega t}, \quad \phi_1 = A_1 e^{kz} \cos kxe^{\mathrm{i}\omega t} \] (2.11)

Now with \( \eta \) the same in both layers, we then have

\[ A_0 = -A_1 = \frac{i\omega a}{k} \] (2.12)

The condition that the pressure be continuous across the interface then gives us the relationship between \( \omega \) and \( k \) in terms of phase velocity \( c = \frac{\omega}{k} \) and wavelength \( \lambda = \frac{2\pi}{k} \). This gives the dispersion relation for waves at boundary between two infinitely deep layers:

\[ c^2 = g \frac{\rho'}{k \rho_0 + \rho_1} = g\frac{\lambda \rho_1 - \rho_0}{2\pi \rho_1 + \rho_0} \] (2.13)

We can now modify these equations and boundary conditions and consider fluids confined between two rigid planes, \( z = h_0 \) and \( z = -h_1 \). The horizontal velocity \( w = 0 \) at these boundaries and the dispersion relation then becomes [Turner, 1973]:

\[ c^2 = \frac{g \rho'}{k}(\rho_0 \coth kh_0 + \rho_1 \coth kh_1)^{-1} \] (2.14)

Of particular interest for our study, is the special case in which \( kh_0 \) is large and \( kh_1 \) is small and thus the case of a layer of dense fluid underlying a layer of deep light fluid. The phase velocity then becomes

\[ c^2 = \frac{\rho'}{\rho_1} g h_1, \] (2.15)

and these waves are non-dispersive. Their phase velocity depends on both the layer
thickness and the density difference and in this limit the group velocity $c_g = c$. We observe waves in a comparable setting in a number of our experiments, for example as pictured above in Figure 2.1.

### 2.2.2 Waves in a continuously stratified fluid

In a continuously, linearly stratified fluid, the stratification supports internal waves. For completeness, we will present the case of these waves here. We start with the equations of motion for a continuously stratified fluid [Kundu, 1990]; [Turner, 1973]; [Baines, 1995]:

$$\rho D\mathbf{u}/Dt = \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = -\nabla p + \rho g + \mu \nabla^2 \mathbf{u}$$ \quad (2.16)

$$\frac{Dp}{Dt} = 0$$ \quad (2.17)

and $\mu$ is the molecular viscosity.

It follows from the Boussinesq approximation that

$$\nabla \cdot \mathbf{u} = 0$$ \quad (2.18)

We can define perturbations about a background state that is in hydrostatic balance $\nabla p_0 = \rho_0 g$. These are $p = p_0 + p'$ and $\rho = \rho_0 + \rho'$, where $p'$ and $\rho'$ are the pressure and density variations, respectively. We now neglect the non-linear terms, assume $\rho = \rho(z)$ and continuous and consider only two-dimensional motions and can thus write

$$\frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial x}$$ \quad (2.19)

$$\frac{\partial w}{\partial t} = -\frac{\partial p'}{\partial z} - g'$$ \quad (2.20)
\[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \]  
(2.21)

\[ \frac{\partial g'}{\partial t} - N^2 w = 0 \]  
(2.22)

Here \( g' = g \frac{\rho'}{\rho_0} \) and

\[ N^2 = -\frac{g}{\rho_0} \frac{\partial \rho}{\partial z} \]  
(2.23)

is the Brunt-Väisälä or buoyancy frequency.

Setting \( \rho = \rho_0 = constant \) using the Boussinesq approximation, we can then cross-differentiate to eliminate the pressure and we obtain

\[ \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) - \frac{\partial g'}{\partial x} = 0, \]  
(2.24)

The equations can be further manipulated (differentiate and combine to eliminate \( g' \) and \( u \)) and we obtain a single equation for \( w \),

\[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) + N^2(z) \frac{\partial^2 w}{\partial x^2} = 0. \]  
(2.25)

One can now assume a wave solution of the form

\[ w = \hat{w}(z) \exp(i(kx - \omega t)). \]  
(2.26)
and derive the dispersion relation by substitution. We obtain

$$\omega = N \left( \frac{k^2}{k^2 + m^2} \right)^{\frac{1}{2}} \tag{2.27}$$

for the two-dimensional case with wavenumber vector $k = (k, m)$. This is the dispersion relation for internal gravity waves in a continuously stratified fluid and it is equivalent to

$$\omega = N \cos \theta, \tag{2.28}$$

where $\theta$ is the angle between the wavenumber vector $k$ and the horizontal.

### 2.3 Instabilities

We here describe shear instabilities in various background flows.

- (i) infinite deep layer separated by density interface
- (ii) finite layer instabilities at an interface
- (iii) instabilities at a finite-thickness interface

We start with a general background as in (i) and then go on to the more specialized case in (ii) which is applicable to our experiments. We then close with describing (iii) and apply the simple result to estimates from our experiments. We focus our discussion on the most prominent of shear instabilities, known as the Kelvin-Helmholtz instability.

#### 2.3.1 Kelvin-Helmholtz instability

For the sake of simplicity we follow again Turner [1973] and we start with a general case, first considering a homogeneous, inviscid two-dimensional flow, with velocity $u(z)$, in x-
direction. Superimpose a disturbance with horizontal wavenumber $k$ and phase velocity $c$, and through the equations of motion obtain the description of the amplitude of the small scale vertical velocity perturbation $w'$,

$$\frac{d^2 w'}{dz^2} = [(u - c)^{-1} \frac{d^2 u}{dz^2} + k^2] w'. \tag{2.29}$$

Three “classes” of instabilities can be described that occur for density interfaces with a velocity shear across them [Turner, 1973]; [Benjamin, 1963]. The first class of instabilities, class “A”, are instabilities in the form of Tollmien-Schlichting waves, which occur near fixed boundaries. These occur where $u = c$ and viscous effects are essential to the growth of these waves in that Reynolds stresses are responsible for transferring momentum from the wall to the critical layer. Class “B” type instabilities are related to disturbances on a free surface, or wind-generated waves. Class “C” type instabilities are the most relevant to our problem. These are Kelvin-Helmoltz instabilities, a type of instability that occurs in a fluid between two layers of different densities and velocities, the heavier fluid at the bottom [Kundu, 1990]. To describe the mechanism, the simplest case to consider is that of a vortex sheet between two deep uniform layers that are moving with different velocities [Turner, 1973]. We can again define velocity potentials, now to include the mean flow as well as the disturbance,

$$\phi_0 = -U_0 x + \phi_0', \phi_1 = -U_1 x + \phi_1', \tag{2.30}$$

where $U_0$ and $U_1$ are the layer velocities in positive $x$-direction. It has been shown that the first disturbances to go unstable at low Richardson number are two-dimensional, so we can assume without loss of generality [Turner, 1973], that the displacements are of the form

$$\eta = a \cos kx e^{i\omega t}. \tag{2.31}$$
The condition that the displacement and pressure must be continuous across the interface leads to the dispersion relation

\[ \rho_0(\omega - kU_0)^2 + \rho_1(\omega - kU_1)^2 = gk(\rho_1 - \rho_0). \] (2.32)

And the phase velocity \( c \) is then

\[ c = \frac{\omega}{k} = \frac{\rho_0 U_0 + \rho_1 U_1}{\rho_0 + \rho_1} \pm \sqrt{\frac{g \rho_1 - \rho_0}{k \rho_1 + \rho_0} - \frac{\rho_0 \rho_1}{(\rho_0 + \rho_1)^2}(U_1 - U_0)^2} \] (2.33)

which reduces to (2.13) when \( U_0 = U_1 = 0 \).

The stability of the interfacial waves depends on the second term in (2.33). The disturbances are exponentially growing and stationary relative to mean velocity rather than oscillatory when the square root is imaginary, which occurs for

\[ (\Delta U)^2 = (U_1 - U_0)^2 > \frac{g \rho_1^2 - \rho_0^2}{k \rho_0 \rho_1}. \] (2.34)

The motion will be unstable and the waves will grow if \( k \) is sufficiently large. In the limiting case of zero density difference, all disturbances are unstable.

For finite layers, equation (2.32) can be generalized

\[ \rho_0 U_0^2 \coth kh_0 + \rho_1 U_1^2 \coth kh_1 = \frac{g}{k} (\rho_1 - \rho_0). \] (2.35)

which gives the condition for stationary, neutrally stable waves on an interface between layers of finite depths \( h_0 \) and \( h_1 \).

We now discuss interfaces of finite thickness. Equation (2.29) can be extended by adding a term proportional to the Richardson number. This makes it suitable for more realistic shear and density distributions. This equation is known as the Taylor-Goldstein
equation
\[ \frac{d^2 w'}{dz^2} = \left[ (u - c)^{-1} \frac{d^2 u}{dz^2} + (u - c)^{-2} \left( \frac{du}{dz} \right)^2 Ri + k^2 \right] w'. \quad (2.36) \]

We consider a layer of finite thickness \( h \) between two deep homogeneous layers where the velocity and density change linearly across the layer (see Turner, 1973, Figure 4.2 (b)). The overall Richardson number can be defined using the total density and velocity changes and depth scale \( h \),

\[ R_{io} = g \frac{\Delta \rho}{\rho} \frac{h}{(\Delta U)^2} \quad (2.37) \]

Here, \( R_{io} \) is also the gradient Richardson number, \( R_i = \frac{N^2}{(\frac{du}{dz})^2} = -\frac{g \frac{d\rho}{dz}}{\rho(\frac{du}{dz})^2} \) across the interface and at low \( R_{io} \), a range of intermediate wavenumbers is unstable. Yet, above \( R_i = \frac{1}{4} \), all small disturbances regardless of wave number are stable. The first wave to go unstable as \( R_i \) falls under \( \frac{1}{4} \) has the following wavelength

\[ \lambda = \frac{2\pi}{k} = 7.5h \quad (2.38) \]

and is given by \( kh = 0.83 \) [Miles, 1961]; [Howard, 1961]; [Turner, 1973].

If velocity and density have a similar tanh-like profile, the most unstable wavenumber is \( kh = 1 \). This most unstable wavelength can vary with the profiles chosen to represent the interface, but for smoothly varying profiles, its range from the value in (2.38) to \( \lambda = 2\pi h = 6.3h \), is small.

Interface thicknesses in our experiments are well-defined and narrow in the experiments that exhibit a quasi two-layer flow regime. For the more turbulent regimes, the interface boundaries become harder to define and are obscured by the strongly overturning flow (Figures 2.1, 2.3). We can estimate an interface thickness for experiments with a sharp density interface as shown in Figure 2.1. We estimate non-dimensional \( h \approx 0.01 \) to 0.05
and thus according to 2.38, the first wave to go unstable has $\lambda \approx 0.075$ to 0.375. The scale of this wave is well resolved by the model resolution in our experiments.

### 2.4 Two-dimensional turbulence

Finally, we close by summarizing results from the theory of two-dimensional (2D) turbulence. We observe several features in our experiments related to 2D turbulence (Figure 2.3). We will here describe some key points from the theory without going into great mathematical detail.

Without presenting detailed derivations - the reader is referred to Davidson [2004], Lesieur [1997], Lesieur [2008] and Vallis [2006] for more background - a key point to mention is that in 2D turbulence, the mean vorticity squared $\langle \zeta^2 \rangle$ declines over time and so the enstrophy $\frac{1}{2}\langle \zeta^2 \rangle$ is bounded by its initial value. Furthermore, in high Reynolds number 2D turbulence, the energy is near-conserved. This leads to the turbulence being long-lived. Davidson [2004] explains this by noting that there is no vortex stretching in 2D turbulence, the “hallmark” of 2D turbulence. In 3D turbulence, the rate of energy dissipation is controlled by the rate at which the larger structures in the flow break down. In 2D flows, however, $\langle \zeta^2 \rangle$ is fixed by the initial conditions. It cannot grow and thus compensate for small viscosities. In the absence of viscosity and external forces, the enstrophy over a closed domain is a conserved quantity. The presence of this quadratic invariant in 2D turbulence arises from the absence of the vortex stretching term. This conservation of enstrophy has important consequences for the flow of energy between scales [Vallis, 2006].

#### 2.4.1 The “inverse energy cascade”

We can think of the energy of the turbulence in these flows being held in eddies [Davidson, 2004]. These eddies grow as the flow evolves and the energy has a tendency to move upscale, e.g. to larger scales, as the flow develops. This phenomenon arises from the integral
constraints of energy and enstrophy conservation [Vallis, 2006]. It is the opposite of 3D turbulence where energy cascades down to smaller scales. The term “inverse energy cascade” was first used to describe this phenomenon in forced 2D turbulence [Kraichnan, 1967]. Davidson [2004] uses it to also describe the case of freely evolving, two-dimensional turbulence, because energy moves to larger scales in both forced and free 2D turbulence.

One mechanism by which eddies may continually grow in size is through “vortex mergers”, the merging of two eddies with vorticity of the same sign [Davidson, 2004]. This type of vortex merging has been observed as early as 1919, by Ayrton, and it is something we observe frequently in our simulations. These vortex mergers are particularly pronounced in the experiments on a free-slip bottom presented in Chapter 4.

An alternative explanation is the combined cascades of energy and enstrophy, by which a patch of vorticity is thinned out until its length scales become small enough for diffusion to set in. We can consider a band of vorticity in a nearly inviscid fluid, where the vorticity of each fluid element is conserved. Random motion will act to elongate the band of vorticity, but since its area must be preserved, the band narrows and the vorticity gradients will increase. The enstrophy thus moves to smaller scales [Vallis, 2006]. Now, the energy in the 2D flow is defined as

\[ \hat{E} = -\frac{1}{2} \int \psi \zeta dA, \]  

(2.39)

where we obtain the streamfunction \( \psi \) through the Poisson equation \( \nabla^2 \psi = \zeta \). We can then imagine that the vorticity is elongated in one direction to preserve area and thus the scale of the streamfunction becomes larger in the direction of stretching, but with no change in the normal direction. If we envision that stretching occurs in all direction, the overall scale of the streamfunction, and thus the energy, increases and the cascade of enstrophy to small scales is accompanied by a transfer of energy to larger scales [Vallis, 2006].

If we look at the conservation of energy and enstrophy we can make another argument.
We can think of the distribution of energy and enstrophy in wavenumber spaces as analogous to the distribution of mass and moment of inertia of a lever, respectively, where wavenumber takes the role of distance from the fulcrum [Vallis, 2006]. Here the rearrangement of mass such that its distribution becomes wider must be accompanied by a move of the center of mass closer to the fulcrum. In analogy, a flow that conserves both energy and enstrophy and that causes their distribution to spread out over wavenumber space - which we expect in a non-linear, overturning flow - will tend to move energy to small wavenumbers and enstrophy to large. We can prove this following Vallis [2006]. We start by writing expressions for average energy and enstrophy as

\[ \bar{E} = \int E(k) \, dk \]  
(2.40)

and

\[ \bar{Z} = \int Z(k) \, dk = \int k^2 E(k) \, dk, \]  
(2.41)

where \( E(k) \) and \( Z(k) \) are energy and enstrophy spectra, respectively. We can define a wavenumber, the centroid, that specifies the spectral location of energy as

\[ k_e = \frac{\int k E(k) \, dk}{\int E(k) \, dk}, \]  
(2.42)

normalized such that the denominator is unity. And we have the width of the energy distribution, \( I \), assumed to increase

\[ I = \int (k - k_e)^2 E(k) \, dk, \quad \frac{dI}{dt} > 0. \]  
(2.43)

Expand the integral and we obtain

\[ I = \int k^2 E(k) \, dk - 2k_e \int k E(k) \, dk + k_e^2 \int E(k) \, dk = \int k^2 E(k) \, dk - k_e^2 \int E(k) \, dk. \]  
(2.44)
With both energy and enstrophy conserved it follows that

\[
\frac{dk^2}{dt} = -\frac{1}{E} \frac{dI}{dt} < 0. \tag{2.45}
\]

This shows that the centroid of the distribution moves to smaller wavenumbers and larger scales. If we correspondingly define a measure for the distribution of enstrophy, we find that it moves to higher wavenumbers instead [Vallis, 2006]. This is easiest if we work with the inverse wavenumber, \( q = \frac{1}{k} \), a direct measure of length. We now assume that the enstrophy distribution is spread out by non-linear interactions in the flow, and in analogy to 2.43, we can write

\[
J = \int (q - q_e)^2 Y(q) \, dq, \tag{2.46}
\]

where \( \int Y(q) \, dq \) is the enstrophy and \( q_e = \frac{\int qY(q) \, dq}{\int Y(q) \, dq} \). Using this and expanding 2.46, we obtain

\[
J = \int q^2 Y(q) \, dq - q_e^2 \int Y(q) \, dq. \tag{2.47}
\]

And now since the energy, \( \int q^2 Y(q) \, dq \), is conserved

\[
\frac{dJ}{dt} = -\frac{d}{dt} q_e^2 \int Y(q) \, dq, \tag{2.48}
\]

and

\[
\frac{dq_e^2}{dt} = -\frac{1}{Z} \frac{dJ}{dt} < 0. \tag{2.49}
\]

As we can see from this equation, the length scale characterizing the enstrophy distribution becomes smaller, the wavenumbers become larger.
Yet another explanation can be given by looking at Batchelor’s self-similar spectrum which implies that there is a build up of energy at large scales [Davidson, 2004]. Vallis [2006] describes the argument of similarity by starting with an initial value problem and posing two conditions: (i) no internally imposed length scale, and (ii) the energy is conserved (which in itself limits the argument to 2D). The energy per unit mass is

\[ \tilde{E} = U^2 = \int E(k,t) \, dk. \]  

(2.50)

Here, \( E(k,t) \) is the energy spectrum, and \( U \) is a measure of the total energy, with units of velocity. It then follows from purely dimensional considerations that

\[ E(k,t) = U^2 L \hat{E}(\hat{k},\hat{t}), \]  

(2.51)

where \( \hat{E}, \hat{k} \) and \( \hat{t} \) are non-dimensional and \( L \) is a length scale. Now from physical considerations, the only parameters to influence the energy spectrum are \( U, t \) and \( k \). The most general form of the energy spectrum, with no dependence on \( L \), then becomes

\[ E(k,t) = U^3 t \tilde{E} = U^3 t g(U kt). \]  

(2.52)

where \( g \) is an arbitrary function. Here the argument of \( g \) is the only non-dimensional grouping of \( U, t \) and \( k \) and we obtain the correct dimensions for \( E \) from \( U^3 t \). Then from conservation of energy, we have the integral

\[ I = \int_0^\infty t g(U kt) \, dk, \]  

(2.53)

which is not a function of time. Define \( \vartheta = U kt \), this holds for \( \int_0^\infty g(\vartheta) \, d\vartheta = constant \). We see that the only dependence of the spectrum on \( k \) is through the combination \( \vartheta = U kt \). Thus as time proceeds, the spectrum moves to smaller \( k \) or larger scales. We see this
explicitly, if we write

\[ k_e = \frac{\int kE(k)dk}{\int E(k)dk} = \frac{\int kU_t g(U kt)dk}{U^2} = \frac{\int \dot{\vartheta}g(\dot{\vartheta})}{U t} d\dot{\vartheta} = \frac{C}{U t}, \quad (2.54) \]

where all the integrals are over \((0, \infty)\) and \(C = \dot{\vartheta}g(\dot{\vartheta}) d\dot{\vartheta}\) is constant. We see here that the wavenumber centroid, \(k_e\), decreases over time, as the scale of the flow, \(\frac{1}{k_e}\), increases. Note that the enstrophy does not enter this argument explicitly, and it is generally not conserved [Vallis, 2006]. It is the restriction on the conservation of energy that limits the argument to two dimensions; if we accept that the energy is conserved, it must be transferred to larger scales.

### 2.4.2 The enstrophy cascade

We conclude our discussion on energy and enstrophy with a brief description of the enstrophy cascade. In 2D homogeneous and isotropic turbulence, the vorticity \(\zeta\) is governed by \(\frac{D\zeta}{Dt} = \nu \nabla^2 \zeta\). Here, vorticity is diffused and advected similar to a passive scalar. Then, in the limit of \(Re \to \infty\) diffusion is small, except in regions of large gradients, and the isovortical lines are continually thinned out by the flow. Davidson [2004] compares this phenomenon to “cream being stirred into coffee”. This results in a vorticity field of thin, spread out filaments of vorticity. The high Reynolds number experiments we present in Chapter 6 exhibit this feature of thinning out of vorticity filaments (see also Figure 2.3). As noted above, this process will amplify the vorticity gradients and the enstrophy becomes increasingly associated with smaller-scale structure [Davidson, 2004]. Vorticity is thinned out to finer and finer scales and enstrophy thus goes from large to small scales, a phenomenon that has been referred to as enstrophy cascade. The rate of filamentation of vorticity is dictated by the large-scale eddies, until scales are small enough for viscosity to act to diffuse vorticity. This is in analogy to 3D turbulence, viscosity plays a passive role, here diffusing enstrophy (energy in 3D turbulence) at the smallest scales at a rate determined by large-scale pro-
cesses. This then concludes that $\frac{d\langle \zeta^2 \rangle}{dt}$ is finite and at large $Re$ independent of viscosity $\nu$.
This is again analogous to 3D turbulence where in the limit of $\nu \to 0$ we have $\frac{d\langle u^2 \rangle}{dt} \sim \frac{u^3}{l}$ (here $l$ is the size of the large eddies, the integral scale).

2.4.3 Coherent vortices

A notable characteristic of 2D isotropic turbulence is the formation of Kelvin-Helmholtz type coherent vortices [Lesieur, 1997]. Vortices of the same sign may “pair”, which is the “merging of two eddies of the same sign which rotate around one another” [Lesieur, 1997]. Vortices that are of opposite sign can form dipoles (Figure 2.3). These vortices may appear through instabilities in the initial velocity field. After their formation, the scale of these structures will grow, this is predominantly due to pairing. As the flow evolves the number of coherent vortices decreases and their size increases Davidson [2004], a feature we observe in our high Reynolds number flow. Furthermore, coherent vortices evident in the late and developed stages of a flow have their origin in small centers of intense vorticity in the initial flow [McWilliams, 1990].

Figure 2.3 Picture of gravity current simulation (density field) exhibiting features of 2D turbulence and vortex dynamics.
Chapter 3

Model Description

3.1 The numerical model

In this chapter, we will describe the model equations and theoretical basis as well as outline the model setup and parameters. More details on the model discretization and implementation, including boundary conditions, are described in Appendix A. The model uses a spectral element solution of the Navier-Stokes equation in streamfunction-vorticity formulation with the Boussinesq approximation. Density is described by an Advection-Diffusion equation.

3.1.1 Non-dimensionalization

The streamfunction-vorticity equations can be non-dimensionalized with the following variables: $L$ for length scale, $U$ for velocity scale and the density anomaly $\rho$ is scaled by $\Delta \rho$. The total density is given by $\rho_{tot} = \rho_0 + \rho$, where $\rho_0$ is a constant. $U/L$ becomes the scale for vorticity, $UL$ for streamfunction and $T = L/U$ for the time scale.
We get the following non-dimensional equations

\[
\dot{\zeta} + \mathbf{u} \cdot \nabla \zeta = \frac{1}{Re} \nabla^2 \zeta - \frac{1}{Fr^2} \rho_x \tag{3.1}
\]

\[
\nabla^2 \psi + \zeta = 0 \tag{3.2}
\]

\[
\rho_t + \mathbf{u} \cdot \nabla \rho = \frac{1}{Pr Re} \nabla^2 \rho \tag{3.3}
\]

where \( \rho \) is the non-dimensional density anomaly, \( \psi \) the streamfunction and \( \zeta \) is vorticity. Here, the non-dimensional parameters are defined as follows: The Reynolds number is,

\[ Re = \frac{UL}{\nu}, \]

which measures the relative importance of advection to viscous dissipation. The Froude number is

\[ Fr = \sqrt{\frac{g \Delta \rho}{\rho_0 L}} \]

which is equal to \( Fr^2 = R_i^{-1} \) where \( R_i \) is the Richardson number. The Froude number measures the relative speed of a current with respect to wave speed (if we identify \( L \) with the depth). Finally, the Prandtl number is defined as

\[ Pr = \frac{\nu}{\kappa} \]

which is the ratio of viscous dissipation to density diffusion.

The linearized equation of state

\[ \rho = \rho_0 (\beta S - \alpha T) \tag{3.4} \]

can be used to relate the above non-dimensional parameters to others that are used in the literature. A change in temperature and salt concentrations will then lead to a change of density of the form:

\[ \Delta \rho = \rho_0 (\beta \Delta S - \alpha \Delta T) \tag{3.5} \]

Disregarding temperature anomalies and concentrating purely on salt anomalies we set \( \alpha = 0 \), then a salt anomaly \( \Delta S \) will induce the following relative density anomaly: \( \beta \Delta S = \frac{\Delta \rho}{\rho_0} \).

The Grashof number is the ratio of the buoyancy force to the viscous force and is

\[ Gr = \frac{g \Delta \rho L^3}{\rho_0 \nu^2} \]
The Rayleigh number is the ratio of the buoyancy force to density diffusion and is

\[ Ra = \frac{g \Delta \rho L^3}{\rho_0 \kappa^2} = Gr Pr^2 \]

The length scale will now be identified with the fluid depth \( L = H \), and the velocity scale will now be associated with the internal gravity wave speed and so we have \( U^2 = g \frac{\Delta \rho}{\rho_0} h_0 \); where \( h_0 \) is the height of the plume. The Rayleigh number can also be defined as [Özgökmen et al., 2004]

\[ Ra = \frac{g \beta \Delta S h_0^3}{\nu^2} = \frac{g \frac{\Delta \rho}{\rho_0} h_0^2}{\nu^2} = \frac{U^2 H^2}{\nu^2} = \frac{h_0^2}{H^2} = Re^2 \frac{h_0^2}{H^2} \]

Then our Froude number will take the following form

\[ Fr^2 = \frac{U^2}{g H} = \frac{g \frac{\Delta \rho}{\rho_0} h_0^2}{\nu^2} = \frac{1}{Re} \frac{\Delta \rho}{\rho_0} \frac{1}{g \frac{\Delta \rho}{\rho_0} h_0} = \frac{h_0}{H} \]

3.1.2 Kinetic energy and enstrophy budgets

We can now derive the equations for kinetic energy and enstrophy. Interpretation of the terms in these equations can help us understand the processes that contribute to mixing in the system and thus shed light on the underlying dynamics of the problem.

**Enstrophy**

The derivation of the enstrophy budget is simple, multiply the vorticity evolution equation by \( \zeta \) and manipulate the derivatives. The PDE for the enstrophy evolution is

\[ \left( \frac{\zeta^2}{2} \right)_t + \nabla \cdot \left( \frac{\zeta^2}{2} \right) = \frac{1}{Re} \nabla \cdot (\zeta \nabla \zeta) - \frac{1}{Re} \nabla \zeta \cdot \nabla \zeta - \frac{1}{Fr^2} \zeta \rho \]

(3.8)
The integral of this evolution equation over a fixed volume $V$ - area $A$ for our two-dimensional domain - is

$$\frac{d}{dt} \left( \int_A \zeta^2 \, dA \right) + \int_{\partial A} u \cdot n \zeta^2 \, dS = \frac{1}{Re} \int_{\partial A} \nabla \zeta \cdot n \, dS - \int_A \left[ \frac{1}{Re} \nabla \cdot \nabla \zeta \cdot \nabla \right] \, dA - \frac{1}{Fr^2} \int_A \zeta \rho_x \, dA$$

(3.9)

The interpretation of the different terms is now easy. The first area integral on the left hand side is the rate of change of the enstrophy budget in $A$, while the surface integral is simply the amount of enstrophy advecting through the boundary $\partial A$. On the right hand side the boundary integral indicates the amount of enstrophy entering $A$ through viscous fluxes, the first area integral is the dissipation of enstrophy through viscous action, and the second area integral is a source/sink of enstrophy through the buoyancy force.

In a setup where the top and bottom boundaries are free-slip, $u \cdot n = 0$ and $\zeta = 0$, and hence do not contribute to the enstrophy budget. (Note, however, that in our study we are mostly concerned with experiments on a no-slip bottom.) At the inlet the flow enters the domain without vorticity, hence $\zeta = 0$ and both boundary integrals (the advective and diffusive flux) vanish also. In this case, only the outflow boundary contributes to the enstrophy budget.

**Energy equation in primitive form**

The derivation of the energy equation from the vorticity evolution equation is a little more involved. For reference we first decide on its form in primitive form (see for example Landau and Lifshitz [1959]; Batchelor [1967]).

$$u_t - u \times (\nabla \times u) + \nabla \left( P + \frac{u \cdot u}{2} \right) = -\frac{1}{Re} \nabla \times \nabla \times u + f$$

(3.10)

where $f = -\frac{1}{Fr^2} \rho k$. Notice that the advection term is written in rotational form, the same is done for the viscous term. Taking the inner product with the velocity vector and using vector identities, the energy evolution equation is then
\[
\left( \frac{\mathbf{u} \cdot \mathbf{u}}{2} \right)_t + \nabla \cdot \left[ \mathbf{u} \left( P + \frac{\mathbf{u} \cdot \mathbf{u}}{2} \right) \right] = \frac{1}{Re} \nabla \cdot [\mathbf{u} \times \nabla \times \mathbf{u}] - \frac{1}{Re} \left( \nabla \times \mathbf{u} \right) \cdot \left( \nabla \times \mathbf{u} \right) + \mathbf{u} \cdot \mathbf{f} \quad (3.11)
\]

or equivalently (after using \( \nabla \cdot \mathbf{u} = 0 \) and subsituting in the buoyancy forcing for \( \mathbf{f} \))

\[
\frac{d}{dt} \left( \frac{\mathbf{u} \cdot \mathbf{u}}{2} + \frac{1}{Fr^2} \rho z \right) + \nabla \cdot [\mathbf{u} P] = \frac{1}{Re} \nabla \cdot [\mathbf{u} \times \nabla \times \mathbf{u}] - \frac{1}{Re} \left( \nabla \times \mathbf{u} \right) \cdot \left( \nabla \times \mathbf{u} \right) + \frac{z}{Fr^2} \frac{d \rho}{dt} \quad (3.12)
\]

The terms \( \frac{\mathbf{u} \cdot \mathbf{u}}{2} \) and \( \frac{\rho z}{Fr^2} \) are the kinetic and potential energy, respectively, of a fluid particle. If we define the total energy \( E = \frac{\mathbf{u} \cdot \mathbf{u}}{2} + \frac{\rho z}{Fr^2} \) the energy equation takes the form

\[
\frac{dE}{dt} + \nabla \cdot [\mathbf{u} P] = \nabla \cdot \left[ \frac{\mathbf{u} \times \nabla \times \mathbf{u}}{Re} + \frac{z \nabla \rho}{RePrFr^2} \right] - \frac{1}{Re} \left( \nabla \times \mathbf{u} \right) \cdot \left( \nabla \times \mathbf{u} \right) - \frac{1}{RePrFr^2} \nabla z \cdot \nabla \rho \quad (3.13)
\]

The contribution of each term to the overall energy budget can be made explicit by integrating equation over an arbitrary region \( A \) to get

\[
\frac{d}{dt} \left( \int_A E \, dA \right) = \int_{\partial A} \mathbf{n} \cdot \left[ \frac{\mathbf{u} \times \mathbf{z} \mathbf{k}}{Re} + \frac{z \nabla \rho}{RePrFr^2} - \mathbf{u}(P + E) \right] \, dS - \int_A \left[ \frac{\mathbf{z}^2}{2Re} + \frac{\nabla z \cdot \nabla \rho}{RePrFr^2} \right] \, dA
\]

\[
= \int_{\partial A} \mathbf{n} \cdot \left[ \frac{\mathbf{u} \times \mathbf{z} \mathbf{k}}{Re} - \mathbf{u}(P + E) \right] \, dS - \int_A \left[ \frac{\mathbf{z}^2}{2Re} - \frac{z \nabla^2 \rho}{RePrFr^2} \right] \, dA \quad (3.14)
\]

The two forms of the energy equation, equations 3.13 and 3.14, are equivalent through the use of the Gauss divergence theorem and the chain rule of differentiation, and they differ by the specific form of the potential-kinetic energy exchange term. The terms in the boundary integral reflect the physical processes affecting the energy budget in area \( A \). These are the advection of total energy through \( S \): \( \mathbf{n} \cdot \mathbf{u} E \) and the pressure work applied at the boundary: \( \mathbf{n} \cdot \mathbf{u} P \). Both terms above are zero if the boundary does not permit normal flow, \( \mathbf{u} \cdot \mathbf{n} = 0 \).

The first term in the boundary integral represent the contribution of a non-zero tangential velocity, viscous stresses, applied at the boundary. At a stationary no-slip boundary this term is zero. It is also zero at boundary with zero vorticity Dirichlet conditions.
Energy equation in $\psi$-$\zeta$ form

The equation for energy derived in the previous section contains the pressure $P$, which is absent from the model equations. To understand the meaning of the contributing terms in the context of the streamfunction-vorticity formulation, and to arrive at an equation that does not contain the pressure, we derive the energy equation in streamfunction-vorticity (see for example Landau and Lifshitz [1959]; Batchelor [1967]).

To derive the kinetic energy (KE) from the vorticity evolution equation, we multiply the latter by $\psi$ and manipulate the different terms. The evolution equation for the mechanical energy is:

$$
- \nabla \cdot (\psi \nabla \psi_t) + \left( \frac{\nabla \psi \cdot \nabla \psi}{2} \right)_t + \nabla \cdot (\psi u \zeta) = \frac{1}{Re} [\nabla \cdot (\psi \nabla \zeta - \zeta \nabla \psi)] - \frac{1}{Fr^2} (\psi \rho)_x 
- \frac{1}{Re} \zeta^2 - \frac{1}{Fr^2} \rho v
$$

(3.15)

Note that $\nabla \psi \cdot \nabla \psi = ||\nabla \psi|| = u^2 + v^2$. Integrating over an area $A$ we get the following equation

$$
\frac{d}{dr} \left( \int_A \frac{\nabla \psi \cdot \nabla \psi}{2} \, dA \right) + \int_{\partial A} (\psi u \zeta - \psi \nabla \psi_t) \cdot \mathbf{n} \, dS = \int_{\partial A} \frac{1}{Re} [\psi \nabla \zeta - \zeta \nabla \psi] \cdot \mathbf{n} \, dS
- \int_{\partial A} \frac{1}{Fr^2} (\psi \rho) \mathbf{n} \cdot i \, dS
- \int_A \left[ \frac{1}{Re} \zeta^2 + \frac{1}{Fr^2} \rho v \right] \, dA
$$

(3.16)

where $i$ is a unit vector in the $x$-direction. The first term in the area integral on the right hand side (the area integral is the third term overall on the right hand side) represents the dissipation of kinetic energy by viscous forces throughout the domain; the rate of dissipation increases with enstrophy. The second term in the integral represents the conversion of potential energy into kinetic energy. The other contributions to the KE budget occurs at the boundary of the domain. The advection contribution is zero whenever $u \cdot n = 0$ or the
boundary vorticity is 0. The boundary contribution to the potential energy occurs solely on the left and right boundaries (the inflow/outflow ones) and is zero on the upper and lower ones.

To complete the derivation, and get an equation for the total energy, we derive the equation for the evolution of the potential energy. Multiplying the density evolution equation by $z$ we obtain, after some minor algebra,

$$
\frac{1}{Fr^2}(z\rho)_t + \frac{1}{Fr^2} \nabla \cdot (u z \rho) - \frac{1}{Fr^2} \rho v = \frac{z}{Fr^2 PrRe} \nabla^2 \rho
$$  (3.17)

Adding the above equation to the mechanical energy evolution equation we get an evolution equation for the total energy (note that the term responsible for the reversible transfer of mechanical and potential energy cancels out):

$$
\left( \frac{\nabla \psi \cdot \nabla \psi}{2} + \frac{z \rho}{Fr^2} \right)_t + \nabla \cdot \left[ u \left( \psi \zeta + \frac{z \rho}{Fr^2} \right) - \psi \nabla \psi_t \right] = \frac{1}{Re} \nabla \cdot (\psi \nabla \zeta - \zeta \nabla \psi) - \frac{1}{Fr^2} (\psi \rho) x
$$

$$
- \frac{1}{Re} \zeta^2 + \frac{z}{Fr^2 PrRe} \nabla^2 \rho
$$  (3.18)

In order to re-write the energy equation in a form that can be mapped one-to-one with that obtained from the primitive equation we need to set $\psi \zeta = \|u\|^2 - \nabla \cdot (\psi \nabla \psi)$ term in the flux term of the left hand side to obtain:

$$
\frac{\partial E}{\partial t} + \nabla \cdot \left[ u \left( u \cdot u + \frac{z \rho}{Fr^2} \right) - u \nabla \cdot (\psi \nabla \psi) - \psi \nabla \psi_t \right] = \nabla \cdot \left[ \frac{1}{Re} (\psi \nabla \zeta - \zeta \nabla \psi) \right] - \frac{1}{Fr^2} (\psi \rho) x
$$

$$
- \frac{1}{Re} \zeta^2 + \frac{z \nabla^2 \rho}{Fr^2 PrRe}
$$  (3.19)

The rate of change of total energy in a control volume (here area $A$) is then given by

$$
\frac{d}{dt} \left( \int_A E \, dA \right) = - \int_A \left[ n \cdot u \left( \psi \zeta + \frac{z \rho}{Fr^2} \right) - \psi n \cdot \nabla \psi_t \right] \, dS + \int_{A \cdot Re} \frac{1}{Re} \left[ \psi \nabla \zeta - \zeta \nabla \psi \right] \cdot n \, dS
$$

$$
- \int_{\partial A} \frac{1}{Fr^2} (\psi \rho) n \cdot i \, dS - \int_A \left[ \frac{1}{Re} \zeta^2 - \frac{z \nabla^2 \rho}{Re Pr Fr^2} \right] \, dA
$$  (3.20)
Here, the first three terms on the right-hand side represent the contribution from the boundaries. Some of these terms vanish for our setup, or only the inlet and/or outlet boundaries may contribute. The last integral on the right-hand side contains the terms that represent viscous dissipation and the contribution from buoyancy. These terms are important for the dynamics of our problem and can be put into context in Chapter 7, when we discuss the energy and enstrophy in the model results.

### 3.2 The spectral element formulation

We now briefly describe the model used in our simulations. Here, we will only mention the major points, aimed at giving the reader an overview of the model and the model philosophy. We refer the interested reader to the more in-depth description contained in Appendix A.

The model is based on a spectral element discretization of the equations of fluid motions using the “classical” continuous Galerkin formulation, and on a Discontinuous Galerkin method for discretizing the density evolution equation. The latter provides the model with just enough implicit dissipation to keep the small scale density features resolved on the computational grid.

The spectral element method combines the high accuracy and fast convergence rates of spectral methods with the geometric flexibility of unstructured-grid finite element methods. Briefly, the spectral element method can be viewed as a finite element method where the interpolation polynomial used inside each element is of high-order, usually 6-15. For smooth solutions, the high-degree polynomials provide exponential convergence rates so that the error can be made very small with few degrees of freedom. The geometric flexibility inherent in the finite element approach allows modelers to discretize complex computational domains, and to use variable resolution grids. Thus, computational resources can be concentrated in those portions of the domain where sharp gradients develop, without having to
uniformly increase the resolution everywhere.

Two important qualities of high-order methods are their high phase fidelity, and the near-absence of numerical dissipation. The latter is an asset and a liability. On the one hand the absence of numerical dissipation gives us confidence in our results in that no spurious damping is introduced by numerical artifacts, and the numerical solution will reproduce faithfully sharp density interfaces, or vorticity gradients anticipated by the continuous equations. The liability facet of a nearly inviscid discretization is that small scale noise, if generated by the dynamics, will not be damped automatically by the scheme unless its growth rate is smaller than the rate of “physical” viscous dissipation. In a strongly nonlinear flow regime, the noise can grow quickly and destabilize the computations. One possible remedy to control noise is to increase the level of physical dissipation; however, this remedy runs the risk of a) putting the simulation outside the targeted flow regime, and b) being overly diffusive since the noise is in general intermittent in space and time. The user must essentially anticipate the minimum amount of dissipation that is required to smear features until they are resolved (and smooth) on the computational grid. An alternative to noise smearing and damping is to locally increase the grid resolution until the sharp feature and noise are well-resolved on the computational grid; this is clearly limited by the availability of computational resources. In general, the approach taken here is a compromise between increased local resolution and scale-selective dissipation. In the present simulation the scale-selective dissipation is activated solely on the density evolution equation, via the discontinuous Galerkin discretization.

3.3 Model setup and parameters

We are interested in studying the effects of temporal variability in forcing on gravity current dynamics and its impact, if any, on the induced mixing. The setting for all of our experiments is a channel where dense fluid is injected at an inlet boundary, allowed to evolve
freely before exiting through an open boundary. This setting is somewhat different from what one is used to with respect to lock exchange experiments where two open boundaries are desirable. However, our attempts to control the structure of the gravity current inside the domain proved difficult when both channel boundaries were open (inflow/outflow at the lateral boundaries is allowed to evolve from model dynamics); furthermore, such a configuration exhibited more numerical noise than one with a single inlet boundary and a single open boundary. For these reasons, we used the single-inlet/single open boundary configuration for all our experiments.

Our matrix of experiments can be classified along several axes: free vs. no-slip bottom, constant vs. varying depth, moderate vs. high Reynolds number and steady vs. temporally varying forcing in either transport or interface height. Tables 3.1 and 3.2 provide a brief summary of all experiments. In each of these experiments, we adjusted the grid resolution so as to capture the flow dynamics as best as we could while keeping the simulation time acceptable.

The free-slip experiments were exploratory in nature and were conducted with $Re = 10^5$, $Pr = 0.7$ and $Fr^{-2} = 2$. The model domain is a sloping (3%) channel geometry, forced at the inlet by a steady vertically uniform transport $Q_0 = 0.1$ and with prescribed density profile. The inlet density has dense fluid, $\rho = 1$, in the lower portion and light fluid, $\rho = 0$, in the upper portion, separated by a sharp interface and intruding into an unstratified medium, $\rho = 0$. The grid for the experiments with a free-slip bottom consists of 190x32 elements of polynomial degree 14x14x13 for $\psi$, $\zeta$ and $\rho$, respectively. In these exploratory experiments, we explored the choice of parameters while familiarizing ourselves with the model behavior for our setup.

Our main results are contained in the no-slip experiments. The vortex generation at the sea-bed increases fluid mixing dramatically and required increasing the resolution to capture the flow dynamics. Two primary regimes were studied, a medium $Re = 15,000$ regime and a high $Re = 50,000$ regime. The Prandtl number was set to $Pr = 7$, while the
remaining parameters were kept fixed. The model slope is set to a 2.5% channel geometry, again forced at the inlet. The initial height of the dense fluid at the inlet \( h_0 \), which had been set to 0.5 in the free-slip case, was changed to \( h_0 = 0.375 \) for the no-slip experiments. The experiments were very costly to run and after exploring the parameter space, we settled on a choice of parameters which we kept fixed for all remaining experiments.

The grid for the no-slip experiments consists of 150x25 elements of polynomial degree 8x8x7 for \( \psi \), \( \zeta \) and \( \rho \), respectively, for \( Re = 15,000 \); and 300x50 elements for \( Re = 50,000 \). The grid resolution is refined in the bottom boundary layer and is coarsened gradually towards the top (Figure 3.1, Figure 3.2). Boundary conditions for all cases are free-slip at the top for \( \psi \) and \( \zeta \), no-flux at top and bottom for \( \rho \) and either free-slip or no-slip at the bottom boundary. \( \zeta \) is zero at the inlet boundary. The right boundary is open (Figure 3.3). More details and mathematical description of the boundary conditions, including the inlet and outflow boundaries, can be found in Appendix A.

We explored forcing periods and amplitudes and settled on the choices shown in Tables 3.1 and 3.2. Computational cost restricted the number of experiments we were able to conduct, and we chose to stay with one forcing amplitude \((a = 0.15)\), and explore the sensitivity of the flow to various forcing periods. These periods were chosen to explore a range of time scales. The time it takes for the dense current to cross the domain is non-dimensional \( t \approx 16 \). We chose forcing periods that are short to very short compared to this value. Due to the limited computational resources available, we were considerably constrained in our freedom to explore longer periods, especially for the runs at \( Re = 50,000 \). At \( Re = 15,000 \), we force with high frequencies \((T_p = 0.2)\), intermediate frequencies \((T_p = 1)\) and lower frequencies \((T_p = 2)\). At higher Reynolds number and with the necessary increased resolution, we were only able to run a select subset of experiments, with forcing period \( T_p = 2 \).

We implemented two types of time-dependent variations in forcing: (i) we modulate the interface height of the dense water at the inlet profile sinusoidally with a period \( T_p \) and amplitude \( a \), (ii) we vary the background transport by changing \( \psi \) at the top of the domain
Figure 3.1  Model grid and a set of partitions (for twelve partitions as indicated by coloring) for $Re = 15,000$. Note the refined grid resolution near the bottom boundary.

Figure 3.2  Model grid for $Re = 50,000$. Note the refined grid resolution near the bottom boundary.
Figure 3.3  Schematic of model setup

as a function of time.

The free-slip experiments were run in parallel (on 24 to 32 processors) on an IBM P655+/P690+ system at the Arctic Region Supercomputing Center (ARSC) in Alaska. The no-slip experiments were run in parallel (on 32 to 96 processors) on the IBM p5-575 cluster at the Center for Computational Science (CCS) at the University of Miami.
Table 3.1  Parameters and experiments for a free-slip bottom. Prandtl number is $Pr = 0.7$.

<table>
<thead>
<tr>
<th>Reynolds number</th>
<th>slope $\theta$</th>
<th>Forcing type</th>
<th>Grid resolution</th>
<th>Polynomial degree</th>
</tr>
</thead>
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<tr>
<td>100,000</td>
<td>3%</td>
<td>N/A</td>
<td>192x30</td>
<td>14x14x13</td>
</tr>
<tr>
<td>100,000</td>
<td>3%</td>
<td>N/A</td>
<td>192x30</td>
<td>14x14x13</td>
</tr>
</tbody>
</table>

Table 3.2  Parameters and experiments for a no-slip bottom. Model slope for these experiments is $\theta = 2.5\%$, Prandtl number is $Pr = 7$.

<table>
<thead>
<tr>
<th>Reynolds number</th>
<th>$T_p$</th>
<th>Forcing type</th>
<th>Grid resolution</th>
<th>Polynomial degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>15,000</td>
<td>0</td>
<td>N/A</td>
<td>150x25</td>
<td>8x8x7</td>
</tr>
<tr>
<td>15,000</td>
<td>0.2</td>
<td>inlet height</td>
<td>150x25</td>
<td>8x8x7</td>
</tr>
<tr>
<td>15,000</td>
<td>1</td>
<td>inlet height</td>
<td>150x25</td>
<td>8x8x7</td>
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<tr>
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</tr>
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<td>300x50</td>
<td>8x8x7</td>
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<tr>
<td>50,000</td>
<td>2</td>
<td>inlet height</td>
<td>300x50</td>
<td>8x8x7</td>
</tr>
<tr>
<td>50,000</td>
<td>2</td>
<td>background transport</td>
<td>300x50</td>
<td>8x8x7</td>
</tr>
<tr>
<td>50,000</td>
<td>4</td>
<td>inlet height</td>
<td>300x50</td>
<td>8x8x7</td>
</tr>
</tbody>
</table>
Chapter 4

Exploratory Experiments - Gravity Current on a Free-Slip Bottom

We start discussing our results by considering the case of a gravity current at high Reynolds number on a free-slip bottom. These are exploratory experiments done at very high model resolution (192x30x15), very high Reynolds number ($Re = 100,000$) and on a free-slip bottom boundary.

We first describe the flow evolution through model density fields illustrating typical flow features. We briefly analyze the features in these exploratory experiments by looking at the time and length scales in the domain. We estimate dilution of water masses by calculating mixed water fractions.

Gravity currents in the environment may be observed under free-slip conditions when there is no rigid boundary, such as in the case of a fresh water plume flowing above salt water. The mixing on the interface between the water masses of different densities plays an important role in the dynamics of the flow [Simpson, 1997]. We discussed mechanisms involved in interfacial mixing in Chapter 2, a dominant mechanism is the Kelvin-Helmholtz instability.

In our experiments of a gravity current on a flat, free-slip bottom boundary, we initially observe a large leading “head”-type vortex (Figure 4.1). As the flow evolves, lighter, ambient fluid is entrained into the head vortex and mixing ensues. We observe small vortices forming at the leading edge of this “head” feature which then merge with the larger...
Figure 4.1 The downstream evolution of the gravity current head is shown. The non-dimensional density anomaly ranges from 0 in the background (purple) to 1 in the dense current at the inlet (orange). Note that only an excerpt of the domain is shown to best illustrate flow evolution. Time evolution is from left to right and top to bottom.

As the head-type vortex propagates into the domain interior, shear instabilities of the Kelvin-Helmholtz type develop in the “tail” region behind the leading vortex. These are associated with mixing as ambient light fluid is entrained into the dense current.

The length scales we observe as the flow propagates across the domain range from the large head vortex ($h \approx 0.5$), to the intermediate scales observed in the tail region ($h \approx 0.25$), to the smallest scale vortices embedded in the larger overturns (Figures 4.1, 4.2). The most notable feature of the flow is the merging of individual vortices of the same sign of vorticity (Figure 4.3). This merging can be observed throughout the evolution of the flow and at various locations across the domain. The coalescing vortices can either be similar in size
Figure 4.2  Density contours on a flat free-slip bottom at time $t = 11.1$ are shown. The normalized density anomaly ranges from 0 in the background (purple) to 1 in the dense current at the inlet (orange). Note that only the bottom part of the domain is shown. All units are non-dimensional.

![Density contours](image)

Figure 4.3  The progression of images illustrates the merging of a set of vortices. Density scale is the same as in Figure 4.2.

or differ considerably in their respective length scales.

We observe the merging of overturns more frequently in the early stages of the flow evolution as the distance between successive vortices is smaller at this stage. We observe the smallest scale vortices near the inlet, intermediate scales in the center of the domain and the largest scales close to the domain outflow boundary. The flow pattern is that of a wave-like succession of overturns with gradually increasing wavelengths. We will attempt to further quantify this behavior in the following sections.

When we introduce a gentle slope in the domain ($\theta = 3\%$), the general flow pattern remains similar (Figure 4.4). The dominant feature of the flow is again the merging of vortices in the tail region and across the domain, where small overturns merge with or are engulfed in larger overturns. This indicates a transfer of energy to larger scales similar to that commonly described in 2D turbulence.
4.1 Time scales - density contours/Hovmoeller diagrams

We investigate how time scales change across the domain and over time by taking time series of density profiles at three different locations across the model domain, at $x = 1$, $x = 2$ and $x = 4.5$ (Figure 4.5). We observe a wave-like pattern at all three locations in the domain. This is consistent with what we see visually from the model density fields: the flow takes on a wave-like pattern with waves at the density interface between the dense current and ambient fluid. The period changes between the measuring “station” at $x = 1$ and $x = 2$, we can observe an increase in time scale. A similar change in time scale is much less pronounced between “stations” $x = 2$ to $x = 4.5$. We further observe that the dense water close to the domain inlet is progressively mixed up into water of lower densities at the measuring points farther downstream in the domain.

We can further illustrate the wave-like pattern and change in scales by calculating the plume height, e.g. the height of the dense water (for $\rho > 0.3$), across the domain and over time (Figure 4.6). The measurement interval starts after the “head” vortex has reached the domain outlet and a continuous flow across the domain is established. The Hovmoeller diagram illustrates a wave-like pattern across the domain with shorter periods closer to the domain inlet and longer periods downstream in the domain. There is a small trend towards an increase in current thickness amplitude across the domain as evident from the more closely spaced and wider “thick” regions (yellow in the figure) closer to the domain outlet.
Figure 4.5  Time series of density profiles at distance across the domain $x = 1$ (top), $x = 2$ (middle) and $x = 4.5$ (bottom) for the case on a flat bottom. Color scale shows non-dimensional density range 0 to 1. All units are non-dimensional.

4.2 Length scales - wavelet spectra

We now investigate the length scales in the flow for experiments on a flat and on a sloping bottom. We calculate wavelet power spectra of density sections along the current at time $t = 15.2$, after the head vortex has exited the domain (Figure 4.7). The wavelet spectrum is a tool used to highlight the energy containing scales in a system. The wavelet spectrum also illustrates how the distribution of energy across scales changes over time.

Density sections were taken at $\Delta z = 5\%$ above the bottom and wavelet power spectra were calculated using a set of Matlab routines provided by C. Torrence and G. Compo at http://paos.colorado.edu/research/wavelets/ [Torrence and Compo, 1998]. We also corrected for bias in the spectrum following Liu et al. [2007]. On the flat bottom (Figure 4.7, top set of figures), the density section shows a gradual increase in length scale across the domain. This is further quantified in the wavelet spectrum; we see the energy contained in small and large scales close to the domain inlet ($x = 0$ to $x = 1$), the trend goes towards larger scales as we cross farther into the domain. The energy transfer to larger scales occurs gradually between $x = 1$ to $x = 2$ with little energy contained in the smaller wavelengths.
Figure 4.6  Plume height for gravity current ($\rho > 0.3$) on a sloping free-slip bottom. We observe wave patterns in the flow and the change in scales across the domain and over time. Color scale is for plume height. Measurements start after the head has reached the domain exit ($t \approx 8.8$), the front associated with the head vortex is still apparent in the lower right corner of the figure.
Figure 4.7  Wavelet power spectra for horizontal density sections (normalized) at $\Delta z = 5\%$ above the bottom. Top set of figures are for flat bottom, bottom figures for sloping bottom. Color scale and y-axis for wavelet spectra are logarithmic. Wavelet spectra y-axis shows length scale $\frac{1}{k}$.
after $x = 2$. Black lines in the contour plots indicate the cone of influence beyond which edge effects can become important. The results reflect what can be observed visually from the model density fields, the distance between two successive vortex cores, and thus the corresponding wavelength, increases across the domain.

The density section on a sloping bottom appears to exhibit a more wave-like pattern downstream in the domain, a progression of bore-type features can be distinguished (Figure 4.7, bottom set of figures). The wavelet spectrum again shows a gradual transfer in energy from small scales close to the domain inlet towards larger scales downstream in the domain. The pattern of distribution of energy across scales is similar between the experiments on a flat and on a sloping bottom.

### 4.3 Mixed water fractions

The previous sections showed that both the flow on the flat bottom and on the gentle slope exhibit a flow regime that can be described as a wave-like succession of vortices with gradually increasing wavelengths. Despite the continuous generation of overturns at the domain inlet, the flow does not show a pronounced increase in current thickness - a proxy for entrainment - across the domain. We attempt to quantify the mixing in the domain by keeping track of the change in water masses. This can be done by calculating fractions of dense water across the domain. We divide the dense water into density bins and then sum over all the points in the domain that contain a certain density bin or water mass (Figure 4.8). Note that we continuously supply dense water of density $\rho = 1$ at the domain inlet, and that once the head vortex reaches the domain outlet, water of various density bins leaves the domain. This needs to be taken into account when interpreting the results. The background density of $\rho = 0$ is excluded from these calculations.

The results show that after the flow has reached the domain exit at time $t \approx 8.8$, water mass fractions appear to reach a quasi steady-state. Several density bins show variations
Figure 4.8  Mixed water fractions for the experiment on a sloping bottom for the duration of the experiment, time $t = 0$ to $t = 18$. Legend shows density bins.
around a mean state, only small trends hint at a decrease in the intermediate dense water \((\rho = 0.6 \text{ to } 0.8)\) and an increase in the two lowest density water masses \((\rho = 0 \text{ to } 0.4)\), which would be indicative of mixing. In the early stages of the flow evolution, passage of the dense head vortex and ensuing instabilities are associated with mixing and entrainment. This is indicated by the rapid increase in water mass of density \(\rho = 0.6 \text{ to } 0.8\) in the early stages of the simulation. These results thus indicate that on a free-slip bottom, after the initial passage of the dense head vortex and the associated instabilities, mixing subsides and the flow reaches a quasi steady-state with little mixing where a wave-like pattern dominates the flow.
Chapter 5

Results - Gravity Current at Reynolds Number $Re = 15,000$

In this chapter, we consider the case of a gravity current at Reynolds number, $Re = 15,000$, on a no-slip bottom. We start by describing the flow evolution through showing model density fields for the different flow regimes we observe. We analyze the flow in the different regimes by looking at dominant time and length scales and quantify mixing and entrainment by investigating the distribution of product waters. Wavelet spectra highlight the dominant time scales in the flow. Turbulent overturning scales, so-called Thorpe scales, are a measure of the turbulent length scales. Transport in density classes illustrates the mixing and entrainment in the gravity current. We start by briefly describing the experiment with steady forcing. We then describe the experiments done with time-dependent forcing and compare the various flow regimes we observe.

5.1 Model fields - density

The flow enters the domain on the left boundary as a dense current. The current descends the slope and a clearly defined gravity current head forms at the leading edge (Figure 5.1). The picture of the head here is notably different from the gravity current head on a free-slip bottom presented in the previous chapter. The friction at the bottom boundary results in generation of vorticity at the bottom and strongly influences the flow structure. The gravity current head for this case is comparable to those observed in laboratory studies.
The downstream evolution of the gravity current head on a no-slip bottom for steady forcing is shown. The non-dimensional density anomaly ranges from 0 in the background (purple) to 1 in the dense current at the inlet (orange). Frames are taken at non-dimensional time $t = 4$, $t = 6$ and $t = 8$ (from left to right, top to bottom).

on a boundary with friction - the tip of the nose is lifted off the bottom and we observe updraft and overturning in the head structure itself [Baines, 1995]. In the wake of the head, the so-called “tail” region, we observe shear instabilities in the form of Kelvin-Helmholtz overturns at the density interface between the dense current and the overlying ambient water. As the flow descends the slope, the shear instabilities behind the head entrain lighter water into the dense current and lead to mixing of the water masses. This mixing is intense in the leading head and the tail region, however, as the head progresses through the domain, the instabilities in the far region die down and a quasi two-layer system is established (Figure 5.1, 5.2).

We implement time-dependent variability in forcing by sinusoidally varying the height of the dense water in the inlet profile. When the flow is forced with very short forcing periods, $T_p = 0.2$, the flow regime appears similar. Once the head has propagated across the domain and the instabilities in the tail region subsided, we observe non-mixing internal
waves on the interface between the dense current and the overlying ambient (Figure 5.2, Second from top). As we increase the forcing period, the flow behavior changes. Forcing with period $T_p = 1$, the flow is strongly overturning and remains turbulent after the head exits the domain (Figure 5.2, Second from bottom). Disturbances are continuously generated through the forcing at the boundary and the flow remains turbulent in the later stages of the flow evolution. No two-layer system is established. The dense water entering the domain at the forcing boundary is mixed up into intermediate water masses and does not reach as far into the domain as in the experiments with shorter forcing periods. When we force with even longer forcing periods, $T_p = 2$, we observe individual “bore” heads propagating downslope, at the period of the forcing (Figure 5.2, Bottom panel). There is strong mixing in the tail region behind each bore head and the densest water is mixed before reaching one third into the domain.

### 5.2 Model fields - density profiles

We contrast the flow regimes for the steady forcing case and the time-dependent forcing case by looking at density and velocity profiles taken at $x = 2$, one third into the domain (Figure 5.3, see top panels for profile location). This is also the location where we take time series measurements of density and velocity for subsequent analysis (see below). A density profile taken from the time-dependently forced flow regime shows strong overturning across the depth of the dense flow. The Richardson number calculated for these profiles (taking into account our non-dimensionalization the Richardson number becomes $Ri = 2 \frac{\rho z}{(u_z)^2}$) reflects the strong overturning and falls below the critical cutoff at $Ri = 0.25$ at several places across the depth of the dense current. The density and velocity profile taken for the steady-forcing regime after the passage of the turbulent head show a stable two-layer flow. The Richardson number falls below the cutoff in both the upper and lower layer (which are essentially unmixed and have low stability), but no overturning is observed.
Figure 5.2  Model density fields for $Re = 15,000$ at time $t = 16$. Top: steady forcing case. Second from top: forcing with forcing period $T_p = 0.2$. Third from top: $T_p = 1$. Bottom: $T_p = 2$. Forcing amplitude $a = 0.15$, for all cases. Color scale is for density, x-axis is distance, all units are non-dimensional (ND).
Figure 5.3  Density and velocity profiles taken at $x = 2$ and $t = 20$, and corresponding Richardson number (Ri) profiles (plotted as $\arctan Ri$). $Ri < \frac{1}{4}$ is indicated by the dashed line. Top panels show density fields with profile location for each regime, steady forcing regime on left, time-dependent forcing with forcing period $T_p = 1$ on right. All units are non-dimensional.
5.3 Length scales - Thorpe scale

The Thorpe scale \( L_{th} \) is a measure of the length scale associated with turbulent overturns in a density profile. The Thorpe scale is defined as the root mean square of the turbulent displacement, \( L_{th} = \sqrt{\langle d^2 \rangle} \) [Thorpe, 1977]; [Özgökmen et al., 2004]. We calculate Thorpe scales for a series of profiles taken at the location \( x = 2 \). We chose this location to maximize the length of the time series while still keeping a distance from the inflow boundary. We obtain a time series of \( L_{th} \) which allows us to demonstrate the evolution of overturning scales over time and for the different flow regimes (Figure 5.4). All experiments show the large overturns associated with the passage of the gravity current head past the point of measurement. In the wake of the head, shorter scales are observed. These are associated with Kelvin-Helmholtz overturns in this region [Özgökmen et al., 2004]. Separate regimes can readily be distinguished. Experiments with steady forcing or forcing at very short forcing periods show no overturns after the head and its turbulent tail have passed through the domain. A stable two-layer system develops. For very short forcing periods, \( T_p = 0.2 \), waves develop at the layer interface at the period of the forcing. In experiments that are forced with longer forcing periods, the flow continues to show turbulent overturning after the head and tail have crossed the domain. A range of overturning scales can be observed at forcing period \( T_p = 1 \). We observe a turbulent flow regime after the head has left the model domain (at time \( t \approx 16 \)). At longer forcing periods \( (T_p = 2) \), we observe periodic bore heads propagating downslope. The time series of \( L_{th} \) illustrates the large scales associated with the gravity current heads which are observed at the period of the forcing and the smaller scales associated with instabilities in the tail region of each head.

5.4 Time scales - wavelet spectra

We now explore the dominant time scales for each flow regime by calculating wavelet spectra. We again computed spectra using the wavelet software provided at
Figure 5.4  Thorpe scales calculated from the density profile taken at $x = 2$ and as a function of time are shown. Top left: steady forcing. Top right: forcing with forcing period $T_p = 0.2$. Bottom left: $T_p = 1$. Bottom right: $T_p = 2$. Forcing amplitude $a = 0.15$ for all cases. All units are non-dimensional.
http://paos.colorado.edu/research/wavelets/ [Torrence and Compo, 1998] and corrected for bias following Liu et al. [2007].

A time series of the downstream velocity taken at $x = 2$ in the velocity core for the steady-forcing case shows the passage of the head and fluctuations due to Kelvin-Helmholtz instabilities in the tail of the current before the system settles into a stable two-layer flow (Figure 5.5). The wavelet power spectrum and global wavelet spectrum show a maximum at periods of $T \approx 0.6$ to $T \approx 0.8$ (Figure 5.5). The flow remains unchanged and at a constant velocity after $t = 20$ (not shown). Scaling estimates using the model density fields and velocity profiles show the period of $T \approx 0.6$ to $T \approx 0.8$ to be that of the large overturns in the wake of the gravity current head. We force our system at periods shorter and longer than this period, ranging from $T_p = 0.2$ to $T_p = 2$. These periods are shorter than the time it takes for the gravity current to cross the domain, $t \approx 16$. Forcing the system with very short forcing periods ($T_p = 0.2$) results in a similar regime and spectrum, where we see a peak in energy associated with the transient head as well as in the large rolls in the tail region (Figure 5.6). Forcing the system with longer forcing periods, $T_p = 1$, we observe power maxima at the forcing period proper. There is a transfer of energy to other scales, mostly multiples of the forcing period. This is an indication of non-linear interactions and is reflected in the observed overturning for this flow regime. At longer forcing periods, $T_p = 2$, the dominant signal is at the forcing period proper, the dominant scale is that of the sequential bore heads propagating downstream at the period of the forcing.

The analysis reveals the strongest non-linearities at the intermediate forcing period, $T_p = 1$. This is the period closest to the natural period of the system, $T \approx 0.8$, found for steady forcing. We expect a strong response in a system forced with a period close to the natural period of the system, which is what we observe in our experiments.
Figure 5.5  (Top three panels) Time series of downstream velocity at $x = 2$ for the steady forcing case (series taken at $z = -0.86$) is shown in the top panel. The contour plot shows the wavelet power spectrum from this velocity time series. The cone of influence (COI) is depicted as a solid black line, values below the COI can be aliased by edge effects. To the right the global wavelet spectrum is shown. (Bottom three panels) As in Top but for time-dependent forcing with a forcing period of $T_p = 0.2$ (series at $z = -0.86$).
Figure 5.6 As Figure 5.5 but for time-dependent forcing with a forcing period of $T_p = 1$ with series taken at $z = -0.85$ (Top three panels) and $T_p = 2$, series at $z = -0.825$ (Bottom three panels).
5.5 Transport in density classes

To quantify the mixing, we calculate transport in density classes across the gravity current at three locations in the domain, $x = 1$, $x = 2$ and $x = 4.5$. We calculate $\sum udz$ for ten density bins spanning the density interval $0 \leq \rho \leq 1$, and then average over four non-dimensional time units (which is also two forcing periods at $T_p = 2$) from time $t = 26$ to $t = 30$. We chose the averaging interval so as to not contain effects of the transient gravity current head. In the steady forcing case, we observe the bulk of the transport - apart from low density background flow - in the densest water class, $\rho = 0.9$ to $\rho = 1.0$, at the location closest to the domain inlet, $x = 1$. There are no significant changes in the distribution of product waters between locations $x = 1$ and $x = 2$. We observe some mixing and dilution towards intermediate density classes as we progress further across the domain to $x = 4.5$.

In the time-dependent forcing regime, at forcing period $T_p = 2$, we still see the bulk of the transport in the densest water class at $x = 1$. As we follow the flow across the domain to $x = 2$, we see a dilution of dense water to intermediate water classes. The transport is distributed over the range of intermediate density water while the transport in the densest water class has been reduced. Farthest across the domain, at location $x = 4.5$, all the water in the densest water class has been mixed into lower density water masses. There is no transport in the densest water class, but an increase in transport over a range of intermediate density waters, with the largest transport in the intermediate light waters. These changes in distribution of product waters illustrate the changes in entrainment between the steady forcing case and the time-dependent forcing. In the case of time-dependent forcing, there is more mixing, e.g. dilution of the densest water to intermediate water masses and we also see a difference in the product waters that are created.
Figure 5.7  Transport in density classes across the domain for $Re = 15,000$. Top three panels: for steady forcing at $x = 1$, $x = 2$, $x = 4.5$. Bottom three panels: same for time-dependent forcing with $T_p = 2$. Numbers in bold give the value for entrainment parameter $E_p$ (as $E_p \cdot 10^3$) as defined in Section 7.4, between $x = 2$ and $x = 4.5$ for each density class.
5.6 Forcing through background transport

The experiments we described in the previous section were forced by varying the thickness of the dense water in the profile at the inflow boundary. In this section, we briefly describe the changes to the flow regime that occur when we introduce time-dependent forcing by variation in background transport. We here only describe a select forcing period, $T_p = 2$, and the distribution of product waters we observe in this regime.

The individual bore heads observed previously at this forcing period are now absent and rather we observe “surge”-like flow across the domain (model animations, not shown). The densest water in the current is more isolated from the less dense overlying ambient and less dilution occurs across the domain. This can be observed in both the model density field (Figure 5.8) and is also reflected in the distribution of product waters (Figure 5.9). The differences between the two forcing-types at location $x = 1$ are small. At $x = 2$, we observe an increase in transport in the low density water classes when forcing through background transport. This is indicative of an increase in the thickness of the interfacial layer, the “light blue” water in Figure 5.8. The distribution at $x = 4.5$ highlights the differences between the forcing regimes. We see a further increase in the thickness of the intefacial layer for the experiment forced through background transport. We also retain the two densest water classes at $x = 4.5$. These were fully mixed into less dense water in the earlier experiment for the same forcing period. Overall, when forcing through background transport with period $T_p = 2$, we see an increase in mixing of the low and intermediate density water classes and a decrease in mixing in the highest density water classes. The densest water remains more isolated from the mixing above in the interfacial layer.
Figure 5.8 Model density field for $Re = 15,000$ for the experiment forced through variation in background transport with $T_p = 2$. Forcing amplitude $a = 0.15$. Color scale is for density.
Figure 5.9 Transport in density classes across the domain for $Re = 15,000$. Top three panels: time-dependent forcing through current thickness at $x = 1$, $x = 2$, $x = 4.5$. Bottom three panels: same for time-dependent forcing through background transport. Forcing period $T_p = 2$ for both cases. Numbers in bold give the value for entrainment parameter $E_p$ (as $E_p \cdot 10^3$) as defined in Section 7.4, between $x = 2$ and $x = 4.5$ for each density class.
Chapter 6

Results - Gravity Current at Reynolds Number $Re = 50,000$

Oceanic gravity currents are in the range of Reynolds numbers of $O(10^8)$. Modeling flows at these Reynolds numbers with a high level of precision and detail requires very high spatial (model grid, spectral order) and temporal (model time step) resolution. The computing resources that were available to us allowed us to conduct a select number of very high Reynolds number experiments, for $Re = 50,000$. We increased the model resolution considerably to resolve the details of the resulting highly inertial flow (see Chapter 3).

As in the previous chapter, we start by describing the experiment with steady forcing. We then describe the experiments done with time-dependent forcing and analyze the flow in the different regimes.

6.1 Model fields - density

The flow at $Re = 50,000$ is highly inertial even in the steady forcing case. We observe a wide spectrum of overturning scales: large overturns associated with the head structure to very small vortices observed throughout the flow (Figure 6.1, top panel). The dense current is thicker than that observed in the $Re = 15,000$ case. There is a trend towards an increase in current thickness downstream, and the densest water does not reach far into the domain but is diluted early by the strong overturning in the flow.
As we introduce time-dependent forcing through varying the height of the inlet profile, changes to the overall flow structure are not visually apparent (Figure 6.1). This is a fundamental difference with respect to the experiments at $Re = 15,000$.

### 6.2 Time scales - wavelet spectra

We calculate wavelet spectra for the flow at $Re = 50,000$ under steady forcing conditions and for time-dependent forcing with period $T_p = 2$. Wavelet spectra for the steady forcing case show the energy distributed over a range of times scales, indicating a strongly non-linear flow (Figure 6.2). With the addition of time-dependent forcing, the dominant signal is at half of the forcing period. This is indicative of strong non-linear interactions in the system.
We observe a peak in energy at the forcing period in a velocity series taken close to the domain inlet, but by the time the flow reaches $x = 2$, the energy is transferred to other scales (Figure 6.3).

### 6.3 Length scales - wavelet spectra

We now calculate wavelet spectra for density sections at two heights in the dense current and compare steady forcing conditions and time-dependent forcing with period $T_p = 2$. These calculations illustrate the dominant wavelengths in the domain at the time the density sections are taken. The sections were taken at $\Delta z = 12.5\%$ and at $\Delta z = 6.25\%$ above the bottom for steady forcing and time-dependent forcing. For each case, we compute the ensemble average of the spectra from ten sections taken at a time interval of $\Delta t = 0.5$ (estimated to be close to the decorrelation time scale of one half the dominant period in the system).

The spectra for the steady forcing regime show the energy distributed over a range of wavelengths close to the domain inlet (Figure 6.4). Farther into the domain, the dominant wavelengths become progressively longer and less energetic. The spectra are similar for both heights above bottom, a dominant length scale emerges at around $L \approx 0.25$ with a secondary peak at half this length. Wavelet power spectra for the experiments with time-dependent forcing show the energy distributed over a wider range of wavelengths close to the domain inlet (Figure 6.5). The dominant scales grow larger across the domain with less energy in the intermediate and short waves. The global wavelet spectra for sections at both heights show a peak at $L \approx 0.25$ with secondary peaks at multiples of this length scale. There is a trend towards larger scales in the time-dependently forced flow which may indicate increased mixing, the energy propagating to large scales in two-dimensional turbulence.

The dominant length scale that emerges, $L \approx 0.25$, can be related to the wave of period
Figure 6.2  (Top three panels) Time-series of downstream velocity taken at $x = 2$ (at $\Delta z = 0.05$ off the bottom) for the steady forcing case is shown in the top panel. The contour plot shows the wavelet power spectrum from this velocity time-series. The cone of influence (COI) is depicted as a solid black line, values below the COI can be aliased by edge effects. To the right the global wavelet spectrum is shown. (Bottom three panels) As in Top but for time-dependent forcing with a forcing period of $T_p = 2$. 
Figure 6.3  (Top three panels) Time-series of downstream velocity for time-dependent forcing with a forcing period of $T_p = 2$ at $x = 0.5$ (at $\Delta z = 0.05$ off the bottom) is shown in the top panel. The contour plot shows the wavelet power spectrum from the velocity time-series. The cone of influence (COI) is depicted as a solid black line, values below the COI can be aliased by edge effects. To the right the global wavelet spectrum is shown. (Bottom three panels) As in Top but for series taken at $x = 1$ (at $\Delta z = 0.05$ off the bottom).
\( T \approx 1 \), found to be the natural period in the system in the moderate Reynolds number experiments. Remnants of these “rolls” observed at \( Re = 15,000 \) appear to remain even in the high Reynolds number experiments. We will discuss these waves and their transition to turbulence via Kelvin-Helmholtz instability in Chapter 8.

6.4 Transport in density classes

The distribution of product waters for the high Reynolds number experiments illustrates the strong mixing. The transport is distributed evenly over density classes at location \( x = 1 \) for both cases (Figure 6.6). The strong mixing in the \( Re = 50,000 \) flow is illustrated by the “loss” of the highest density class and the creation of intermediate water masses by \( x = 2 \).

We observe differences in distribution of transport between the steady forcing and time-dependently forced experiments. In the steady forcing case, at \( x = 2 \), we see the highest transport in intermediate waters of non-dimensional density \( \rho = 0.4 \) to \( \rho = 0.8 \), whereas for the time-dependently forced case this maximum is shifted to lower density water, namely \( \rho = 0.1 \) to \( \rho = 0.6 \). The transport maxima remain as we cross the domain to \( x = 4.5 \). The three densest classes have vanished this far into the domain and the water has been mixed into intermediate density classes. Furthermore, the unsteady case exhibits larger transports in the intermediate light waters, indicative of stronger mixing. Qualitatively, the behavior between the two experiments is similar, and we observe much greater mixing than in the experiments with \( Re = 15,000 \).

When forcing with longer forcing periods, \( T_p = 4 \), the overall flow structure remains similar. The transport distribution is close to that of the experiment forced with \( T_p = 2 \). The water is distributed almost evenly over the density classes that remain at \( x = 4.5 \) with a shift in transport towards the very lightest water masses (Figure 6.7).
Figure 6.4  (Top two panels) Ensemble average of wavelet spectra from horizontal density sections (ten sections taken at time intervals $\Delta t = 0.5$) across the domain (at $\Delta z = 12.5\%$ above the bottom) for the steady forcing case is shown in the top panel. The contour plot shows the wavelet power spectrum, length scale is $L = \frac{1}{k}$. The cone of influence (COI) is depicted as a solid black line, values below the COI can be aliased by edge effects. To the right the global wavelet spectrum is shown. (Bottom panels) As in Top but at $\Delta z = 6.25\%$ above the bottom.
Figure 6.5  (Top two panels) Ensemble average of wavelet spectra from horizontal density sections (ten sections taken at time intervals $\Delta t = 0.5$) across the domain (at $\Delta z = 12.5\%$ above the bottom) for time-dependent forcing with a forcing period of $T_p = 2$ is shown. The contour plot shows the wavelet power spectrum, length scale is $L = \frac{1}{k}$. The cone of influence (COI) is depicted as a solid black line, values below the COI can be aliased by edge effects. To the right the global wavelet spectrum is shown. (Bottom panels) As in Top but at $\Delta z = 6.25\%$ above the bottom.
Figure 6.6  Transport in density classes across the domain for $Re = 50,000$. Top three panels: for steady forcing at $x = 1$, $x = 2$, $x = 4.5$. Bottom three panels: same for time-dependent forcing with $T_p = 2$. Numbers in bold give the value for entrainment parameter $E_p$ (as $E_p \cdot 10^3$) as defined in Section 7.4, between $x = 2$ and $x = 4.5$ for each density class.
Figure 6.7  Transport in density classes across the domain at $x = 1, x = 2, x = 4.5$, for $Re = 50,000$ for time-dependent forcing with forcing period $T_p = 4$ (transport is averaged over two forcing periods). Numbers in bold give the value for entrainment parameter $E_p$ (as $E_p \cdot 10^3$) as defined in Section 7.4, between $x = 2$ and $x = 4.5$ for each density class.
Figure 6.8  Model density field for $Re = 50,000$ for the experiment forced through variation in background transport with $T_p = 2$, at $t = 23$. Forcing amplitude $a = 0.15$. Color scale is for density.

6.5  Forcing through background transport

As for the experiments at $Re = 15,000$, we conducted an experiment at $Re = 50,000$ where we introduce time-dependent forcing by variation in background transport, with forcing period $T_p = 2$. Unlike in the experiments at lower Reynolds number, where forcing through variation in background transport hinted at an increase in the thickness of the interfacial layer, at $Re = 50,000$ changes to the overall flow structure are not visually evident. We observe a highly inertial flow and dilution of dense water occurs early in the domain (Figure 6.8). The distribution of product waters highlights the differences between forcing regimes. Forcing through background transport leads to an increase of transport in intermediate density water (Figure 6.9). The distribution of transport in density classes is similar to the distribution in the steady forcing case, the maximum transport occurs in slightly denser classes than for the experiment forced through variation of current thickness. The experiment forced through background transport shows less mixing in the very lightest water masses.
Figure 6.9  Transport in density classes across the domain for $Re = 50,000$. Top three panels: time-dependent forcing through current thickness at $x = 1$, $x = 2$, $x = 4.5$. Bottom three panels: same for time-dependent forcing through background transport. Forcing period $T_p = 2$ for both cases. Numbers in bold give the value for entrainment parameter $E_p$ (as $E_p \cdot 10^3$) as defined in Section 7.4, between $x = 2$ and $x = 4.5$ for each density class.
Chapter 7

Energy and Enstrophy

In the following sections, we will investigate a number of variables that can help us better understand the mixing and dissipation happening in the different flow regimes. Vorticity fields can provide a measure of the overturns near the bottom boundary and at the shear interface and calculating enstrophy allows us to highlight regions of high dissipation in the flow. Energetics can help us understand the contribution from the different terms, potential energy or kinetic energy, to the overall balance.

7.1 Vorticity and enstrophy

We start by presenting fields of vorticity and enstrophy for flow regimes at Reynolds numbers $Re = 15,000$ and $Re = 50,000$ under steady and time-dependent forcing. At lower Reynolds number, under steady forcing, the vorticity is concentrated near the bottom wall and the top of the dense current. The vorticity is positive at the density interface and negative at the bottom wall.

We observe a well defined boundary with low to zero vorticity between these two zones of opposing vorticity (Figure 7.1, Top). With time-dependent forcing, we see well-defined vortices spin up near the inlet boundary, negative close to the bottom wall, positive at the interface at the top of the dense fluid. These form vortex pairs very close to the inlet
boundary. Farther downstream the vorticity field shows vortices merging and thinning of vortex filaments, indicative of the mixing in the flow. We still see mainly negative vorticity close to the bottom boundary, and vorticity of the opposite sign in the fluid on the shear interface at the top of the dense current (Figure 7.1, Bottom). The enstrophy, defined as \( \frac{\zeta^2}{2} \), is a measure of the viscous dissipation in the flow (Equation 3.14). The fields at \( Re = 15,000 \) show dissipation to occur mainly in the interfacial layer and at the bottom boundary for the experiments with steady forcing (Figure 7.2, Top). The dissipation is highest closest to the inlet boundary. Similarly, for time-dependent forcing, we see the highest enstrophy values in the vortex pairs close to the inlet boundary. Downstream in the flow, we see most of the dissipation in the vortices associated with the large head ‘bores’ and in a thin layer close to the bottom wall (Figure 7.2, Bottom).

When we look at the vorticity and enstrophy fields for the high Reynolds number flow, at \( Re = 50,000 \), we again see a very different regime (Figures 7.3, 7.4). The vorticity field for the experiments with steady forcing shows a strongly overturning flow with high vorticity, both positive and negative, in individual vortex pairs throughout the flow. Vortex pairs spin up close to the inlet boundary and some continue upwards and into the domain before merging or getting engulfed in the larger scale flow. The field exhibits vigorous overturning and vorticity filaments are thinned out as a result of the mixing (Figure 7.3, Top). We see more coherent vortices and vortex pairs closer to the inlet boundary and less farther downstream. Closer to the inlet, only a thin layer close to the bottom exhibits predominantly negative vorticity, the dense current above the bottom layer exhibits a range of both positive and negative vorticity, indicative of the strong overturning in this location. Farther into the domain, the vorticity appears predominantly negative in the bottom half of the dense current, and we observe thin filaments of both positive and negative vorticity in the flow above.

Time-dependent forcing in the high Reynolds number regime results in a comparable vorticity field. Vortex pairs spin up close to the inlet and continue as coherent structures
in the domain (Figure 7.3, Bottom). Some of these vortex pairs reach far upwards into the ambient fluid and survive as coherent structures for some time before merging again with the dense current. Vorticity is high in individual overturns throughout the flow, downstream in the bottom half of the plume we observe predominantly negative vorticity, as in the steady forcing case.

Enstrophy fields for the steady forcing experiments at this Reynolds number show the dissipation is high throughout the flow, concentrated in overturns across the dense current. We observe the highest dissipation in the first half of the domain, closer to the inlet boundary, the dissipation decreases farther into the domain. Dissipation is high near the bottom boundary and throughout the flow in vortex filaments and merging vortices (Figure 7.4, Top). With time-dependent forcing, the dissipation across the flow exhibits a similar picture. High enstrophy values close to the inlet, with the dissipation concentrated in overturns and in thin vortex filaments throughout the flow (Figure 7.4, Bottom). Enstrophy values and thus dissipation decrease in the current downstream in the domain.

7.2 Energy and enstrophy spectra

We now compute energy and enstrophy spectra for velocity sections close to the bottom boundary and in the current. This allows us to put our experiments in the context of the theory available for 2D turbulence by comparing the shape of the spectra.

The total energy $E$ of a signal $y(x)$ is defined as $E = \int_{-\infty}^{\infty} |y(x)|^2 \, dx$. If the signal energy is finite, the Fourier transform of the signal exists, $Y(k) = \int_{-\infty}^{\infty} y(x)e^{-2\pi ikx} \, dx$ [Emery and Thomson, 2001]. The square of the modulus of the Fourier transform for all frequencies, $|Y(k)|^2$, is then the energy spectral density. Following Parseval’s Theorem, the total energy of a signal in the space domain equals the total energy in the wavenumber domain, for a
Figure 7.1  Vorticity fields for $Re = 15,000$ at (non-dimensional) time $t=20$ - Top: steady forcing, Bottom: time-dependent forcing, $T_p = 2$. Color scale is for vorticity.
Figure 7.2  Enstrophy fields for $Re = 15,000$ at time $t=20$ - Top: steady forcing, Bottom: time-dependent forcing, $T_p = 2$. Color scale is for enstrophy.
Figure 7.3  Vorticity fields for $Re = 50,000$ - Top: steady forcing at time $t=23.2$, Bottom: time-dependent forcing, $T_p = 2$, at time $t=41.3$. Color scale is for vorticity.
Figure 7.4  Enstrophy fields for $Re = 50,000$ - Top: steady forcing at time $t=23.2$, Bottom: time-dependent forcing, $T_p = 2$, at time $t=41.3$. Color scale is for enstrophy.
discrete signal

\[ \Delta x \sum_{x=1}^{2N} |y(x)|^2 = \Delta k \sum_{k=0}^{2N-1} |Y(k)|^2 \]  

(7.1)

where \( Y(k) \) is the Discrete Fourier transform of a variable \( y(x) \), both of length \( 2N \), and \( \Delta k = \frac{1}{2N \Delta x} \) [Emery and Thomson, 2001]. We use a simple routine provided at MATLAB Central and part of the “EzyFit Toolbox” by Frédéric Moisy to calculate the energy and enstrophy spectra. The routine defines the energy spectrum as follows

\[ E(k) = \frac{2[abs(fft(y))]^2}{(2N)^2 \Delta k} \]  

(7.2)

where \( E(k) \) is the energy spectrum, \( N \) is half the signal length and \( \Delta k \) the wavenumber increment, given by \( \frac{1}{2N \Delta x} \).

The total kinetic energy of the flow is defined as \( E_{total} = \frac{1}{2} \int_{-\infty}^{\infty} (u^2 + w^2) \, dx \, dz \). However, the flow in our experiments is highly non-homogeneous in the vertical direction. We chose to treat the velocity components individually, to make the spectra easier to interpret. We here present spectra of velocity (downstream component, \( u \)) and vorticity of a horizontal section across the domain. We compute the ensemble average of spectra from ten sections taken at a time interval of \( \Delta t = 0.5 \) and at \( \Delta z = 12.5\% \) and \( \Delta z = 6.25\% \) above the bottom and contrast steady and time-dependent forcing.

Here, we are primarily interested in the shape of the spectrum, and how the slope compares to turbulence theory.

The spectra calculated from velocity, representing the energy spectrum, show a slope in the inertial range very close to the predicted \( k^{-3} \) for two-dimensional turbulence from Batchelor’s self similarity theory [Davidson, 2004] (Figure 7.5, 7.6). In the enstrophy spectra for the sections taken away from the boundary (Figure 7.7), the slopes are close to the predicted value of \( k^{-1} \) from Batchelor’s theory [Batchelor, 1969]; [Dritschel and Scott, 2009]. The slope for the spectra from sections closer to the bottom is slightly steeper.
Figure 7.5  Energy spectra - from $u$, lower in the current.

(Figure 7.8). We observe no clear differences between steady and time-dependent forcing in both energy and enstrophy spectra.

Note that turbulence theory is defined for homogenous, isotropic turbulence [Lesieur, 2008]. Our flow can not be considered homogeneous or isotropic. This has to be taken into account in the interpretation of our spectra. In particular, close to the bottom, the flow is anisotropic due to the presence of the wall. We can claim that features very small in comparison with the distance from the wall can be considered isotropic.

7.3  Energy

The model data allows us to compute energy in the system over time, which we can use to investigate the contribution of the individual terms to the overall balance. The open
Figure 7.6  Energy spectra - from $u$, higher in the current.
Figure 7.7  Enstrophy spectra - from vorticity, lower in the current.
Figure 7.8  Enstrophy spectra - from vorticity, higher in the current.
boundaries at the domain inlet and the outlet make it difficult to close the energy budget. However, we can still compute bulk potential energy, kinetic energy and total energy in the system over time. These are presented in this section.

In the experiments at $Re = 15,000$, at steady forcing, we see an initial increase in potential, kinetic and total energy (due simply to the gravity current progressively filling the domain), until the time when the head reaches the domain exit around $t \approx 16$ (Figure 7.9). After the head exits the domain, the flow reaches an equilibrium with little change in potential, kinetic or total energy. When we add time-dependent forcing with very short forcing periods, $T_p = 0.2$, we observe a similar overall picture. After reaching steady-state, potential and total energy oscillate around the equilibrium with a period close to that of the forcing. At time-dependent forcing with $T_p = 1$ and $T_p = 2$, forcing periods that resulted in strongly mixing regimes, we see the flow reach a quasi-equilibrium after the head exits the domain at $t \approx 16$. Potential, kinetic and total energy oscillate around the equilibrium at the respective forcing periods. Here, energy hints at a low frequency modulation in the system, especially for the experiment with $T_p = 1$, the forcing period that excited the strongest mixing response.

When forcing through variation in background transport, at this Reynolds number, the flow reaches the domain exit again at $t \approx 16$. Both potential and kinetic energy, as well as total energy reach an equilibrium. The kinetic and total energy oscillate around a steady-state at the period of the forcing.

At high Reynolds number, $Re = 50,000$, at steady forcing, we see an initial increase in potential, kinetic and total energy until the head reaches the domain outlet at $t \approx 16$. The flow trends towards an equilibrium after the head exits the domain (Figure 7.10). After we start time-dependent forcing with $T_p = 2$, we see the kinetic and total energy oscillate with a period close to the forcing period. We observe an increase in potential energy accompanied by a slight dip in kinetic energy, indicative of mixing in the system.

With time-dependent forcing through variation in background transport, at $Re = 50,000$
and $T_p = 2$, we observe the head exiting the domain at $t \approx 16$. The system then trends towards a decrease in kinetic energy and increase in potential energy, consistent with the strong mixing observed in this flow regime.

Overall, the changes in energy that we observe confirm what we have seen in the analysis so far. We see a mixing response when adding time-dependent forcing at the inlet boundary at $Re = 15,000$, except for very short forcing periods. When forcing through variation in background transport we observe less of a response at this Reynolds number. At $Re = 50,000$, the flow is highly inertial and strongly mixing even without the addition of time-dependent forcing. Here, we see mixing in the case of steady forcing, time-dependent forcing at the inlet and time-dependent forcing through variation in background transport.

### 7.4 Entrainment parameters based on transport in density classes

A frequently reported measure for entrainment and mixing is the entrainment parameter $E_p$. This parameter can be considered a measure of bulk entrainment and is defined as $E_p = \frac{w_e}{U_0}$, where $w_e = \frac{dQ}{dx}$ [Morton et al., 1956][Turner, 1986] and $U_0 = \sqrt{\frac{g(h_0^2)}{2}}$ ($U_0$ can be taken as $U_0 \approx 1$ for our case). The results from most of our numerical experiments are not easily comparable to the two layer system upon which the definition of this parameter is based. Furthermore, even for observations of oceanic gravity currents, it has been reported that the necessity to fit the dense current to a simple layered model makes estimates of entrainment rates difficult and the interpretation of the resulting estimates have to be considered semiquantitative [Baringer and Price, 1997b].

Taking into account these caveats, we can still attempt to calculate entrainment parameters for our experiments and we report these here. We calculate the entrainment parameter $E_p$ as the “detrainment” in light water ($\Delta \rho$ from 0 to 0.1) for the entire domain, between $x = 0$ and $x = 6$. 
Figure 7.9  Evolution of bulk potential energy (PE), kinetic energy (KE) and total energy (TE) in the domain. $Re = 15,000$ - top row: steady forcing (left), forcing through profile thickness $T_p = 0.2$ (right); second row: forcing through profile thickness $T_p = 1$ (left) and $T_p = 2$ (right); bottom: forcing through background transport $T_p = 2$. 
Figure 7.10  Evolution of bulk potential energy (PE), kinetic energy (KE) and total energy (TE) in the domain. $Re = 50,000$ - top row: steady forcing (left), forcing through profile thickness $T_p = 2$ (right); bottom: forcing through background transport $T_p = 2$. 
Table 7.1  Entrainment parameter calculated as $E_p = w_e = \frac{dQ}{dx}$ (for $U_0 = 1$) for experiments at low and high Reynolds number, steady forcing versus time-dependent forcing and for three different density intervals. Here, we calculate $E_p$ as the “detrainment” in class $\Delta \rho$ from 0 to 0.1 between $x = 0$ and $x = 6$.

At low Reynolds number, our current most closely resembles the layered model that the definition of $E_p$ is based upon, and the results are consistent with our observations of mixing in the different regimes. The entrainment parameter shows an increase as we add time-dependent forcing though variation at the inlet boundary. When forcing though variation in background transport, the entrainment parameter is slightly reduced compared to forcing through the inlet profile height. This reflects what we found from the transport in density classes, the dense water is more isolated when forcing through variation in background transport and we see less mixing overall.

In the very inertial, strongly overturning high Reynolds number flow regimes, we see no significant increase in the entrainment parameter as we add time-dependent forcing. The entrainment is comparable for all high Reynolds number regimes.
Chapter 8

Transition of Waves to Turbulence

In this chapter, we will attempt to put our numerical results into the context of hydrodynamic stability theory. Our numerical experiments at $Re = 15,000$ under steady forcing and with time-dependent forcing at very short forcing period ($T_p = 0.2$) result in a quasi two-layer regime (Figure 5.2). We consider the parameters of the interfacial waves that theory predicts for this system and see how these compare to the waves we observe in our simulations. The experiments also show a transition from waves to turbulence in the wake of the gravity current. We observe strong overturning in the head and near tail region of the gravity current, and a transition to a two-layer flow in the far tail region. We can estimate the flow conditions that are expected for the occurrence of such a transition from waves to turbulence and compare the predicted values to the data from our experiments.

8.1 Interfacial waves

We start by looking at the case of waves at the density interface between two fluids. We observe such waves in our experiments at $Re = 15,000$ in conditions of steady forcing and when forcing with very short periods (Figure 5.2). We discussed in Chapter 2 the special case in which $kh_0$ is large and $kh_1$ is small, where $k$ is the wavenumber and $h_0$ is the thickness of the upper layer of fluid and $h_1$ of the lower layer. This case of a layer of
dense fluid underlying a layer of deep light fluid most closely resembles what we observe in our experiments at \( Re = 15,000 \). As shown in Chapter 2, the phase velocity in our non-dimensional notation is

\[
e^2 = F_r^{-2} h_1,
\]

(8.1)

These are non-dispersive waves. Their phase velocity depends on both the layer thickness and the density difference and in this limit the group velocity \( c_g = c \). We can now estimate the phase velocity we would expect in our system and see if it coincides with our observations. We see from observations in the flow that at \( x = 2 \), \( h_1 \) can be estimated to be \( h_1 \approx 0.9 - 0.8375 = 0.0625 \), from the bottom to the largest density gradient in the profile, which corresponds to half the interfacial layer (Figure 5.3). We then obtain a value for the phase velocity \( c = \sqrt{2 \cdot 0.0625} = 0.35 \). We estimate the wavelength \( \lambda \) of the stable waves in the two-layer system, in the wake of the gravity current, from Figure 5.2 as \( \lambda \approx 0.25 \) to 0.3 (between \( x = 3 \) and \( x = 5 \) in Figure 5.2, top panel). We can then estimate the wave period \( T \) from the relationship \( c = \frac{\lambda}{T} \approx \frac{0.3}{T} \approx 0.35 \), and we obtain \( T \approx 0.8 \). This is also the period we found to be the dominant period in the system from wavelet spectral analysis for this flow regime (see Chapter 5, Section 4). Also, the velocity profile at this location shows a current velocity of \( u \approx 0.62 \) at the density interface defined above, the mean layer velocity, the velocity average over the dense fluid, is \( u \approx 0.6 \). Thus, the phase velocity of the interfacial waves (\( c \sim O(0.4) \)) appears to be a bit less than the mean layer velocity of the dense fluid \( u \approx 0.6 \). This flow is slightly supercritical and the waves are advected downstream by the mean current.

In the head and near tail region of the gravity current and when we add time-dependent forcing to the system or increase the Reynolds number, we observe a more turbulent regime and we can no longer fit the flow to a layered system. Under steady forcing and with \( Re = 15,000 \), we observe Kelvin-Helmholtz instabilities in the far tail region of the gravity current before the flow settles into a stable two-layer regime. Below, we will discuss some aspects of the theory that apply to this transition from stable interfacial waves to instability.
8.2 Transition to turbulence

We note that the density interface in our experiments is of finite thickness and not infinitesimal as in the theory pertaining to the above estimates. We can revisit the theory for an interface of finite thickness. This case is governed by the Taylor-Goldstein equation (2.36) as presented in Chapter 2,

\[
\frac{d^2 w_t}{d z^2} = [(u - c)^{-1} \frac{d^2 u}{d z^2} + (u - c)^{-2} \left(\frac{d u}{d z}\right)^2 R_i + k^2] w_t. \tag{8.2}
\]

In general, the criteria sufficient for the stability of a heterogeneous shear flow are given in Miles [1961] and are \( R_i > \frac{1}{4} \) for \( \frac{d u}{d z} \neq 0 \), where \( R_i \) is the local gradient Richardson number,

\[
R_i = -\frac{g \frac{d \rho}{d z}}{(\frac{d u}{d z})^2}. \tag{8.4}
\]

The semicircle theorem states that the phase velocity of the unstable modes, when \( R_i < \frac{1}{4} \), must satisfy

\[
\left[ \text{Re}(c) - \frac{1}{2}(u_{\text{max}} + u_{\text{min}}) \right]^2 + \text{Im}(c)^2 \leq \left[ \frac{1}{2}(u_{\text{min}} - u_{\text{max}}) \right]^2. \tag{8.3}
\]

The maximum growth rate for an instability is described in Howard [1961]. Miles’ theorem and the semicircle theorem put a bound on the Richardson number and complex wave velocity that is accessible to unstable modes. Howard derives an upper bound on the maximum growth rate, \( k c_i \), where \( c_i \) is the imaginary part of the phase velocity. The upper bound on the growth rate is given as

\[
\omega_i^2 = k^2 c_i^2 \leq \max \left[ \frac{1}{4} \left( \frac{d u}{d z} \right)^2 - g \frac{d \rho}{d z} \rho \right] = \max \left[ \left( \frac{d u}{d z} \right)^2 \left[ \frac{1}{4} - R_i \right] \right], \tag{8.4}
\]

and should give the correct order of magnitude of the maximum growth rate in most cases. As \( k \to \infty, k c_i \to 0 \) and there is a critical wavelength below which all disturbances are stable [Howard, 1961].
As mentioned above, at Richardson numbers $Ri > \frac{1}{4}$, all disturbances are stable. Drazin [1958] for an exponential density distribution and a tanh velocity profile, showed that the critical Richardson number ($\frac{1}{4}$) occurs at $kh = \sqrt{2}$, where $Ri$ and $h$, here the interface thickness, are based on the gradients at the center of the interface and the total velocity change. Choosing the interfacial thickness, $h \approx 0.04$, we obtain an estimate for $\lambda \approx 0.2$. As noted in Chapter 2, the first wave to go unstable - the fastest growing disturbance - is estimated to occur in the range of $\lambda = 6.3h$ to $\lambda = 7.5h$, for smoothly varying continuous density and velocity profiles and propagates at the mean layer velocity [Turner, 1973]; [Drazin, 1958]. We can again use our interfacial thickness $h$ to estimate this wave to have the wavelength $\lambda \approx 6.3 \cdot 0.04 = 0.25$ to $\lambda \approx 7.5 \cdot 0.04 = 0.3$. These results are close to what we observe in our simulations, where we can estimate the wavelengths of the waves in the tail of the gravity current at around $\lambda \gtrsim 0.25$ (Figure 5.2). (The dominant length scale in the high Reynolds number flow, found from wavelet spectra as well as from energy and enstrophy spectra, also appears to be in this range.) If we take again $c \approx 0.4$, we confirm the dominant wave period as $T \approx 0.8$, from $c = \frac{\lambda}{T} \approx 0.3 \cdot 0.4 \approx 0.4$.

As noted in Chapter 2, the overall Richardson number (which is also the gradient Richardson number across the interface) can also be defined using the total density and velocity changes and a length scale $h$, $R_{io} = \frac{\Delta \rho}{\rho} \frac{h}{(\Delta U)^2}$. In our non-dimensional notation, this becomes $R_{io} = Fr^{-2} \frac{h}{(\Delta U)^2}$. At $x = 2$ for the steady forcing case and a stable two-layer system, we can estimate $R_{io}$ across the interface as $R_{io} \approx 2 \frac{0.04}{(0.6)^2} \approx 0.22$, which is just below the critical cut-off, consistent with the results shown in Figure 5.3.

We observe such low Richardson numbers in the critical range below $\frac{1}{4}$ in regions of high shear and low stability for the steady forcing regime at $Re = 15,000$, even in regions where the flow has reached a stable two-layer stratification. We believe that this may be due viscous effects acting in this Reynolds number regime. We note that the theoretical results reviewed above are based on theory that describes stability of a stratified shear flow of an inviscid, incompressible fluid. In general, the stability of a density interface is well
described by inviscid theory [Turner, 1973], with the exception - in an unbounded fluid - of flow at low Reynolds numbers where viscous damping can act to reduce the growth rate of disturbances.
Chapter 9

Summary and Conclusions

Results from an investigation of the tidal signal in the Red Sea outflow [Matt and Johns, 2007] as well as a scaling argument on the downstream advection of variability in overflows [Peters et al., 2005] suggested rapid dissipation of any time-dependent disturbances. However, due to the limited temporal resolution of the observational data, the mechanisms involved remained unresolved. In our work, we attempt to address questions that are relevant to this problem, and attempt to quantify the impact of time-dependent forcing on mixing and entrainment in a dense current.

Our main research tool here has been a high-order non-hydrostatic spectral element model configured to simulate gravity currents dynamics and mixing processes at all resolvable spatial and temporal scales. The geometry consisted of a gently sloping channel where a dense plume flowed into an unstratified environment. The intruding dense plume was subjected to time-dependent disturbances and the ensuing effects on the mixing were investigated. Time-dependent disturbances can affect mixing in gravity currents in several ways:

• (i) Baroclinic forcing: Non-local time-dependent disturbances can propagate along the sharp density interface at the top of the dense current. This can result in the formation of bores or hydraulic jumps which can become unstable and lead to mixing
and entrainment in the flow.

- (ii) Barotropic forcing: Local forcing acting on the overflow plume can act to increase the speed of the entire water column and impact the overflow plume through the bottom stress.

These two mechanisms can act in concert, or in the case of a rapid dissipation and dispersion of the baroclinic signal, barotropic effects can still play a role in the mixing of the dense current. We implemented both baroclinic and barotropic time-dependent forcing. Within the scope of this study, we treated these forcing mechanisms only separately. We investigated flow at two different Reynolds numbers \( Re = 15,000 \) and \( Re = 50,000 \) and summarize our findings below for each case separately.

### 9.1 Summary of results

At \( Re = 15,000 \), a stable two-layer system emerges in the steady forcing case with very little mixing. When forcing at very short forcing periods, \( T_p = 0.2 \), the regime converges to the steady forcing case. Here, after the passage of the head with its strongly overturning tail, we observe non-mixing internal waves on the density interface. In contrast, time-dependent forcing at period \( T_p = 1 \) results in a strongly turbulent flow regime with a wide range of time and length scales. The natural time scale in the system is found to be close to this forcing period and the strong mixing response can be related to non-linear interactions in the flow. At long forcing periods, individual heads appear and enhance mixing. Finally, when forcing through background transport, we find a thicker interfacial layer and more transport in the intermediate light water masses.

Wavelet spectra and Thorpe scales allow us to estimate both the time and length scales associated with the flow, respectively. We readily distinguish different flow regimes related to the forcing period: the internal wave regime, the strongly turbulent overturning regime and the bore regime. These separate regimes are also reflected in the entrainment
parameters calculated for the various experiments. The values for the entrainment parameter $E_p \approx 3 \cdot 10^{-3}$ to $10^{-2}$ are in the general range observed in previous numerical and laboratory studies for gravity current flows [Turner, 1986]; [Özgökmen et al., 2004].

The experiments at $Re = 50,000$ show a strongly mixing regime even in the steady forcing case. Time-dependent forcing affects the flow dynamics less, and changes to the flow structure are not visually apparent. Nevertheless, the density class transport shows enhanced mixing in the intermediate light water. We thus find that at high Reynolds number, forcing with periods much shorter than the time it takes for the gravity current to cross the domain does not significantly affect the overall mixing. The forcing seems to impact the distribution of water masses. Forcing through background transport yields a similar result, no immediately observable changes in flow “appearance” but changes in product waters.

## 9.2 Quantifying the mixing

Establishing a robust and consistent metric to quantify the mixing for the different flow regimes proved to be a challenging task. We have attempted to inter-compare the different runs based on energy budgets, entrainment velocity, and on spectral analysis. None of these proved to be completely satisfactory for reasons we discuss below.

### 9.2.1 Energetics

An elegant way to quantify mixing in a density-stratified Boussinesq fluid is to calculate the available potential energy (APE) of the flow [Winters et al., 1995]; [Tseng and Ferziger, 2001]. Changes in the APE are due to irreversible mixing in the fluid, and its time rate of change is a direct measure of the mixing rate. However, the presence of open boundaries made it impossible to close the budgets since potential energy was incoming and leaving through the open boundaries at a variable rate determined by the internal dynamics of the flow. Furthermore, even if we had kept a time record of the boundary sources and
sinks at every time step, it would have required a time-integration of the energy budgets to account for the energy’s time-evolution. Similarly, when calculating density fractions of each relative product water, the inflow and outflow boundaries complicate the accounting tremendously.

### 9.2.2 Entrainment parameter

Alternatively, a frequently reported measure is the entrainment parameter following the definition by Morton et al. [1956] and Turner [1986] given in Chapter 8. It represents a bulk measure of how fast the light fluid is entrained into the plume to expand it and dilute it. The observed rates in laboratory experiments and in field observations vary widely from $10^{-4}$ to $10^{-2}$; the values reported here fall within that range. A drawback of this parameter is that it is only a global average and does not reflect local mixing rates, nor the details of the exchanges between particular density classes. Furthermore, it maps best to a gravity current consisting of a two-layer system with a well-defined density interface, where this entrainment parameter can be associated roughly with the vertical velocity. In our case the plume’s interfacial layer is quite thick and exhibits strong density gradients in both vertical and horizontal directions making the entrainment velocity’s identification with a vertical velocity tenuous. For our case, the best estimate of mixing proved to be that associated with the *loss* of the lightest density water to all denser classes.

### 9.2.3 Spectra

We have attempted to use spectra to discriminate between the different simulations, and to quantify dominant time and length scales in the domain. We have calculated both wavelet spectra and traditional Fourier spectra. Wavelet spectra are shown for time-series of velocity and for spatial sections of density fields. The results indicate that the dominant period in the lower Reynolds number flow is associated with Kelvin-Helmholtz billows in the tail of the gravity current that manifest in overturning “rolls” seen throughout the flow field. Rem-
nants of these organized features are apparent even in the strongly inertial high Reynolds number flow.

The wavelet spectra for the density sections from the higher Reynolds number experiments illustrate the changes in the flow across the domain. Small length scales are important close to the domain inlet where we see strong overturning over a wide range of scales, whereas longer scales become apparent farther downstream in the domain where large overturns are the predominant flow feature.

Traditional Fourier spectra calculated for velocity and vorticity sections for the higher Reynolds number flow show little differences between steady and time-dependent forcing. The slopes of the energy and enstrophy spectra are consistent with the spectral slopes expected from theory of 2D turbulence.

### 9.3 Computational demands

The spectral element model used in this study is a state-of-the-art high-order numerical model that requires considerable computational resources. Single-processor runs at the lower Reynolds number take several weeks to complete for each experiment. At higher Reynolds number, running the model in parallel on tens of processors became a necessity. In the initial stages of the study, our computational resources were very limited and this constrained our domain size and restricted our freedom with the choice of experiments and exploration of parameter space.

More recently, the considerable computing resources offered at the Center for Computational Science at the University of Miami became available to us. This allowed us to complete our experiments and to investigate the effect of time-dependent forcing on the high Reynolds number flow. After we completed the main set of experiments, we were able to run a number of cases at high Reynolds number ($Re = 50,000$) and with time-dependent forcing at even longer forcing periods ($T_p = 4$ and $8$). These experiments appear similar to
the high Reynolds number flow under steady and time-dependent forcing at period $T_p = 2$
and changes to the overall flow structure are not visually apparent. We have not included the
case of $T_p = 8$ in the main analysis presented here, as we did not have a long enough record
to average over two forcing periods (for our calculation of transport in density classes).

Due to computational constraints imposed by the high-order formulation of our model
and the high resolution required for the simulations, the size of the domain used in the
present study is limited. Nonetheless, the domain size allows us to capture well the initial
adjustment of the gravity current as well as a subsequent stabilization of the flow. Other
studies simulating gravity currents with a high-order numerical model have used similar
domain size and parameters as presented here [Ö zgökmen et al., 2004].

9.4 Limitations

9.4.1 2D versus 3D

The model used in this study solves the streamfunction-vorticity formulation of the Navier-
Stokes equations. It is a 2D model without rotation. The use of a 2D numerical model
to simulate gravity currents has been justified previously by Ö zgökmen et al. [2003] for
simulations of the Northern branch of the Red Sea outflow. The Northern Channel of
the Red Sea outflow is about 80 km long and 5 km wide, which constrains the flow in the
spanwise direction. Furthermore, the Rossby radius of deformation is reported as $R_f =
50 km$, whereas the channel at the overflow depth is only 3 km wide, which justifies the use
of a 2D model without rotation [Ö zgökmen et al., 2003]. The study found good agreement
between the model and results from observations in the outflow plume.

Furthermore, the first instabilities to occur in a shear flow, the Kelvin-Helmholtz insta-
bilities, are two-dimensional in nature. Kelvin-Helmholtz instabilities are considered to be
the dominant mechanism responsible for mixing in the tail of the gravity current. Gravity
currents in three dimensions (3D) exhibit spanwise instabilities manifested in “lobes and
clefts” structures in the current. These instabilities are not present in our 2D simulation. Previous studies comparing gravity current mixing for 2D versus 3D simulations, found that 2D simulations qualitatively capture well the main features of the current [Özgökmen et al., 2004]. However, 2D simulations tend to overestimate the mixing when compared to full 3D. The formulation of our model restricted us to the use of a 2D setup. In addition, we note that the computational cost of running high-order high-resolution 3D simulations, in a setup comparable to the one presented here, would have been largely prohibitive.

9.4.2 Rotation

Our model setup also neglects the influence of rotation on the flow. The time scales in our simulations, see 9.5, are well below the time scale at which rotation would become important.

9.5 Application to oceanic gravity currents

Here, we attempt to redimensionalize our model results to relate them to specific oceanic overflows. The natural time scale we find in our system from spectra is about non-dimensional $T=1$ and it takes the current about 16 non-dimensional time units to cross the domain. The forcing periods were chosen to be of the order of the natural time scale in the system but shorter than the advection time scale across the domain.

In Chapter 3, we have seen that our non-dimensional time scale becomes $T^2 = \frac{L}{Fr^2 g}$. Furthermore, in our definition of the Froude number $Fr$, we have $Fr^2 = \frac{U^2}{g \bar{H}} = \frac{h_0}{\bar{H}} = \frac{1}{2}$ (where $h_0$ is the depth of the dense plume). This corresponds to the Froude number associated with the total depth $H$ rather than the Froude number of the dense plume, $Fr_{plume} = \frac{U}{\sqrt{g h_0}}$, which is assumed to be 1 in our definition.

We can now redimensionalize using typical scales found in major oceanic overflows. We chose observational data available from stations located in strongly mixing areas of the
Outflow | $U_{\text{plume},m^{-1}}$ | $\rho_0,kg\,m^{-3}$ | $\rho,kg\,m^{-3}$ | $g',m\,s^{-2}$ | $h_0,m$ | $H,m$ | $\frac{h_0}{H}$
---|---|---|---|---|---|---|---
Red Sea | 0.8 | 1025.75 | 1027.6 | 0.018 | 100 | 350 | 0.3
AASC | 1.25 | 1027.4 | 1027.9 | 0.005 | 200 | 500 | 0.4
Med | 0.9 | 1027.2 | 1028.8 | 0.016 | 150 | 500 | 0.3

Table 9.1 Observational values used in our redimensionalization taken from studies of different oceanic overflows, the Red Sea outflow (Red Sea), the Antarctic Slope Currents (AASC) and the Mediterranean outflow (Med). For reference, we also included the velocity scale for each overflow at the corresponding station location.

| Outflow | $T,s$ | $L,m$ | $U,m\,s^{-1}$ | Domain length,km |
---|---|---|---|---|
Red Sea | 197 | 350 | 1.8 | 2.1 |
AASC | 447 | 500 | 1.1 | 2.5 |
Med | 253 | 500 | 1.9 | 2.5 |

Table 9.2 Dimensional scales for our simulations found from the redimensionalization using data from different oceanic overflows as shown in Table 9.1.

respective overflow plumes. In the Mediterranean and the Red Sea outflow, bulk Froude numbers for the dense plume have been reported for these station locations and are close to 1 [Peters and Johns, 2005];[Baringer and Price, 1997a].

Red Sea Outflow

Using the example of the Northern Channel of the Red Sea outflow, we can set the total depth scale to $H = L = 350m$ (close to the “beginning” of the Northern Channel; see Table 9.1) and the density difference to $\Delta \rho = 1.85\,kg\,m^{-3}$ [Matt, 2004].

The reduced gravity becomes $g' = 0.018\,m\,s^{-2}$ and the dimensional time scale $T = 197s = 3.3min$. The corresponding velocity scale is $U = \frac{L}{T} = 1.8\,m\,s^{-1}$. The domain length is six times the depth scale, in this case 2.1km. In the Red Sea outflow plume in the Northern Channel at station 35, the ratio of dense current to total depth is $\frac{h_0}{H} = 0.3$.

The density and length scales used in the redimensionalization are summarized in Table 9.1, the redimensionalized time and length scales shown in Table 9.2.
**Mediterranean Outflow**

Similarly, using values found in the Mediterranean outflow by Baringer and Price [1997a] at their station 75, we can estimate a dimensional time scale of $T = 253s = 4.2min$ and velocity scale $U = 1.9m s^{-1}$. The ratio of dense current to total depth here is $\frac{h_0}{H} = 0.3$.

**Antarctic Slope Currents**

And, using data from the density currents found on the Antarctic Slope from Visbeck and Thurnherr [2009] (at their station number 82), we find a dimensional time scale $T = 447s = 7.5min$ and $U = 1.1m s^{-1}$. The ratio of dense current to total depth is $\frac{h_0}{H} = 0.4$.

The length and time scales found from the redimensionalization point to the limited domain size and short time scales. Our simulations capture well the initial head structure of the gravity current and the Kelvin-Helmholtz instabilities in the tail of the current which are well-known from laboratory experiments. The flow in our simulations reaches a quasi-equilibrium state after the head feature leaves the model domain but further adjustment may still occur after very long times or for very large domains.

Because of the limited domain size - and also the 2D model formulation - a direct comparison to larger-scale oceanic gravity currents such as the Mediterranean or Red Sea outflows that span many tens of kilometers in length and width, and have time scales on the order of days or longer, must remain guarded. The results presented here may be more applicable to more transient oceanic overflows such as the Antarctic Slope Currents and possibly gravity currents in sub-marine canyons [Thurnherr, 2006].

### 9.6 In conclusion

The results presented in this work suggest that in very high Reynolds number, oceanic scale gravity currents, time-dependent disturbances of a scale shorter than the advection time
scale of the current over several kilometers do not significantly impact the bulk mixing and entrainment.

High-frequency disturbances could consist of internal waves generated at topographic features or by turbulent interactions in the wind-mixed layer [Bell, 1978]. These upward or downward propagating internal waves are likely to encounter critical layers in the interfacial layer of the overflow plume [Peters and Johns, 2005];[Peters et al., 2010].

At Reynolds numbers lower than the oceanic scale (but still much larger than generally used in laboratory experiments), we find that when the time scale of the disturbance is of the scale of the Kelvin-Helmholtz billows in the tail of the gravity current, a strong mixing response is excited. Disturbances at longer scales lead to "bore heads". Internal bores have been observed in oceanic gravity currents, notably the tidal bore in the Mediterranean outflow [Richez, 1994].

At very high Reynolds numbers, the time scale of the Kelvin-Helmholtz billows found from the lower Reynolds number simulations remains a dominant signal in the system. Any disturbance introduced at this time scale - which is at the scale of the turbulence - and also at slightly longer time scales appears to be dissipated early in the domain by the strong mixing and non-linear interactions in the flow. The time-dependent disturbances still appear to influence the properties of the product waters.

In oceanic overflows, the product waters, created by the mixing with the overlying oceanic ambient as the dense overflows descend the continental slope, determine the depth at which the outflow water equilibrates. Ultimately, it is these product waters that enter the large-scale circulation. Thus, by affecting the properties of the product waters, high-frequency temporal variability in overflow plumes may have the potential to influence how these oceanic gravity currents interact with the large-scale circulation tied to the global climate system.

To our knowledge, no other studies investigating gravity current mixing with direct numerical simulations have run a high-order model, at the resolution and with the detail
presented here, and at Reynolds numbers above $Re = 10,000$. Furthermore, we are aware of no other studies that present such simulations with the addition of time-dependent forcing and with an open outflow boundary that allows the observation of the flow after the initial transient stage associated with the gravity current head. The results from this study are expected to improve our understanding of mixing in gravity currents and may be useful for its parameterization in large scale ocean models.
Appendix A

The Numerical Model - Spectral Element Ocean Model (SEOM)

In this chapter, we outline the basis of the modeling system used for the discretization, both for the Continuous Galerkin method, and briefly for the Discontinuous Galerkin one. We highlight in particular the essential features that distinguish the spectral element method from its low-order finite element cousins. We close the chapter by discussing the boundary conditions imposed on our gravity current calculations. Much of the present material is inspired from Dr. Mohamed Iskandarani’s course notes and from Karniadakis and Sherwin [1999].

A.1 The spectral element formulation

A.1.1 The weak formulation

The classical continuous Galerkin spectral element formulation starts with the weak form of the governing equations. To derive these equations for our case we multiply the vorticity evolution equation, and the Poisson equation with arbitrary but continuous weight functions $\Phi$ and $\Psi$ and integrate over our computational domain, to obtain after a few
integration by parts:

\[
\int_A \frac{\partial \zeta}{\partial t} \Phi \, dA - \int_A \left[ (\zeta u - \frac{\nabla \zeta}{R_e}) \cdot \nabla \Phi + f \times \nabla \Phi \right] \, dA
\]

\[
= \int_{\partial A} \Phi \left[ (\zeta u + \frac{\nabla \zeta}{R_e}) \cdot \mathbf{n} + \mathbf{n} \times f \right] \, dS \quad (A.1)
\]

\[
\int_A \nabla \psi \cdot \nabla \Psi \, dA = \int_A \zeta \Psi \, dA + \int_{\Gamma_0} \Psi u_b \times \mathbf{n} \, dS \quad (A.2)
\]

Here, the buoyancy force is \( f = (0, -\frac{\rho}{R_e}) \), and \( \mathbf{n} \) is the unit outward normal to the boundary. The vorticity advection term in equation A.1 has been integrated by parts so that inflow/outflow boundary conditions can be imposed weakly. Likewise, the Stokes theorem was used to integrate by part the buoyancy term in case the latter is discontinuous. In addition to requiring the continuity of \( \Phi \) and \( \Psi \), we require that these functions, vanish on all portions of the boundaries where vorticity and streamfunction values are specified, respectively. That is \( \Phi = 0 \) on those portions of the boundary where the vorticity is specified, and \( \Psi = 0 \) on those portions where \( \psi \) is specified.

### A.1.2 The Galerkin discretization

The discrete equations are derived from the continuous weak forms, equations A.1 and A.2, by restricting the weight functions to a finite set, and assuming an expansion for the solution in a particular basis set. In a Galerkin approach the weight functions and the basis set functions are identical and are chosen to span, as best as possible, the vector space in which the solution “lives”. The sole restriction, again, is that this basis set be continuous and vanish on the boundaries where the solution is specified. Prior to presenting the details pertaining with the spectral element procedure, we illustrate the derivation of the algebraic equations symbolically for the streamfunction equations. We thus assume a solution of the form:

\[
\psi(x) = \sum_{j=1}^{M} \hat{\psi}_j \phi_j(x) \quad (A.3)
\]
where \( \phi_j(x) \) are the (yet to unspecified) basis functions and the \( \hat{\psi}_i \) are unknown coefficients that must be determined so that the weak formulation is satisfied as best as possible. We thus have \( M \) unknown coefficients. Furthermore, and as per the Galerkin requirement we choose the weight functions to be:

\[
\Psi = \phi_i, \quad i = 1, 2, \ldots, M
\]  

(A.4)

Replacing the above expressions into the continuous weak form, equation A.2, and, pulling the summation sign out of the gradient and integral and inner product operator (since the equation is linear) we obtain

\[
\sum_{j=1}^{M} \int_A \nabla \phi_i \cdot \nabla \phi_j \, dA \hat{\psi}_j = \int_A \zeta \phi_i \, dA + \int_{\Gamma_0} \phi_i u_b \times n \, dS
\]  

(A.5)

The above equation can be recast into a matrix equation for the streamfunction by introducing the following definitions:

\[
\mathbf{K} \hat{\psi} = \mathbf{b}
\]  

(A.6)

\[
K_{ij} = \int_A \nabla \phi_i \cdot \nabla \phi_j \, dA
\]  

(A.7)

\[
b_i = \int_A \zeta \phi_i \, dA + \int_{\Gamma_0} \phi_i u_b \times n \, dS
\]  

(A.8)

Notice that the matrix \( \mathbf{K} \) is symmetric \((K_{ij} = K_{ji})\) and positive definite regardless of the particular choice of basis function \( \phi_i \). Given, the vorticity we can compute the load vector \( \mathbf{b} \) and solve the matrix system.

A similar procedure can be applied to the vorticity, we thus set:

\[
\zeta = \sum_{i=1}^{M} \hat{\zeta}_j(t) \phi_j
\]  

(A.9)

\[
\Phi = \phi_i, \quad i = 1, 2, \ldots, M
\]  

(A.10)
where we have shown explicitly the time-dependence of the coefficients $\hat{\zeta}_i$ (the same is actually presumed for $\psi$). Replacing these expressions in the weak form of the vorticity equation we obtain

$$
\sum_{j=1}^{M} \int_{A} \phi_i \phi_j \, dA \, \frac{d\hat{\zeta}_j}{dt} = \sum_{j=1}^{M} \int_{A} \left[ \left( \phi_j \mathbf{u} - \frac{\nabla \phi_j}{Re} \right) \cdot \nabla \phi_i \, dA \right] \hat{\zeta}_i + \int_{A} \phi_i \mathbf{f} \times \nabla \phi_i \, dA \\
+ \int_{\partial A} \phi_i \left[ \left( -\zeta \mathbf{u} + \frac{\nabla \zeta}{Re} \right) \cdot \mathbf{n} + \mathbf{n} \times \mathbf{f} \right] \, dS \quad \text{(A.11)}
$$

The matrix form can be written down with the following definitions for the various matrices that appear:

$$
\mathbf{M} \frac{d\hat{\zeta}}{dt} = (\mathbf{A} - \mathbf{D}) \hat{\zeta} + \mathbf{c} \quad \text{(A.12)}
$$

$$
M_{ij} = \int_{A} \phi_i \phi_j \, dA \quad \text{(A.13)}
$$

$$
A_{ij} = \int_{A} \phi_j \mathbf{u} \cdot \nabla \phi_i \, dA \quad \text{(A.14)}
$$

$$
D_{ij} = \int_{A} \frac{1}{Re} \nabla \phi_i \cdot \nabla \phi_j \, dA \quad \text{(A.15)}
$$

$$
c_i = \int_{A} \phi_i \mathbf{f} \times \nabla \phi_i \, dA + \int_{\partial A} \phi_i \left[ \left( -\zeta \mathbf{u} + \frac{\nabla \zeta}{Re} \right) \cdot \mathbf{n} + \mathbf{n} \times \mathbf{f} \right] \, dS \quad \text{(A.16)}
$$

where the matrices $\mathbf{M}$, $\mathbf{A}$ and $\mathbf{D}$ are the mass matrix, the discrete advection matrix, and the viscous dissipation matrix, respectively.

**Remarks:**

- The mass and diffusion matrices are symmetric regardless of the choice of $\phi_i$.
- The matrix $A_{ij}$ is not symmetric.
- When $Re$ is constant we have $\mathbf{K} = Re \mathbf{D}$.
- It is important that the mass matrix be easily invertible since even an explicit time stepping scheme requires $\mathbf{M}^{-1}$.
- There is no need to explicitly construct the matrices $\mathbf{K}$ and $\mathbf{A}$ if explicit integration is performed, only their action on the vector $\hat{\zeta}$ is required, and the result is simply a
vector that can be folded into the definition of c. This is the choice made here.

We time-march the vorticity using an explicit scheme for the integration. If we use superscripts to denote time-levels, then the solution at time level $n+1$ can be computed as follows using a third-order Adams-Bashforth (AB3) scheme

$$\hat{\zeta}_{n+1} = \hat{\zeta}_n + \Delta t M^{-1} \left[ \frac{23}{12} c^n - \frac{16}{12} c^{n-1} + \frac{5}{12} c^{n-2} \right]$$  \hspace{1cm} (A.17)

where the $c^m$ refers to the right hand side term c at time level m. At each time-step, the streamfunction and the density need to be updated to synchronize all the variables. Notice also that AB3 requires a start-up procedure to jump start the calculations. One choice is to use a third-order Runge-Kutta scheme and perform 2 steps before switching to AB3. The latter takes the form:

$$\hat{\zeta}^{(1)} = \hat{\zeta}_n + \Delta t M^{-1} c^n$$  \hspace{1cm} (A.18)

$$\hat{\zeta}^{(2)} = \frac{3}{4} \hat{\zeta}^{(1)} + \frac{1}{4} \left[ \hat{\zeta}^{(1)} + \Delta t M^{-1} c^{(1)} \right]$$  \hspace{1cm} (A.19)

$$\hat{\zeta}_{n+1} = \frac{1}{3} \hat{\zeta}^{(1)} + \frac{2}{3} \left[ \hat{\zeta}^{(2)} + \Delta t M^{-1} c^{(2)} \right]$$  \hspace{1cm} (A.20)

where $\hat{\zeta}^{(1,2)}$ are provisional values and $c^{(1,2)}$ are right hand side vectors obtained with these provisional values. Notice that the RK3 schemes requires 3 evaluations of the vector $c$, along with 3 updates of the density and streamfunction, per time step, whereas the AB3 scheme requires one evaluation only.

A.1.3 The spectral element choices

The spectral element method makes particular choices for the basis functions $\phi$, for the evaluation of the various integrals, and for the geometric discretization. The latter traditionally relies on quadrilateral elements to describe the geometry of the domain. These mapped quadrilaterals could have curved edges to describe complicated boundary shapes;
it is usually enough to rely on straight edges in the domain interior. Inside each element, the
solution is expanded in terms of Lagrange interpolation polynomials of (relatively) high de-
gree; this choice permits the error in the spectral element method to decrease spectrally fast
for smooth solutions. In contrast, traditional finite element methods use linear or quadratic
interpolation and achieve second or third order convergence rates. To illustrate this process,
it is best if the interpolation and discretization are described for one-dimensional problems.

**One-dimensional SE discretization**

In 1D the elements are line-segments that, for element $k$, span the interval $x_{k-1} \leq x \leq x_k$.
To simplify the computations, this element is mapped into a computational space $|\xi| < 1$:

$$
x = \Delta x_k \frac{\xi + 1}{2} + x_{k-1} \text{ or } \xi = 2 \frac{x - x_{k-1}}{\Delta x_k} - 1, \quad \Delta x_k = x_k - x_{k-1}
$$

(A.21)

In order to define a Lagrangian interpolation, one need only specify the interpolation, alternatively called collocation, points where the solution will be calculated: $\xi_i$, given $N$ such points we can define the Lagrange polynomial a follow:

$$
h_i(\xi) = \prod_{j=1, j \neq i}^{N} \frac{\xi - \xi_j}{\xi_i - \xi_j} = \frac{(\xi - \xi_1)(\xi - \xi_2)\ldots(\xi - \xi_{i-1})(\xi - \xi_{i+1})\ldots(\xi - \xi_{N-1})(\xi - \xi_N)}{(\xi_i - \xi_1)(\xi_i - \xi_2)\ldots(\xi_i - \xi_{i-1})(\xi_i - \xi_{i+1})\ldots(\xi_i - \xi_{N-1})(\xi_i - \xi_N)}
$$

(A.22)

By construction $h_i(\xi)$ is a polynomial of degree $P = N - 1$ which when evaluated at one
of the collocation point $\xi_k$ yields either 1 if $k = i$ or 0 if $k \neq i$, or $h_i(\xi_k) = \delta_{ik}$ where $\delta_{ik}$ is
the Kronecker $\delta$. If the basis function $\phi_i$ are now identified with this polynomial, we have
a simple interpretation for the coefficients $\hat{\psi}_i$, namely they are the function values at the
collocation point $\xi_i$:

$$
\psi(\xi_i) = \sum_{j=1}^{N} \hat{\psi}_j h_j(\xi_i) = \sum_{j=1}^{N} \hat{\psi}_j \delta_{ij} = \hat{\psi}_i
$$

(A.23)

The hat superscript on the coefficients $\psi_i$ will be omitted for simplicity since the coeffi-
cients are nothing but the value of the solution at the collocation point.
Figure A.1  Distribution of Gauss-Lobatto points for a fourth-degree polynomial and sketch of the first 3 Lagrangian interpolants. The remaining ones are mirror images about the center.

Figure A.2  Illustration of the continuity preservation across element boundaries (the mid point is actually shared by the two elements). Since the Lagrangian polynomials of the left element don’t contribute to the interpolation on its right-edge point ($\xi = 1$) except for $h_N(\xi)$ we have $\psi(\xi = 1) = \sum_{i=1}^{N} \psi_l h'_l(1) = \psi_l^f$. On the right element we have $\psi(\xi = -1) = \sum_{i=1}^{N} \psi_r h'_r(1) = \psi_r^f$. Hence continuity is guaranteed if we carry one degree of freedom for the node shared by the two elements, that is $\psi^f_N = \psi^r_1$. In contrast the discontinuous Galerkin procedure permits discontinuous interpolation across element so that $\psi^f_N \neq \psi^r_1$. 
The spectral element method relies on an uneven distribution of the interpolation points within the interval $|\xi| \leq 1$ in order to avoid the Runge oscillations that plague uniformly spaced high-order interpolation. Instead the collocation points are the Gauss-Lobatto roots of the Legendre polynomials, i.e. the roots of

$$\left(1 - \xi_i \right)^2 P_{N-1}'(\xi) = 0 \quad \text{(A.24)}$$

where $P_{N-1}$ is the Legendre polynomial of degree $(N - 1)$. The interpolation points and the Lagrangian polynomials are illustrated in figures A.1 and A.2 for the case $N = 5$. The clustering of the roots near the end points $\pm 1$ is critical to defeat the Runge oscillations. The choice of Legendre roots is further motivated by their association with an accurate quadrature rule that is exact for polynomials of degree $2N - 3$ or less, that is

$$\int_{-1}^{1} f(\xi) \, d\xi = \sum_{i=1}^{N} f(\xi_i) w_i + R, \quad \text{(A.25)}$$

where $w_i$ are the quadrature weights and $R$ is the error; $R = 0$ is $f(\xi)$ is a polynomial of degree less or equal to $(2N - 3)$.

**2D Spectral element**

Our 2D spectral elements are based on mapped quadrilateral elements. For straight edged elements the mapping between physical and computational space can rely on a simple bilinear map:

$$x(\xi, \eta) = \frac{1 - \xi}{2} \left( \frac{1 - \eta}{2} x_1 + \frac{1 + \eta}{2} x_3 \right) + \frac{1 + \xi}{2} \left( \frac{1 - \eta}{2} x_2 + \frac{1 + \eta}{2} x_4 \right) \quad \text{(A.26)}$$

where $x_i$ denotes the coordinates of the four corner points, figure A.3, and where $(\xi, \eta)$ are the computational coordinates. Thus the side $\eta = 1$ in computational space corresponds to
Figure A.3 Mapping of a quadrilateral between the unit square in computational space (left) and physical space (right).

the line segment defined by
\[ \frac{x_3 + x_4}{2} + \xi \frac{x_4 - x_3}{2}. \]

Thus \((\xi = 1, \eta = 1)\) corresponds to the corner point \(x_4\) and \((\xi = -1, \eta = 1)\) to the point \(x_3\).

The 2D interpolation in the computational plane is constructed with tensor-product of the 1D interpolation:
\[
\psi(\xi, \eta) = \sum_{i=0}^{N} \sum_{j=0}^{N} h_i(\xi)h_j(\eta)\psi_{ij} \tag{A.27}
\]

The Lagrangian interpolation property holds since if \((\xi_k, \eta_l)\) is one of the Gauss-Lobatto points we have
\[
\psi(\xi_k, \eta_l) = \sum_{i=0}^{N} \sum_{j=0}^{N} h_i(\xi_k)h_j(\eta_l)\psi_{ij} = \sum_{i=0}^{N} \sum_{j=0}^{N} \delta_{ik}\delta_{lj}\psi_{ij} = \psi_{kl}. \tag{A.28}
\]

Furthermore, continuity across elements is maintained. To show this consider two elements that are adjacent along their boundary \(\xi = 1\) for the left element, and \(\xi = -1\) for the right element. For the element on the left the variation along the \(\eta\) direction is
\[
\psi^L(1, \eta) = \sum_{j=1}^{N} h_j(\eta)\psi_{Nj}^L, \text{ and for the element on the right we would have } \psi^R(-1, \eta) = \sum_{j=1}^{N} h_j(\eta)\psi_{1j}^R. \]

Hence continuity is guaranteed solely by requiring the two elements to share the same degrees of freedom along their common edge, that is if \(\psi_{Nj}^L = \psi_{1j}^R\).
The derivative of the function $\psi$ inside the element can be computed with the help of the collocation derivative:

\[
\psi_\xi(\xi_k, \eta_l) = \sum_{i=0}^{N} \sum_{j=0}^{N} h_i'(\xi_k) h_j(\eta_l) \psi_{ij} = \sum_{i=0}^{N} h_i'(\xi_k) \psi_{il}
\]  
\[
\psi_\xi(\xi_k, \eta_l) = \sum_{i=0}^{N} \sum_{j=0}^{N} h'_i(\xi_k) h_j(\eta_l) \psi_{ij} = \sum_{j=0}^{N} h'_j(\eta_l) \psi_{kj}
\]  

(A.29)  
(A.30)

where $h_i'(\xi_k)$ is the derivative of the $h_i(\xi)$ evaluated at $\xi_k$; and where the second equality holds whenever $\eta_l$ or $\xi_k$ are one of the collocation points. The derivative of a function $\psi$ in physical space can be expressed in terms of derivatives in computational space by using the chain rule of differentiation; in matrix form this can be expressed as:

\[
\begin{pmatrix}
\psi_x \\
\psi_y
\end{pmatrix}
= \begin{pmatrix}
x_x & \eta_x \\
\xi_y & \eta_y
\end{pmatrix}
\begin{pmatrix}
\psi_\xi \\
\psi_\eta
\end{pmatrix}
= \frac{1}{J} \begin{pmatrix}
y_\eta & -y_\xi \\
-x_\eta & x_\xi
\end{pmatrix}
\begin{pmatrix}
\psi_\xi \\
\psi_\eta
\end{pmatrix}, \quad J = x_\xi y_\eta - x_\eta y_\xi
\]  

(A.31)

Here $J$ denotes the Jacobian of the mapping.

The weak formulation requires the evaluation of integrals over the entire domain. These can be broken up into integrals over individual elements, and evaluated in computational space as follows:

\[
\int_A f(x) \, dA = \sum_{e=1}^{E} \int_{A_e} f(x) \, dA = \sum_{e=1}^{E} \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) |J| \, d\xi d\eta
\]  

(A.32)

where $E$ is the number of elements. It is convenient to evaluate the element integrals with quadrature. The customary choice is to use Gauss-Lobatto quadrature of order $Q$, so that

\[
\int_{E} f(x) \, dA = \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) |J| \, d\xi d\eta \approx \sum_{p=1}^{Q} \sum_{q=1}^{Q} f(\xi_p^Q, \eta_q^Q) |J| \omega_p^Q \omega_q^Q
\]  

(A.33)

where $(\xi_p^Q, \eta_q^Q)$ are the quadrature points (which may be different from the interpolation points) and $\omega_p^Q$ are the weight of the quadrature. Again, the integral is exact if the integrand
is a $2Q - 3$ polynomial in $\xi$ or $\eta$. Tremendous saving in computations can be had if the
interpolation and quadrature points are the same, for then all that is needed is to evaluate the
function at the interpolation points and no further computations are required. We illustrate
this for the important case of the mass matrix where we have

$$
M_{ij,kl} = \int_{-1}^{1} \int_{-1}^{1} h_i(\xi) h_j(\eta) h_k(\xi) h_l(\eta) |J| d\xi d\eta \\
\approx \sum_{p=1}^{Q} \sum_{q=1}^{Q} h_i(\xi_p^Q) h_j(\eta_q^Q) h_k(\xi_p^Q) h_l(\eta_q^Q) |J|_{pq} \omega_p^Q \omega_q^Q
$$

$$
= \sum_{p=1}^{Q} \sum_{q=1}^{Q} h_i(\xi_p^Q) h_j(\eta_q^Q) h_k(\xi_p^Q) h_l(\eta_q^Q) |J|_{pq} \omega_p^Q \omega_q^Q
$$

$$
= \sum_{p=1}^{Q} \sum_{q=1}^{Q} \delta_{ip} \delta_{jq} \delta_{kp} \delta_{lq} |J|_{pq} \omega_p \omega_q = \delta_{ik} \delta_{jl} |J|_{kl} \omega_k \omega_l
$$

which means that the element mass matrix is **diagonal**. The globally assembled mass
matrix $M = \sum_{e=1}^{E} M_{i,j,kl}^e$ is hence also diagonal. The mass matrix inversion appearing in the
explicit time-stepping scheme is then computationally trivial to perform.

Once the interpolation and quadrature rules are set, it is straight-forward to compute the

### A.1.4 Solution of the Discrete Poisson Equation

Once the vorticity is updated via numerically integrating A.12, the discrete Poisson equation A.6 is solved for the associated streamfunction. The matrix $K$ is symmetric positive
definite of size $P \times P$ where $P$ is the total number of streamfunction nodes. Its storage, of
$O(P^2)$ memory units, and inversion, of $O(P^3)$ work unit, become prohibitive for large-size
problem. The approach followed here is to solve the system iteratively via Preconditioned
Conjugate Gradient (PCG) iterations. However, in order to limit the number of iterations
one needs to make to reduce the error, we reduce the size of the total system by relying on
static condensation to eliminate those degrees of freedom that are local to an element. The
resulting matrix system, the Schur complement, couples only the edge nodes of a spectral element; once the solution on the edge nodes is obtained the interior values are obtained by solving small local problems in parallel for each element. The reduction in the number of unknowns depends on the geometry of the problem; for a rectangular problem with \( E_x \times E_y \) elements in the \( x \) and \( y \) directions, respectively, and is

\[
\frac{(E_x P + 1)(E_y P + 1)}{(E_x P + 1)(E_y P + 1) - E_x E_y (P - 1)^2} = \frac{P}{2} \left[ 1 + \frac{E_x + E_y}{E_x E_y P} + \frac{1}{E_x E_y P^2} \right] \]  
(A.38)

which asymptotes to \( P/2 \) for large \( E_x \) and large \( P \). The Schur complement system is solved via PCG iterations, using the diagonal of the Schur matrix as a preconditioner.

### A.2 Discretization of the density equation

Our interest here is in simulating flows with high Reynolds and Peclet numbers, where density advection dominates diffusion. There are well-known difficulties in solving such an equation using centered-type schemes, such as the classical spectral element method, when sharp fronts are present. Discontinuous Galerkin Methods [Cockburn and Shu, 1998] are much more suitable then. The density evolution equation in DGM form is:

\[
\int_{A_e} \chi \frac{\partial \rho}{\partial t} \, dA = \int_{A_e} \left( u \rho - \frac{q}{P R_e} \right) \cdot \nabla \chi \, dA + \int_{\partial A_e} \left( u \tilde{\rho} + \frac{1}{P R_e} \tilde{q} \right) \cdot n \chi \, dS \tag{A.39}
\]

\[
\int_{A_e} \Phi q \, dA = \int_{\partial A_e} \Phi \tilde{T} n \, dS - \int_{A_e} T \nabla \Phi \, dA \tag{A.40}
\]

where \( \chi \) are the discontinuous shape and weight functions, \( \tilde{\rho} \) refers to the density taken from the upstream of edges \( \partial A_e \), and \( \tilde{q} \) refers to the diffusive fluxes computed as the weak gradient of the function \( T \). For the diffusive fluxes we adopt the local discontinuous flux presented in Cockburn and Shu [1998] and Shu [2001]. The Galerkin discretization of the density equation proceeds by dividing the flow domain into quadrilateral spectral element
and representing the solution in terms of Lagrangian polynomials that are discontinuous on inter-element boundary. We use a nodal basis for the polynomials whose collocation points are the Gauss roots of the $N$-degree Legendre polynomials. The associated quadrature weights yield exact integration for polynomials of degree less or equal to $2N + 1$. This yields a local diagonal mass matrix for each element, see for example Giraldo et al. [2002], and does not incur any approximation when the elements are rectangular. Note that the density representation uses one collocation less than that for the vorticity and streamfunction.

### A.3 Boundary conditions for gravity current simulation

#### A.3.1 The top boundary

We treat the surface as a stress-free rigid-lid where no density flux is permitted. The rigid-lid approximation precludes the simulation of surface waves, but allows us to simplify the formulation so we don’t have to deal with a moving domain. These physical boundary conditions can be translated directly into conditions on the streamfunction and vorticity. Since $\nu = -\psi_x = 0$ along $z = 0$, the streamfunction must be constant along the rigid-lid and must be equal to the total transport through the domain:

$$\psi(x, 0, t) = Q(t)$$  \hspace{1cm} (A.41)

The stress condition can be turned into a condition on the vorticity:

$$\zeta(x, 0, t) = v_x(x, 0, t) - u_y(x, 0, t) = -u_y(x, 0, t) = -\frac{\tau(x, t)}{\nu}$$  \hspace{1cm} (A.42)

where $\tau$ is the wind stress at the surface. Here we don’t consider the effects of winds, and so $\tau = 0$. Finally, the no-density flux condition is $\nabla \rho \cdot \mathbf{n} = 0$. 

A.3.2 The bottom boundary

The bottom boundary can be treated similarly to the rigid-lid boundary except as far as the density flux, and the streamfunction are concerned:

$$\psi = 0, \text{ and } \nabla \rho \cdot n = 0.$$  (A.43)

If free-slip boundary conditions are applied, then the vorticity at the sea-bottom is also zero. A minor complication arises if no-slip conditions are imposed, since it seems that two boundary conditions can be imposed on the streamfunction at that boundary and none on the vorticity. This is taken up in section A.3.3.

A.3.3 The no-slip boundary condition

It is well-known that imposing the no-slip boundary condition in a streamfunction-vorticity formulation requires special care [Roache, 1982]. On a boundary aligned with the x-axis for example, the normal and tangential boundary conditions yield two conditions on the streamfunction: 1) the normal flow condition $v = -\psi_x = 0$ or $\psi = \text{constant}$, and 2) the tangential flow condition $u = \psi_y = 0$, and none on the vorticity. The solution is to enforce equation 3.2 inside the computational domain and on its boundary, thereby providing a value for the boundary vorticity. Weinan and Liu [1996] and Liu and Weinan [2000] pointed out that, provided the vorticity equation is integrated explicitly, the update of the vorticity and streamfunction can be decoupled regardless of the underlying spatial discretization. Their procedure can be applied as follows here:

1. Integrate explicitly the vorticity equation to update all interior vorticity nodes.
2. Solve the Poisson equation subject to streamfunction Dirichlet boundary conditions. The value of the vorticity on the boundary is immaterial for this step since $\Psi = 0$ on those boundaries, and hence the boundary vorticity is not needed. The boundary
integral on the right hand side of A.2 need not be computed either.

3. Once the streamfunction is known everywhere equation A.2 can be used to determine the boundary vorticity. Equation A.2 is then viewed as a definition for the vorticity with weight functions associated with boundary vorticity unknowns. The boundary integral must be included in the computations whenever the tangential velocity at the boundary is non-zero. Equation A.2 need not be applied in the interior of the domain, but only on elements with boundary edges. The entire procedure involves only the inversion of the mass matrix which, in our case, is diagonal.

A.3.4 Open boundary conditions

Open boundary conditions are an artifact of having to truncate large computational domains to a practical size so that the computations would not consume an undue or impractical amount of resources (CPU time and storage). The PDEs require us to supply information at these artificial boundaries to make the problem well-posed. Open boundaries are those portions of the domain borders where fluid is allowed to leave or enter the domain. At the inlet boundaries, the properties of the incoming flow must be specified, whereas at the outlet boundaries, fluid must be allowed to leave the domain with as few “constraints” as possible. The chief difficulties concern the specification of the incoming flow properties, when the latter are not known at priori, or when they actually depend on the flow conditions inside the domain, for example when flow reversal occurs in a time-dependent flow. When flow reversal occurs, it is easy to imagine that the density that just left the domain should be allowed back in; hence the open boundary density cannot be specified arbitrarily.

Inlet conditions

To simplify our tasks, and to minimize the opportunity for “noise” generation we select the left boundary to be an inlet boundary at all times. All flow properties are imposed: velocity
profiles and density profiles are specified: $v(0,y,t) = 0$ and $u(0,y,t) = Q/h$ where $h$ is the depth at the inlet-boundary and $Q$ is the total transport. These conditions on the velocity can be translated into boundary conditions on the vorticity and streamfunction

$$\psi(0,y,t) = \int_{-h}^{y} \psi_y \, dy = \int_{-h}^{y} u \, dy = \frac{Q}{h} y \quad (A.44)$$

$$\zeta(0,y,t) = v_x - u_y = v_x = 0, \quad (A.45)$$

where we have assumed that $v_x = 0$; this is tantamount to requiring the incoming flow to enter the domain without vorticity. The density profile at the inlet is specified as:

$$\rho(0,y,t) = \frac{\rho_{\text{max}} + \rho_{\text{min}}}{2} - \frac{\rho_{\text{max}} - \rho_{\text{min}}}{2} \tanh \left( \frac{y - y_c(t)}{\delta} \right) \quad (A.46)$$

where $y_c(t)$ controls the height of the transition from light to dense water, and $\delta$ controls the width of the transition zone. For $(y - y_c) \gg \delta$, $\rho \to \rho_{\text{min}}$, while $(y_c - y) \gg \delta$, $\rho \to \rho_{\text{max}}$ with a smooth transition of characteristic width $\delta$ in-between.

**Inlet/Outlet boundary conditions**

The out-flow boundary is considerably more delicate to set up. Our goal is to impose the weakest possible constraints to allow the flow to evolve freely. We begin by imposing a weak constraint on the vorticity where we require its normal gradient to be zero:

$$\nabla \zeta \cdot n = 0, \text{ on } x = L \quad (A.47)$$

The streamfunction profile at the open boundary is a little more complicated. One could impose $\psi_x = v = 0$ on the open boundary, but this is actually too strong of a constraint since $v \neq 0$ when vortex rolls exit the domain. Instead we impose the normal gradient of the vertical velocity be zero at the open boundary: $\nabla v \cdot n = v_x = 0$. The vorticity definition reduces then to $\zeta = v_x - u_y = -u_y = -\psi_{yy}$ at the open boundary. This equation can be
solved locally for the streamfunction if the boundary vorticity is available, and the velocity profile of the exiting fluid is free to evolve under the control of local dynamics, and with reasonable assumptions. Finally, for the density equation, and in light of the very high Peclet numbers used in our experiment we neglect the diffusive operator at the open boundary and deal exclusively with the advective fluxes of density. If the fluid is leaving the domain, the dense water is allowed to exist. If it is entering the domain a density value, \( \rho_o(y,t) \), is imposed, whose value depends on the “memory” of the system:

\[
\rho_o(y,t) = \rho_r + (\rho_e(L,y,\tau) - \rho_r)r^{\frac{r-1}{\Delta t}} \tag{A.48}
\]

where \( \rho_r \) is a reference density, \( \tau \) is the last time since the fluid exited the domain, \( u(L,y,t) > 0 \), with density \( \rho_e \), \( \Delta t \) is the time step, and \( r = 1 - \epsilon \) is a correction damping factor. Note that \( \frac{t-\tau}{\Delta t} = n \) is just the number of time steps since the fluid last exited at that level. The following behavior is easy to verify: a short time after the flow reverses \( n \) and hence \( r^n \) are close to 1, and the last exited value is restored with a small nudging. If the fluid keeps entering for a long time then \( n \gg 1 \) and \( r^n \approx 0 \), and the restored value reverts to the reference \( \rho_r \). In most of our experiments we have used \( r = 0.9999 \) so that the half life of the correction is 7,000 time steps. This amounts to a very weak nudging to the reference density.

### A.4 Overall solution procedure

The following points are worth mentioning regarding the discretization and integration procedures. First, the vorticity equation is integrated explicitly using a third-order Adams-Bashforth (AB3) scheme, and all interior vorticity values are updated. Second, the streamfunction is updated by solving equation A.2. A substructing scheme is used to consolidate the system of equations into the degrees of freedom on the edges of elements, and the latter system is solved with conjugate gradient iterations using diagonal preconditioning. Third, the density field is time-integrated using an AB3 scheme.
Bibliography


