Anomalous Structures in Oceanic Turbulence: Dynamics, Energetics, Transport

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UNIVERSITY OF MIAMI

ANOMALOUS STRUCTURES OF OCEANIC TURBULENCE: DYNAMICS, ENERGETICS, TRANSPORT

By

Mykhailo Rudko

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ANOMALOUS STRUCTURES OF OCEANIC TURBULENCE: DYNAMICS, ENERGETICS, TRANSPORT

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In this thesis, anomalous structures of oceanic mesoscale turbulence, as exemplified by coherent vortices and zonally-elongated transient flows (ZELTs), are considered. All simulations of mesoscale turbulence flows are performed in a two-layer, quasi-geostrophic model.

The first chapter explores stability of and transport by baroclinic vortices on the $\beta$-plane. The study adapts a wave-mean flow formalism and examines interactions between the axisymmetric flow (“the vortex”) and residuals (“the waves”). Unlike baroclinically unstable vortices on the $f$-plane, such vortices on the $\beta$-plane can be also vulnerable to barotropic instability as revealed by the globally integrated energy balance analysis. The spatial structure of energy fluxes shows the energy leakage inside the vortex core when its breakdown occurs. Mixing by stable and unstable vortical flows is quantified through the computation of Finite-Time Lyapunov Exponent (FTLE) maps. Depending on the strength of wave radiation, the upper-layer FTLE maps of stable vortices show either an annulus or volute ring of vigorous mixing inside the vortex interior. This ring region is disrupted when the vortex becomes unstable. Both stable and unstable vortices show the wavy patterns of FTLE in the near- and far-fields. Despite the fact that the initial vortex resides in the top layer only, significant FTLE patterns are observed in the deep layer at later times. La-
Grangian analysis of the vortex-induced change of large-scale tracer gradient demonstrates significant effects of vortex instability in the top layer and the importance of the wave-like anomalies in the bottom layer.

The second chapter explores the phenomenology of zonally-elongated transients (ZELTs) in the ocean and the sensitivity of their properties to changes in several environmental factors. ZELTs explain a major part of anisotropy in mesoscale turbulent flow. Calculations are performed in a two-layer, quasi-geostrophic model. Empirical Orthogonal Functions (EOF) decomposition allows for the separation of ZELTs from the background turbulent flow as several leading EOF modes. The leading Extended EOF reveals that ZELTs propagate westward at the speed of \( \sim 1 \text{ cms}^{-1} \). The decrease in the planetary vorticity gradient and increase in the bottom drag coefficient each leads to flattening of the variance spectrum, isotropization of the leading EOF and fast decay of the autocorrelation function of its corresponding Principal Component.

The third chapter deals with the underpinning mechanisms of ZELTs formation. As evidenced by spatial Fourier spectrum, spatial structure of the leading EOF and the autocorrelation function of its corresponding Principal Component, simulations in reduced-dynamics models with completely removed eddy-eddy interactions show no presence of ZELTs, thereby suggesting that the physical mechanism based on energy cascade arguments is more plausible for the formation of ZELTs. The energy exchanges produced by baroclinic-baroclinic and mixed-mode interactions are the major cause of the emergence of ZELTs as revealed in simulations with both moderate and high values of bottom drag. Barotropic-barotropic interactions, which entail energy cascade in barotropic mode, play a secondary role in ZELTs development in
the moderate-drag simulation, and these interactions have no impact on the dynamics of ZELTs in the high-drag simulation.
to my family
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CHAPTER 1

Introduction

1.1 Anomalous structures of oceanic turbulence

Ocean general circulation is a complex physical system with the broadband kinetic energy spectrum. Mesoscale eddies, the flow structures with horizontal scale around the first baroclinic Rossby deformation radius, account for the peak in this spectrum [112]. Arising due to barotropic or baroclinic instabilities of large-scale currents, these flow features play an important role in regulating ocean circulation. Mesoscale eddies can maintain western boundary and equatorial currents. Additionally, mesoscale eddies redistribute material concentration throughout the oceanic interior. Even in simplified models of ocean circulation mesoscale eddies can induce decadal climate variability [14]. Among weak turbulent eddies one can clearly distinguish two types of anomalous structures: long-lived, coherent vortices and low-frequency, zonally-elongated flow patterns. These flow structures introduce anomalous scalings in geophysical turbulence [107, 108] and cause inhomogeneous [65] and anisotropic distribution of various tracers [37, 64, 100]. Sufficiently resolved simulations of comprehensive climate models suggest the inability of parameterization schemes to correctly represent the feedback of mesoscale eddies on the large-scale
climate [62]. Vortices and zonally-elongated flow patterns can be the causes of inefficiency of a diffusive approach to parametrize the effects of mesoscale eddies. Proper understanding of dynamics, energetics and transport properties of vortices and ZELTs is a necessary step towards developing reliable parameterizations.

Much of our current understanding on the predictability, dynamics and structure of turbulent flows comes from the spectral theory of turbulence, which makes a prediction of how much energy is contained at each scale of motion in a statistically steady state [63]. For three dimensional isotropic turbulent flows this theory predicts the existence of inertial range between forcing and dissipation scales within which the flux of turbulent kinetic energy $\epsilon$ is constant and directed from larger to smaller scales; the energy spectrum $E(k) = C_k \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$ depends on the rate of energy flux $\epsilon$ and wavenumber $k$, $C_k$ is a constant [63]. The conservation of additional invariant, enstrophy, in two-dimensional turbulent flows leads to the existence of two inertial ranges. The energy in such a fluid system cascades upscale with energy spectrum $E(k) = C_k \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$ in the energy inertial range, while the enstrophy cascades downscale with energy spectrum given by $E(k) = C_k \eta^{\frac{2}{3}} k^{-3}$ in the enstrophy inertial range. However, numerical simulations of decaying two-dimensional turbulence reveal the abundance of spatially localized, coherent vortices [74] with distinct statistics as expected from classical turbulence theory [50]. These flow structures are responsible for the deviant slope of energy spectrum in forced-dissipative turbulence [16].

Dynamics of mesoscale turbulent flows is different from those in classical three- or two-dimensional turbulence in that it is affected by stratification and the Earth’s rotation. This dynamics is governed by quasi-geostrophic equations [81]. The studies on energy transport in stratified, rotating, quasi-geostrophic fluid suggest that total
energy injected into the fluid system at large horizontal scale is transferred from higher baroclinic modes to the first baroclinic mode and then it is converted into the kinetic energy of barotropic mode [47, 88]. Non-uniform stratification inhibits the energy transfers between the first baroclinic and barotropic modes, so that the energy accumulates near the first baroclinic Rossby deformation radius [38, 99]. Even though the energy transfers are inhibited, the energy still makes its way to the first barotropic mode, where it experiences upscale cascade. The upscale energy cascade is slowed down by the $\beta$-effect, although the ultimate terminus of energy cascade occurs by bottom friction. Such anisotropization of the barotropic inverse energy cascade creates a conducive environment for the emergence of zonally-elongated flow patterns [85]. The estimates of kinetic energy transport from altimetry data suggest that this transport is directed towards larger scales [93]. Since the first baroclinic mode is more pronounced than the barotropic mode at the sea surface this inevitably suggests that kinetic energy in the baroclinic mode also experiences upscale cascade, which can be 10 times more intense than the same cascade in barotropic mode [91]. Finally, the $\beta$-effect has also a strong impact on the propagation of coherent vortices [33] and their lifecycle in two-dimensional turbulence [69]. Although several studies addressed the problem of joint influence of stratification and rotation on the dynamics of vortices [19, 76], such dynamics is still poorly understood. Quantifying material transport induced by vortices in a stratified non-uniformly rotating fluid is a challenging problem with far-reaching applications.

In this thesis we address the following topics: 1) dynamics of and transport by vortices and 2) statistical properties and dynamics of zonally-elongated flow patterns in a stratified, rotating environment.
1.2 Thesis structure

In this section, we provide a general description of the content of each chapter and formulate the specific research problems covered in this work. The governing equations of quasi-geostrophic dynamics and the numerical model used in this work are described in chapter 2. We discuss numerical schemes used to solve the governing equations, specific type of forcing, boundary and initial conditions. We also mention several limitations and advantages of quasi-geostrophic model to studying oceanic turbulence. In chapter 3 we review the stability of vortices on the $f$-plane, discuss several approaches to study mixing by vortical flows and provide some estimates of large-scale transport by these flow structures from previous studies. Specific questions targeted in this chapter are the stability properties of baroclinic vortices on the $\beta$-plane, the impact of their breakdown on mixing and large-scale transport. In Chapter 4 we introduce the notion of zonally-elongated transient flows (ZELTs) and discuss some difficulties arising from extracting ZELTs from the background flow. We also study their sensitivity to several environmental parameters. Chapter 5 deals with the dynamics of ZELTs. We discuss two major mechanisms of zonal flows formation and elaborate on their relevance to ZELTs. Chapter 6 concludes our work and offers several directions for future research.
CHAPTER 2

Numerical Model Configuration

2.1 Governing equations

The numerical model used in this work is based on a traditional quasi-geostrophic approximation [81] supplemented with lateral viscosity and bottom friction. The stratification is represented by two stack isopycnal layers and the nonuniform Earth’s rotation is approximated by the $\beta$-effect.

The governing equations are the conservation of potential vorticity in each layer:

\begin{align}
\frac{\partial q_1}{\partial t} + J(q_1, q_1) &= F + \nu \nabla^4 \psi_1 \\
\frac{\partial q_2}{\partial t} + J(q_2, q_2) &= -\gamma \nabla^2 \psi_2 + \nu \nabla^4 \psi_2
\end{align}

(2.1)

where $\psi_n$ is the streamfunction in layer $n$, $n = 1, 2$ (hereafter, indices 1 and 2 refer to the top and bottom layers, respectively), $J(a, b) = \frac{\partial a \partial b}{\partial x \partial y} - \frac{\partial a \partial b}{\partial y \partial x}$ is the Jacobian operator, $\nu$ is a lateral eddy viscosity and $\gamma$ is a bottom drag coefficient, $F$ is a forcing function, the concrete expression of which will be discussed later. Potential vorticity is related to the streamfunction in each layer by
\[ q_1 = \nabla^2 \psi_1 + S_1(\psi_2 - \psi_1) + \beta y \]

\[ q_2 = \nabla^2 \psi_2 + S_2(\psi_1 - \psi_2) + \beta y \]

The stratification parameters \( S_n \) are given by \( S_n = \frac{f_n^2}{H_n g'} \). Here \( H_n \) stands for the thicknesses of the layers at rest, \( g' = g \frac{\Delta \rho}{\rho} \) is a reduced gravity acceleration. The first baroclinic Rossby deformation radius \( R_d = \sqrt{\frac{1}{s_1 + s_2}} = \frac{1}{f_0} \sqrt{g' \frac{H_1 H_2}{H_1 + H_2}} \) is an important length scale that determines properties of baroclinic flow.

Periodic boundary conditions are applied on each boundary. The nonlinear terms are discretized with the Arakawa energy, enstrophy and symmetry conserving scheme [1]. The time integration is performed with Adams-Bashford 3rd-order scheme.

### 2.2 Initial conditions, parameter range, the structure of forcing and examples of output

#### 2.2.1 Baroclinic vortex on the \( \beta \)-plane

For studies in chapter 3, the model is initialized with \( \psi_1 = a \exp(-\frac{r}{b})^2 \) and \( \psi_2 = 0 \), where \( r = \sqrt{x^2 + y^2} \). The parameters \( a \) and \( b \) are defined as \( a = V_{\text{max}} R_{\text{max}} \exp(2) \) and \( b = \sqrt{2} R_{\text{max}} \), where \( V_{\text{max}} \) is the maximum swirling velocity and \( R_{\text{max}} \) is the radius of the maximum swirling velocity. A Gaussian profile is arguably not the best approximation for oceanic vortices [18], but it simplifies the comparison of our results with previous studies and guides us in choosing the corresponding parameters. The
governing equations are solved on the square domain of 2560 km in each direction with the spatial resolution of 5 km. The values of all parameters used in this paper are summarized in Table 2.1 and chosen from the stability diagram in [77].

The snapshots of the upper-layer streamfunction for the vortex with $R_{\text{max}} = 100$ km are shown in Fig. 2.1. Initially propagating in the north-west direction and slowly fading due to wave radiation, the vortex becomes contorted at day 37. It tends to restore its shape after this day (see Fig. 2.1d). At day 59 the vortex deforms again and spawns two vortices at day 100. This is an example of vortex breakdown. The same vortex with $R_{\text{max}} = 70$ km also propagates in the north-west direction with the waves observed in its wake (not shown), yet it remains coherent during the entire period of its evolution.

The tracks of stable and unstable vortices also reveal the effects of the instability (Fig. 2.2). While stable vortices ($R_{\text{max}} = 60$ km and $R_{\text{max}} = 70$ km) move to the northwest along a straight line, except for the vortex with $R_{\text{max}} = 80$ km which develops small oscillations along its trajectory, the unstable vortices ($R_{\text{max}} = 90$ km–120 km) have looping trajectories.

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<th>CASE</th>
<th>$V_{\text{max}}, \text{ms}^{-1}$</th>
<th>$R_{\text{max}}, \text{km}$</th>
<th>$\beta, \text{m}^{-1}\text{s}^{-1}$</th>
<th>$R_d, \text{km}$</th>
<th>$\delta = \frac{n_1}{H_2}$</th>
<th>$\nu, \text{m}^2\text{s}^{-1}$</th>
<th>$\tau, \text{s}^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stable Vortex</td>
<td>0.8</td>
<td>60</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>Stable Vortex</td>
<td>0.8</td>
<td>70</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>Stable Vortex</td>
<td>0.8</td>
<td>80</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>Unstable Vortex</td>
<td>0.8</td>
<td>90</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>Unstable Vortex</td>
<td>0.8</td>
<td>100</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>Unstable Vortex</td>
<td>0.8</td>
<td>110</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>Unstable Vortex</td>
<td>0.8</td>
<td>120</td>
<td>$1.7 \times 10^{-11}$</td>
<td>45</td>
<td>0.1628</td>
<td>100</td>
<td>$10^{-7}$</td>
</tr>
</tbody>
</table>

Table 2.1: Model parameters
2.2.2 Zonally-elongated transient flows

For anisotropic turbulence simulations, we use the forcing function $F$ in the form of baroclinic velocity shear $\Psi_i = -U_i y, \psi_i \rightarrow \Psi_i + \psi_i$ [42]. The eastward directed background flow was restricted to the upper layer only, $U_1 = U = 6 \text{ cms}^{-1}$ and $U_2 = 0 \text{ cms}^{-1}$. Such a background flow is subject to baroclinic instability [81]. The governing equations can be rewritten as

$$\frac{\partial \xi_1}{\partial t} + J(\psi_1, \xi_1) + (\beta + S_1 U) \frac{\partial \psi_1}{\partial x} = -U \frac{\partial \xi_1}{\partial x} + \nu \nabla^4 \psi_1$$

(2.3)

$$\frac{\partial \xi_2}{\partial t} + J(\psi_2, \xi_2) + (\beta - S_2 U) \frac{\partial \psi_2}{\partial x} = -\gamma \nabla^2 \psi_2 + \nu \nabla^4 \psi_2$$

(2.4)

where

$$\xi_1 = \nabla^2 \psi_1 - S_1 (\psi_1 - \psi_2)$$

$$\xi_2 = \nabla^2 \psi_2 - S_2 (\psi_2 - \psi_1)$$

Isotropic turbulence simulations were conducted on the $f$-plane

$$\frac{\partial \xi_1}{\partial t} + J(\psi_1, \xi_1) = F + \nu \nabla^4 \psi_1$$

(2.5)

$$\frac{\partial \xi_2}{\partial t} + J(\psi_2, \xi_2) = -\gamma \nabla^2 \psi_2 + \nu \nabla^4 \psi_2$$

(2.6)

with a forcing function of the following form

$$F(x, y, t) = Re\left(\sum_{m=1}^{N_x} \sum_{n=1}^{N_y} (A - \frac{1}{\sqrt{\delta t}}w_{mn} e^{i\phi(t)}) e^{i(k_m x + l_n y)}\right)$$

where $A$ is a constant amplitude, $\sqrt{\delta t}$ is a square root of the time step, $\phi(t)$ is a randomly chosen phase from the interval $[0, 2\pi]$ of a uniform distribution, $k_m = \frac{m}{L_x}$ and $l_n = \frac{n}{L_y}$ are the wavenumbers in the zonal and meridional directions, $L_x$ and
$L_y$ are the length of the domain in the zonal and meridional directions. We use the “ring-shaped” forcing function for which

$$w_{mn} = \begin{cases} 
1, & |K - K_f| \leq 3 \\
0, & |K - K_f| > 3.
\end{cases}$$

(2.7)

$K = \sqrt{k_m^2 + l_n^2}$, $K_f = 33$. This is a typical forcing function used in many studies of two-dimensional turbulence [25].

The model was integrated in the rectangular domain ($L_x \times L_y$, $L_y = 3600$ km and $L_x = 7200$ km) and in the square domain ($L_x \times L_y$, $L_x = 3600$ km and $L_y = 3600$ km). The number of grid points in zonal and meridional directions were chosen to be $N_x = 512$ and $N_y = 256$ for the rectangular domain, and $N_x = N_y = 256$ for the square domain. This gives the resolution of approximately 14 km for both cases. The first baroclinic Rossby deformation radius is $R_d = 25$ km for all cases. Although the minimum resolvable length scale ($2 \times 14$ km = 28 km) is larger then $R_d$, the energy injection scale ($\sim 7 \times R_d$) still falls within resolvable range. The model is integrated for for 70000 days, or equivalently, 192 years.

Examples of model output are shown in Fig. 2.3. Simulated anisotropic turbulent flow (Fig. 2.3a) consists of multiple zonal jets and numerous eddies. Isotropic turbulent flow contains eddies propagating around the domain and interacting with each other (Fig. 2.3b). Unlike the anisotropic turbulent flow, the isotropic flow does not have a preference for zonal direction and is not expected to have ZELTs.
2.3 Advantages and limitations of the numerical model

Quasi-geostrophic model approximates the dynamics of mesoscale currents. In this work we use the simplest possible configuration of this model. Several other environmental parameters can affect the dynamics of anomalous structures. First, it is widely known that vortices can experience catastrophic fate upon encountering topographic ridges. Topography itself changes the gradient of the ambient potential vorticity, which in turn may have serious implications for the statistics of turbulent flow field. Second, the numerical model represents stratification in the simplest possible way. As indicated in [96], the higher baroclinic modes have a notable impact on the routes of energy circulation. Among the advantages of this model is its computational efficiency, which allows an efficient exploration of its parameter space. However, this model does not differentiate between cyclonic and anticyclonic vortices, the stability properties of which can be substantially different ([6, 28]). Additionally, this model does not capture the dynamics of submesoscale currents. Marginally controlled by rotation and stratification, these currents might be one of the significant conduits for energy dissipation. They also have relatively large vertical velocities, thus inducing large vertical material fluxes.
Figure 2.1: Evolution of unstable vortex with $R_{max} = 100$ km at a) 1 day b) 20 day c) 37 day d) 45 day e) 59 day f) 100 day. Shown is streamfunction in the upper layer. Units are $\frac{m^2}{s} \times 10^4$
Figure 2.2: Tracks of vortices with different radii of maximum swirling velocity within 140 days.

Figure 2.3: Upper-layer snapshots of the velocity streamfunction $\times 10^4$ of a) anisotropic and b) isotropic turbulent flows.
CHAPTER 3

Stability of baroclinic vortices on the $\beta$-plane and implications for transport

3.1 Background

Mesoscale coherent vortices are found in every part of the World Ocean [20, 56]. Observational records suggest that such vortices have a baroclinic structure, and many exhibit a very long lifecycle [78]. However, some of these vortices have been found to break down shortly after generation [90]. Under the influence of the Earth’s rotation, these structures propagate in the north- or south-west directions depending on the sense of their own spin [33]. Oceanic vortices can generate regions of anomalous mixing [82] and contribute to large-scale transport [29]. The stability and transport properties of baroclinic vortical flows on the $\beta$-plane are the main topics to be addressed in this chapter.

Laboratory experiments [41, 106] and numerical simulations [35, 44, 51] of baroclinic vortices on the $f$-plane suggest that the vortices with radius larger than the Rossby deformation radius are vulnerable to the wave-number 2 instability as revealed by normal-mode analysis. This goes at odds with observations and several attempts have been pursued in order to resolve this paradox. One factor which stabilizes
vortices in the upper layer is the magnitude and sense of the flow in the lower layer; a co-rotating flow in the lower layer strengthens the barotropic component of the flow, thereby inhibits baroclinic release of energy by its barotropic gain [27,28,61]. [5] argues that the parametric range of baroclinic vortex stability can be extended even further if the flow in the lower layer has uniform potential vorticity. In view of the absence of radial shear in the potential vorticity profile of the basic state, the lower layer flow does not support a counter propagating Rossby-vortex wave that can resonate with the one in the upper layer, which is necessary for the structure of the most unstable disturbance.

On the other hand, simulations of two-layer geostrophic turbulence on the $\beta$-plane show that the vortices tend to be depth-compensated rather than having significant flow in the lower layer [12]. Within some range of parameters, not coinciding with the stability diagram on the $f$-plane, an initially depth-compensated, baroclinic vortex on the $\beta$-plane undergoes the leakage of potential vorticity out of its core [77]. We use the term breakdown to describe this process. Nevertheless, the vortex becomes stable if the simulation is carried out in the equivalent-barotropic model. This model does not have a mechanism of the conversion of the basic state available potential energy into the kinetic energy of disturbances, and thus, baroclinic instability seems at first to be the main cause of the vortex breakdown. However, the lack of the mixed-mode eddy-momentum fluxes in the equivalent-barotropic model partially precludes the release of the kinetic energy of the basic state, which is essential for barotropic instability.

Inhomogeneous mixing areas in the ocean can partially be explained by the fact that fluid can be trapped inside vortex cores for long times [26,82]. Analyzing Finite-
Size Lyapunov Exponent (FSLE) maps computed from altimetry data, [68] show that an Agulhas ring has a core with weak mixing and the spirals of FSLE encircling its core. Finite-Time Lyapunov Exponent (FTLE) maps of two-dimensional turbulence confirm this structure of the mixing patterns within the vortex interior, but also reveal outgoing FTLE filaments from one vortex which end in the close proximity of another vortex [66]. Similarly, the evolution of the tracer concentration produced by stable barotropic vortical flow on the $f$-plane shows the formation of spirals inside the vortex interior unless the initial concentration of the released tracer has the same form as potential vorticity or streamfunction profile of the vortex [86]. These spirals eventually disappear due to the action of turbulent diffusion. The evolution of the same vortex on the $\beta$-plane exhibits spiraling and a high-gradient strip peeling off the vortex [4]. It remains unknown, however, as to how the baroclinic structure of the vortex, its potential breakdown and radiation of waves change these mixing patterns.

The $\beta$-induced drift of the vortices and inhibited permeability of their interior for material fluxes make these structures to be one of the major drivers of global meridional transport [29]. [3] demonstrate the importance of Agulhas rings in changing thermohaline properties of the Atlantic waters. Stable North Brazil current rings contribute up to 9.3 Sv/year of water transport from tropical to subtropical gyres in the Atlantic Ocean [40, 57]. In their estimates of vortex-induced transport, these authors assume that the flows generated by vortices in the deeper layers and radiated waves do not contribute significantly to the total transport. Additionally, the transport estimates of those studies are based on an assumed shape of vortices which is assumed to be constant in time. While these assumptions seem be reasonable for stable vortices, unstable vortices generate strong flows in the deeper layers, do not
preserve their shape and radiate waves that might also change the distribution of tracers in the vicinity of vortex, as well as in the far field.

We present an analysis of globally integrated and local energy balances to find the mechanism of the instability of baroclinic vortices on the $\beta$-plane. We then utilize Finite-Time Lyapunov Exponent (FTLE) maps to compare mixing regimes of stable and unstable vortical flows. Finally, we estimate the large-scale transport produced by vortices from the latitudinal change of Lagrangian particle concentration.

\section*{3.2 Energetics}

\subsection*{3.2.1 General discussion of vortex energetics}

The problem of vortex instability on the $f$-plane is typically approached by linearizing the governing equations around an initial vortical profile and subsequently calculating eigenvalues and eigenvectors of a time-independent, linearized dynamical operator. The eigenvalues and corresponding eigenvectors (e.g. normal modes) provide information about the growth rate and structure of infinitesimal disturbances initially impressed on the vortex. On the $\beta$-plane, however, the same initial vortical profile is not a stationary solution of the governing equations; the linearized dynamical operator becomes time-dependent, and the concept of normal modes does not apply \[30\]. A more comprehensive technique to tackle the instability problem is to analyze the energy exchanges between the basic-state flow and disturbances. We analyze globally integrated and local energy balances for unstable vortical profiles with parameters listed in Table 2.1. Before proceeding to the discussion of the energy balance and the results of calculations several definitions are required. First,
the term “vortex” further down in this section stands for the azimuthally-averaged flow in the polar reference frame co-moving with the vortex center. Second, the term “waves” refers to the asymmetric motions defined as the difference between total and azimuthally-averaged (“vortex”) flows in the co-moving system of reference. Finally, we adopt the term “instability” to describe the process of vortex breakdown, although no linearization has been performed. Thus, we do not impose any prior assumption on the smallness of the wave field magnitude compared to the magnitude of the basic state.

The energy balance derived in the Appendix A shows that the energy of the vortical flow is partitioned between the barotropic and baroclinic modes. The barotropic mode contains only kinetic energy while the baroclinic mode has both kinetic and available potential energies. The vortex releases its energy when the integrated energy tendency is negative, or gains energy if the integrated energy tendency is positive. Except for the frictional dissipation, the energy redistribution is given by the energy fluxes which are on the right hand side of the energy balance equations. To facilitate the analysis we combine all energy flux terms associated with the $\beta$-effect, co-moving reference frame and dissipation with the left-hand side, so that our energy balance takes the following form:

$$\int_0^\infty \partial_t <E_{BT}^* > r dr = \int_0^\infty \frac{1}{\alpha_T} <\psi_{BT} > < J(\psi_{BT}', \dot{q}_{BT}') > + <\psi_{BT} > < J(\psi_{BC}', \dot{q}_{BC}') > r dr \quad (3.1)$$
\[
\int_0^\infty \frac{\partial \langle E_{BC}^s \rangle }{\partial t} r dr = \int_0^\infty \langle \psi_{BC} \rangle \langle J(\psi_{BC}', \xi_{BC}') \rangle + \langle \psi_{BC} \rangle \langle J(\psi_{BC}', q_{BC}') \rangle \\
- (S_1 + S_2) < \psi_{BC} > < J(\psi_{BC}', \psi_{BT}') > \\
+ \alpha_C < \psi_{BC} > < J(\psi_{BC}', q_{BC}') > r dr
\] 

(3.2)

where the left-hand sides of these equations defined by:

\[
\int_0^\infty \frac{\partial \langle E_{BT}^s \rangle }{\partial t} r dr = \int_0^\infty \langle \psi_{BT} \rangle \frac{\partial \langle q_{BT} \rangle }{\partial t} r dr \\
- \int_0^\infty \langle \psi_{BT} \rangle \beta \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) < \psi_{BT}' \cos \theta > \\
+ \langle \psi_{BT} \rangle \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) (U(t) < q_{BT}' \cos \theta > + V(t) < q_{BT}' \sin \theta > ) \\
- < \psi_{BT} > < D_{BT} > r dr
\]

\[
\int_0^\infty \frac{\partial \langle E_{BC}^s \rangle }{\partial t} r dr = \int_0^\infty \frac{\partial \langle E_{BC} \rangle }{\partial t} dr \\
- \int_0^\infty \langle \psi_{BC} \rangle \beta \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) < \psi_{BC}' \cos \theta > \\
+ \langle \psi_{BC} \rangle \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) (U(t) < q_{BC}' \cos \theta > + V(t) < q_{BC}' \sin \theta > ) \\
- < \psi_{BC} > < D_{BC} > r dr
\]

The impact of the energy fluxes associated with fluxes the $\beta$-effect, co-moving reference frame and dissipation can be understood by comparing how linear versus nonlinear dynamics affect the vortex energy evolution. The time series of globally integrated energy of barotropic and baroclinic modes for different vortical profiles are
shown in Figure 3.1. Baroclinic energy exceeds barotropic energy by approximately two orders of magnitude and thus dominates the energy content of the vortex. After initial decline, barotropic energy increases dramatically reaching its first maximum at around day 45. The second dip in the magnitude of barotropic energy is observed between days 60 and 70 for different cases. For the cases of vortices with $R_{max} = 90$ km and $R_{max} = 100$ km, barotropic energy rises again and then subsequently decreases by the end of the vortex evolution. Barotropic energy of the bigger vortices ($R_{max} = 110$ km and $R_{max} = 120$ km) shows more irregular behaviour with several sharp jumps. In contrast, the initial and final trends of baroclinic energy remain nearly flat suggesting that the vortices remain in a nearly stationary state. The intermediate evolution of baroclinic energy for the vortices with $R_{max} = 90$ km and $R_{max} = 100$ km consists of two stages: it slightly declines up to the day 55 and then decreases more steeply up to the day 70. The baroclinic energy decays steadily for the vortex with $R_{max} = 110$ km and two periods of rapid decrease are observed for the vortex with $R_{max} = 120$ km before the energy levels off for both cases. The linear dynamics of vortices, in which all nonlinear energy fluxes are absent, shows that radiation from the barotropic mode occurs very efficiently, while the baroclinic mode remains relatively stiff ([34,76]). This prompts us to claim that the hollow parts of the curves of barotropic vortex energy and the flat parts of the curves of baroclinic vortex energy are explained by the energy drain caused mostly by the $\beta$-effect and partially by friction. The decrease of baroclinic vortex energy during the intermediate period of vortex evolution and sharp peaks in the evolution of barotropic vortex energy can be attributed to the action of nonlinear energy fluxes. We, therefore, argue that the energy fluxes associated with the $\beta$-effect, co-moving reference frame and dissipation
do not affect the rate of change of baroclinic vortex energy. The energy production of barotropic mode can be affected at the times when the nonlinear forcing is small. This, however, does not change the conclusions of this study.

The main goal of this section is to address the role of the kinetic and available potential energy fluxes as potential drivers of vortex instability. In table 3.1 we summarize all terms in energy balance, their acronyms and physical meaning. The acronyms follow the convention that “EC” indicates energy change, “W” indicates work, “REY” indicates the Reynolds stress forcing, “FORM” stands for the form stress forcing, “BT”, “BC” and “MX” are the shorthands for the words barotropic, baroclinic and mixed-mode, respectively. The terms WREY_BT, WREY_BC, WREY_MX, WREY_BC1 convert kinetic energy of the vortex and are associated with wave momentum fluxes; the term WFORM is responsible for the transformation of available potential energy of the vortex and associated with wave buoyancy fluxes. If the vortex loses its energy, these two different conversion mechanisms are typically associated with barotropic and baroclinic instabilities, respectively. The vortex can also strengthen at the expense of the wave energy.

3.2.2 Global energy balance analysis

We first present the time history of globally integrated energy balance based on equations 3.1 and 3.2. We use similar notations for all terms of energy balance as listed in table 3.1, but add the subscript $g$ which means the globally integrated quantity. During the entire period of vortex evolution (100 days), all cases indicate that the baroclinic mode energy experiences two periods of significant loss marked by the vertical dashed lines (Fig. 3.2). The first major loss of baroclinic energy is caused
by the WFORM\textsubscript{g} term meaning that the vortex loses energy through baroclinic exchanges. The WREY\textsubscript{MX}\textsubscript{g} and WREY\textsubscript{BC1}\textsubscript{g} terms act to increase baroclinic energy at this time but their contribution is almost negligible. However, all vortices stabilise after this time as indicated by the ascending trend of EC\textsubscript{BC}\textsuperscript{*}. For the vortices with $R_{\text{max}} = 90$ km and $R_{\text{max}} = 100$ km the loss of energy gradually reaches zero, while for the vortices with $R_{\text{max}} = 110$ km and $R_{\text{max}} = 120$ km the energy tendency becomes positive followed by a smooth decline. A more interesting behaviour is observed during the time of the vortex breakdown, when the second significant energy loss is observed. While the vortices with $R_{\text{max}} = 90$ km and $R_{\text{max}} = 120$ km release their energy due to WFORM\textsubscript{g} term, the vortices with $R_{\text{max}} = 100$ km and $R_{\text{max}} = 110$ km loose their energy through WREY\textsubscript{MX}\textsubscript{g} term.

The barotropic mode gains energy during the entire period and has two well-defined maxima barring the case of vortex with $R_{\text{max}} = 120$ km (Fig. 3.3). The energy gain is primarily supported by WREY\textsubscript{BT}\textsubscript{g} energy flux while WREY\textsubscript{BC}\textsubscript{g} flux plays a secondary role.

Summing everything up, we provide a physical interpretation of the calculations above. Since the dominant part of vortex energy is concentrated in the baroclinic mode, the energy production of this mode sets the energy production of the entire vortex. The baroclinic energy production of every unstable vortex becomes strongly negative twice and it is during the second period that the actual vortex splitting occurs. At this time, both energy fluxes associated with form stress forcing (WFORM\textsubscript{g}) and Reynolds stress forcing (WREY\textsubscript{MX}\textsubscript{g}) can cause this strong negative maximum in energy tendency. For the cases of vortices with $R_{\text{max}} = 90$ km and $R_{\text{max}} = 120$ km the main contribution to this energy loss is provided by energy fluxes associated with
wave buoyancy ($\text{WFORM}_y$), thereby the vortices are baroclinically unstable. For the cases of vortices with $R_{\text{max}} = 100$ km and $R_{\text{max}} = 110$ km this second negative maximum of energy change is supported by energy fluxes associated with the part of Reynolds stress forcing ($\text{WREY}_\text{MX}_y$), so that the vortices are barotorpically unstable. We also note a compensating role by the barotropic component, as previously suggested in [27]. For moderate and small ratios of $R_{\text{max}}$ to Rossby deformation radius the content of energy in barotropic and baroclinic modes is comparable in magnitude, thereby the loss in baroclinic energy is balanced by the gain in barotropic energy keeping the vortex stable. Increasing this ratio makes the baroclinic mode stronger, thereby leading to the vortex breakdown.

### 3.2.3 Local energy balance analysis

The integrated energy balance analyzed above provides information on total energy tendency and energy fluxes at a particular time. Such an analysis, however, does not show the location of maximum energy growth or decay nor does it give the insight into the way the energy fluxes set the energy rate distribution. We next present the analysis of local energy balance at the time of two most significant energy transformations marked by the vertical dashed lines in Fig. 3.2. We show the radial structure of the tendency and flux terms in figures 3.4-3.7. To quantify the relative role of each energy flux in the redistribution of energy, we compute the spatial correlation between the local energy fluxes and energy tendency. Table 3.2 summarises the results of these calculations. The notations $c_{\text{r}_{\text{br}}}$, $c_{\text{r}_{\text{bc}}}$ stand for the correlation coefficient between $\text{WREY}_\text{BT}$ and $\text{EC}_\text{BT}^*$, $\text{WREY}_\text{BC}$ and $\text{EC}_\text{BT}^*$, respectively.
The notations $cr_f$, $cr_{mx}$, and $cr_{bc1}$ stand for the correlation coefficient of each of WFORM, WREY_MX, WREY_BC1 and EC_BC*, respectively.

Figure 3.4 shows the radial profiles of energy fluxes and energy rate for the barotropic mode at the time of vortex deformation. For all cases, the barotropic mode gains energy within the vortex core with the maximum gain at the vortex center. The correlation coefficients between EC_BT* and energy fluxes indicate that the gain of the barotropic energy is primarily supported by WREY_BT flux while a supportive role of the WREY_BC flux is of less importance (Table 3.2). A similar result is observed at the time of vortex breakdown for all cases, with the exception of the vortex with $R_{max} = 120$ km which releases its energy at the center and receives it at around the distance of 90 km away from the center (Fig. 3.5). WREY_BT flux once again dominates the development EC_BT* while WREY_BC flux resists it (Table 3.2).

The shape of the EC_BC* curve at the time of vortex deformation indicates almost uniform energy loss within the vortex core (Fig. 3.6). All cases show that the WREY_MX and WFORM fluxes correlate positively with EC_BC*, whereas the WREY_BC1 has a negative correlation with EC_BC* (Table 3.1). For the cases of vortices with $R_{max} = 110$ km and $R_{max} = 120$ km, the correlation of WFORM and EC_BC* is the largest (0.8825 and 0.9209, respectively), meaning that this flux dominates the development of EC_BC*. On the contrary, the correlation coefficients for the case of vortex with $R_{max} = 90$ km show the prevailing role of WREY_MX (0.7982) in the modification of the baroclinic energy, while the change of the baroclinic energy of the vortex with $R_{max} = 100$ km is set by both WFORM (0.6493) and WREY_MX (0.6269). At the time of the vortex breakdown, except for the case of the
vortex with \( R_{\text{max}} = 120 \text{ km} \), the shape of the EC_{BC} profile is set by WREY_{MX} flux as can be seen by comparing the blue and green lines in Fig. 3.7, and also by correlation coefficient in Table 3.1. The profile of EC_{BC} exhibits a sharp drop inside the \( r = R_{\text{max}} \) for the cases of vortices with \( R_{\text{max}} = 110 \text{ km} \) and \( R_{\text{max}} = 120 \text{ km} \), the development of similar local minima near \( R_{\text{max}} \) for smaller vortices with \( R_{\text{max}} = 90 \text{ km} \) and \( R_{\text{max}} = 100 \text{ km} \) is also present (Fig. 3.7). This is consistent with the definition of vortex breakdown as a rupture of the vortex core with the outflow of energy to the vortex exterior.

Comparing the results from the globally integrated and local energy analyses, we note that the distributions of the local energy tendency and energy fluxes for the barotropic mode show the expected effect of the dominance of local WREY_{BT} flux in changing the barotropic energy. The results of computations for the baroclinic mode can be counterintuitive: the leading role of the globally integrated flux of wave buoyancy (WFORM\(_g\)) at the time of vortex deformation is not always consistent with the same flux locally (see the cases of vortices with \( R_{\text{max}} = 90 \text{ km} \) and \( R_{\text{max}} = 100 \text{ km} \)). During the time of vortex breakdown, the predominant role of the local WREY_{MX} flux for the case of vortex with \( R_{\text{max}} = 90 \text{ km} \) goes at odds with the secondary role of the globally integrated WREY_{MX\(_g\)} flux. Additionally, as opposed to the globally integrated analysis, the impact of the local WREY_{MX} in the case of vortex with \( R_{\text{max}} = 120 \text{ km} \) is also considerable. It suffices to say that the local energy analysis can provide a deeper and more complete view of the mechanism of instability.
3.3 Transport

3.3.1 Overview of kinematics

In this study we use a Lagrangian approach to characterize the transport by vortices. The equations governing the trajectories of neutrally buoyant Lagrangian particles are

\[ \frac{dx(t)}{dt} = u(x, t), \quad x(t_0) = X, \quad t \geq t_0 \]

(3.3)

where \( x(t) \) is a vector of particle positions at time \( t \), \( X \) is a vector of particle positions at the initial time \( t = t_0 \), and \( u \) is the geostrophic velocity. This is a set of non-autonomous, ordinary differential equations. Since the geostrophic velocity may be expressed through the streamfunction we can rewrite the system of equations 3.3 as

\[ \frac{dx_i}{dt} = -\frac{\partial \psi_i(x, y, t)}{\partial y}, \quad \frac{dy_i}{dt} = \frac{\partial \psi_i(x, y, t)}{\partial x} \]

(3.4)

where \( i \) is a number of the layer, \( \psi_i \) is the corresponding streamfunction. This system represents a so-called Hamiltonian system, where the streamfunction is the Hamiltonian. The 4-th order Runge-Kutta method is used to solve (3.4) with the streamfunction spatial derivatives calculated at each particle location using linear interpolation. The application of higher-order interpolation does not change the final position of the particles, but slows down the model integration considerably.

3.3.2 Mixing

Two methods are typically invoked in quantifying mixing by turbulent flows: the advection of tracer concentration by turbulent velocities and computation of the stretching rate between two nearby trajectories of Lagrangian particles advected by
the velocity field. While both methods have their own advantages and disadvantages, the advection of tracer concentration requires the introduction of some type of diffusion. The apparent choice of diffusion largely remains uncertain, not to mention the fact that the mixing by vortical flows strongly depends on it [36, 86]. Furthermore, numerous studies have shown that effective turbulent friction coefficient is spatially inhomogeneous and anisotropic; the models usually assume the opposite. The stretching rate of two Lagrangian particle trajectories can be measured either by FTLE or FSLE, however the latter is ill-posed and produces spurious ridges in its maps [60]. We therefore use the FTLE metric to characterize mixing.

The FTLE can be introduced (see, for instance, [43]) by noting that the system of equations (3.3) admits a flow map $F_{t_0}^t(x_0) := x(t; t_0, x_0)$, where $x(t; t_0, x_0)$ is a trajectory which starts at $x = x_0$ at time $t = t_0$. We then can define the Cauchy-Green strain tensor as

$$C_{t_0}^t(x_0) = (\nabla F_{t_0}^t(x_0))^T (\nabla F_{t_0}^t(x_0)) \quad (3.5)$$

The tensor $C_{t_0}^t(x_0)$ is positive definite and symmetric, so it admits two positive real eigenvalues ($\lambda_{\min}(C_{t_0}^t(x_0))$ and $\lambda_{\max}(C_{t_0}^t(x_0))$) and corresponding eigenvectors. FTLE is then defined as

$$\sigma_{t_0}^t(x_0) = \frac{1}{2(t - t_0)} \ln \lambda_{\max}(C_{t_0}^t(x_0)) \quad (3.6)$$

The FTLE maps in the upper layer for stable and unstable vortices are shown in Fig. 3.8. We show the maps only at day 100 because the patterns at earlier times do not provide any additional information. The highest values of FTLE for stable vortices are observed within the ring region which is almost continuous for $R_{\max} = 60$ km and has a volute structure for the vortex with $R_{\max} = 80$ km. In contrast, the cases of unstable vortices with $R_{\max} = 100$ km and $R_{\max} = 120$ km
show the development of additional eddies and filaments connecting these eddies, 
and indicating an increasingly complex picture of the mixing pattern. Additionally, 
we also observe the wavy patterns of FTLE field in the close proximity of the vortex 
as well as outside of it.

To separate the role of the primary vortex from waves and secondary vortices, we 
run an idealized experiment in which the FTLE map is computed from the synthetic 
flow produced by the initial vortical profile moving along the same trajectory as in 
the full experiment. As shown in Fig. 3.9, the wavy patterns in the near- and far-
fields are completely absent. However, the bigger vortices also have trailing spirals, 
although the spirals are broader than in the case of the full experiment.

Lower layer FTLE maps do not show any presence of a coherent structure (Fig. 
3.10). Instead, a wave-like pattern is observed for the stable vortical flows ($R_{\text{max}} = 
60 \text{ km}$ and $R_{\text{max}} = 80 \text{ km}$), while the development of irregular filaments is seen for 
unstable cases ($R_{\text{max}} = 100 \text{ km}$ and $R_{\text{max}} = 120 \text{ km}$). These patterns indicate the 
presence of significant mixing in the lower layer associated with asymmetric waves 
that are forced by motions in the upper layer through quasi-geostrophic dynamics.

To conclude, we discuss the differences between the structure of FTLE on the $f$- 
and $\beta$-planes based on the considerations derived from dynamical system theory. 
Consider a frictionless, $f$-plane case when an axisymmetric vortex is a stationary 
solution of the equations of motion and it remains so for all time. Such a flow 
is known to be not chaotic [79] with FTLE maps having perfectly continuous ring 
structure (not shown). The introduction of the $\beta$-effect makes the initial profile 
to be not a stationary solution of the governing equations. The dynamical system 
governing the evolution of particle trajectories has a right-hand side which depends
on time and is, therefore, exposed to some degree of chaoticity. The vortex with \( R_{\text{max}} = 60 \text{ km} \) is weakly affected by the \( \beta \)-effect, and so, the spiraling within the ring region is insignificant (Fig. 3.8a). As the radius of the vortex becomes larger, more intense radiation leads to higher variability of the streamfunction; the dynamical system becomes more chaotic and the volute structure of the FTLE ring becomes more manifest.

### 3.3.3 Large-scale transport

In this section we quantify large-scale meridional transport produced by stable and unstable vortices as the change in time of the gradient of a specified tracer. The tracer distribution is represented by Lagrangian particles as follows: the domain is divided into \( N \times N \) (N=40) square cells, each of which has meridional \( k \) \((k = 1 \ldots N)\) and zonal \( l \) \((l = 1 \ldots N)\) indices. At initial time, 40 particles are released into each cell and each \( i \)-th particle within a particular \( kl \)-th cell is assigned a marker value \( c_i = k, k = 1, \ldots, N \). We then define the tracer concentration \( C_{kl}(t) \) in each \( kl \)-th square cell at time \( t \) as an average marker value of all particles within this cell:

\[
C_{kl}(t) = \frac{\sum_{i} c_i n_i}_{kl} \quad \text{(3.7)}
\]

where \( n_i \) is a number of particles with marker value of \( c_i \), and the summation is taken over the number of particles within a particular \( lk \)-th cell at time \( t \). The initial distribution of \( C_{kl} \) is equal to \( k \) \((C_{kl} = k, k = 1, \ldots, N)\), and is equivalent to the distribution of tracer with a constant meridional gradient. An example of a composite map consisting of \( C_{kl} \) values of each subdomain at day 100 for stable \((R_{\text{max}} = 60 \text{ km})\) and unstable \((R_{\text{max}} = 120 \text{ km})\) vortices is shown in Fig.3.11. The
initial gradient is clearly modified in both upper and lower layers as a result of vortex propagation and disintegration. We quantify the large-scale meridional transport at any arbitrary time $t$ as the difference between a zonally averaged tracer concentration at time $t$ and initial time i.e. $\sum_t(C_{kl}(t) - C_{kl}(0))$

As in the previous section, we perform full and idealized experiments to distinguish the relative role of vortex and asymmetric motions. The results of calculations are shown in Fig.3.12. The general trend present in all cases in the upper layer is the increase of the average concentration change in the wake of the moving vortex and its decrease ahead of the vortex. In the case when the vortex is small ($R_{max} = 60$ km) and wave activity is weak, the upper layer average concentration change in the full and idealized experiments bear little differences. However, for the larger vortex corresponding to stronger radiation ($R_{max} = 80$ km), the magnitude of the average concentration change behind the vortex becomes smaller while the area swept by the vortex-induced flow becomes wider. The instability of the vortices gives rise to even stronger differences between full and idealized experiments. For example, pinched off vorticity patches in the full simulation cause either a considerable reduction of the average concentration between $y = 0$ km and $y = 300$ km ($R_{max} = 100$ km) or the change of its sign at around 600 km ($R_{max} = 120$ km). The dips in the shape of the average concentration profile in idealized experiments with large vortices ($R_{max} = 100$ km and $R_{max} = 120$ km) are due to the looping trajectories of these vortices.

The structure of the average concentration change in the lower layer is characterized by two distinct features. First, the its magnitude in the lower layer is roughly half of that in the upper layer. Second, the distribution for the unstable cases is
more irregular than the distribution for the stable ones. Since FTLE maps and tracer concentration field in the lower layer show no presence of a coherent structure we conclude that this large-scale transport is entirely caused by the waves and turbulent wave-like eddies excited when the vortex breaks apart.

<table>
<thead>
<tr>
<th>TERM</th>
<th>DEFINITION</th>
<th>MEANING</th>
</tr>
</thead>
<tbody>
<tr>
<td>EC_BTC</td>
<td>$&lt; \psi_{BT} &gt; \frac{\partial &lt; q_{BT} &gt;}{\partial t}$</td>
<td>change of barotropic vortex energy</td>
</tr>
<tr>
<td>EC_BTC*</td>
<td>$&lt; \psi_{BT} &gt; \left[ \frac{\partial &lt; q_{BT} &gt;}{\partial t} \right.$</td>
<td>modified by energy fluxes associated with Coriolis force, moving reference frame and friction</td>
</tr>
<tr>
<td></td>
<td>$- \beta (\frac{1}{r} + \frac{\partial}{\partial r}) &lt; \psi'_{BT} \cos \theta &gt;$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ (\frac{1}{r} + \frac{\partial}{\partial r}) (U(t) &lt; q'_{BT} \cos \theta &gt;$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ V(t) &lt; q'_{BT} \sin \theta &gt;$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\left. - &lt; D_{BT} &gt; \right]$</td>
<td></td>
</tr>
<tr>
<td>WREY_BTC</td>
<td>$\frac{1}{\alpha r} &lt; \psi_{BT} &gt; &lt; J(\psi_{BT}', q_{BT}') &gt;$</td>
<td>flux of barotropic kinetic energy due to barotropic-barotropic interactions</td>
</tr>
<tr>
<td>WREY_BC</td>
<td>$&lt; \psi_{BT} &gt; &lt; J(\psi'<em>{BC}, q'</em>{BC}) &gt;$</td>
<td>change of baroclinic vortex energy</td>
</tr>
<tr>
<td>EC_BC</td>
<td>$&lt; \psi_{BC} &gt; \frac{\partial &lt; q_{BC} &gt;}{\partial t}$</td>
<td>change of baroclinic vortex energy</td>
</tr>
<tr>
<td>EC_BC*</td>
<td>$&lt; \psi_{BC} &gt; \left[ \frac{\partial &lt; q_{BC} &gt;}{\partial t} \right.$</td>
<td>modified by energy fluxes associated with Coriolis force, moving reference frame and friction</td>
</tr>
<tr>
<td></td>
<td>$- \beta (\frac{1}{r} + \frac{\partial}{\partial r}) &lt; \psi'_{BC} \cos \theta &gt;$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ (\frac{1}{r} + \frac{\partial}{\partial r}) (U(t) &lt; q'_{BC} \cos \theta &gt;$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ V(t) &lt; q'_{BC} \sin \theta &gt;$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\left. - &lt; D_{BC} &gt; \right]$</td>
<td></td>
</tr>
<tr>
<td>WREY_MX</td>
<td>$&lt; \psi_{BC} &gt; &lt; J(\psi'<em>{BT}, q'</em>{BT}) &gt;$</td>
<td>flux of baroclinic kinetic energy due to mixed-mode interactions</td>
</tr>
<tr>
<td>WFORM</td>
<td>$(S_1 + S_2) &lt; \psi_{BC} &gt; &lt; J(\psi'<em>{BC}, \psi'</em>{BT}) &gt;$</td>
<td>flux of available potential energy</td>
</tr>
<tr>
<td>WREY_BC1</td>
<td>$\alpha_C &lt; \psi_{BC} &gt; &lt; J(\psi'<em>{BC}, q'</em>{BC}) &gt;$</td>
<td>flux of baroclinic kinetic energy due to baroclinic-baroclinic interactions</td>
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Table 3.1: Energy balance terms
Table 3.2: Contribution of local energy fluxes

<table>
<thead>
<tr>
<th>$R_{max}$, km</th>
<th>PROCESS</th>
<th>$cr_{br}$</th>
<th>$cr_{bc}$</th>
<th>$cr_f$</th>
<th>$cr_{mx}$</th>
<th>$cr_{bc1}$</th>
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<td>90</td>
<td>Vort. deformation</td>
<td>0.9984</td>
<td>0.8601</td>
<td>0.4637</td>
<td>0.8093</td>
<td>-0.8674</td>
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<tr>
<td>90</td>
<td>Vort. breakdown</td>
<td>0.9831</td>
<td>-0.8495</td>
<td>0.0963</td>
<td>0.9812</td>
<td>0.1904</td>
</tr>
<tr>
<td>100</td>
<td>Vort. deformation</td>
<td>0.9946</td>
<td>0.8044</td>
<td>0.6493</td>
<td>0.6269</td>
<td>-0.8726</td>
</tr>
<tr>
<td>100</td>
<td>Vort. breakdown</td>
<td>0.9953</td>
<td>-0.6373</td>
<td>-0.9409</td>
<td>0.9516</td>
<td>0.6337</td>
</tr>
<tr>
<td>110</td>
<td>Vort. deformation</td>
<td>0.9824</td>
<td>0.7692</td>
<td>0.8825</td>
<td>0.4109</td>
<td>-0.8602</td>
</tr>
<tr>
<td>110</td>
<td>Vort. breakdown</td>
<td>0.9896</td>
<td>-0.3019</td>
<td>-0.7789</td>
<td>0.8369</td>
<td>0.8798</td>
</tr>
<tr>
<td>120</td>
<td>Vort. deformation</td>
<td>0.9652</td>
<td>0.7288</td>
<td>0.9209</td>
<td>0.4189</td>
<td>-0.8708</td>
</tr>
<tr>
<td>120</td>
<td>Vort. breakdown</td>
<td>0.9667</td>
<td>-0.3245</td>
<td>0.9628</td>
<td>0.6752</td>
<td>-0.3534</td>
</tr>
</tbody>
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Figure 3.1: Time series of globally integrated a) barotropic b) baroclinic vortex energy.
Figure 3.2: Integrated energy balance for the vortices with a) $R_{\text{max}} = 90$ km b) $R_{\text{max}} = 100$ km c) $R_{\text{max}} = 110$ km d) $R_{\text{max}} = 120$ km. Shown are time series of energy balance components for the baroclinic mode. Two vertical dashed lines denote two minima in energy tendency.

Figure 3.3: Integrated energy balance for the vortices with a) $R_{\text{max}} = 90$ km b) $R_{\text{max}} = 100$ km c) $R_{\text{max}} = 110$ km d) $R_{\text{max}} = 120$ km. Shown are time series of energy balance components for the barotropic mode.
Figure 3.4: Local energy balance for the vortices with a) $R_{\text{max}} = 90$ km b) $R_{\text{max}} = 100$ km c) $R_{\text{max}} = 110$ km d) $R_{\text{max}} = 120$ km. $R_{\text{max}}$ is denoted by the dashed line. Shown are radial profiles of energy balance components for the barotropic mode at the time of vortex deformation (see the text).
Figure 3.5: Local energy balance for the vortices with a) $R_{\text{max}} = 90$ km b) $R_{\text{max}} = 100$ km c) $R_{\text{max}} = 110$ km d) $R_{\text{max}} = 120$ km. $R_{\text{max}}$ is denoted by the dashed line. Shown are radial profiles of energy balance components for the barotropic mode at the time of vortex breakdown (see the text).
Figure 3.6: Local energy balance for the vortices with a) \( R_{\text{max}} = 90 \text{ km} \) b) \( R_{\text{max}} = 100 \text{ km} \) c) \( R_{\text{max}} = 110 \text{ km} \) d) \( R_{\text{max}} = 120 \text{ km} \). \( R_{\text{max}} \) is denoted by the dashed line. Shown are radial profiles of energy balance components for the baroclinic mode at the time of vortex deformation (see the text).
Figure 3.7: Local energy balance for the vortices with a) $R_{max}=90$ km b) $R_{max}=100$ km c) $R_{max}=110$ km d) $R_{max}=120$ km. $R_{max}$ is denoted by the dashed line. Shown are radial profiles of energy balance components for the baroclinic mode at the time of vortex breakdown (see the text).
Figure 3.8: Upper layer FTLE maps at day 100 for the vortices with a) $R_{\text{max}} = 60$ km b) $R_{\text{max}} = 80$ km c) $R_{\text{max}} = 100$ km d) $R_{\text{max}} = 120$ km.
Figure 3.9: FTLE maps of idealised experiment at day 100 for the vortices with a) $R_{max} = 60$ km b) $R_{max} = 80$ km c) $R_{max} = 100$ km d) $R_{max} = 120$ km.
Figure 3.10: Lower layer FTLE maps at day 100 for the vortices with a) $R_{max} = 60$ km b) $R_{max} = 80$ km c) $R_{max} = 100$ km d) $R_{max} = 120$ km.
Figure 3.11: Tracer concentration $\times 10^5$ at day 100 for the vortices with a) $R_{max} = 60$ km, upper layer b) $R_{max} = 60$ km, lower layer c) $R_{max} = 120$ km, upper layer d) $R_{max} = 120$ km, lower layer. Contours of streamfunction are plotted on top.
Figure 3.12: Average concentration change at day 100 for the vortices with a) $R_{\text{max}} = 60$ km b) $R_{\text{max}} = 80$ km c) $R_{\text{max}} = 100$ km d) $R_{\text{max}} = 120$ km. The vertical dashed line stands for $y$-coordinate of the vortex centre.
CHAPTER 4

Zonally-elongated transient flows: phenomenology and sensitivity analysis

4.1 Background

Satellite observations and numerical models output reveal a series of zonally-elongated transient flow patterns populating every basin of the World Ocean [72]. We refer to these flow structures as zonally-elongated transients or ZELTs [59]. Anisotropic and non-uniform distribution of Eulerian [48, 92, 102] and Lagrangian [59, 67, 87] statistics further suggests that ZELTs exhibit a significant spatial variability over different parts of the World Ocean. The present article focuses on two aspects of ZELTs: first, their proper identification and separation from the background flow, and second, their sensitivity to and dependence on environmental parameters.

Multiple, predominantly zonal oceanic flows are detected in the time-mean sea surface height [73] and Argo data [94] measurements. These flow structures are claimed to be the counterparts of stationary jets observed in the atmospheres of giant gas (e.g., Jupiter and Saturn) and terrestrial (e.g., Earth, Venus and Mars) planets. However, most of the kinetic energy at the mesoscale in the ocean is contained in the time-evolving flow anomalies [113]. Averaged over 18 – 200 weeks, these flow anomalies
reveal the presence of low-frequency zonally-elongated flow patterns [72]. On the other hand, even randomly distributed westward propagating vortices can appear as zonally-elongated flow patterns in time-averaged fields [89]. Although [17] and [21] argue that randomly distributed propagating eddies cannot explain the entire signal associated with zonally-elongated flow patterns, the time-averaging does not seem to be a reliable tool for identifying ZELTs.

Zonally-elongated patterns can also be educed from the turbulent flow by applying spatial Fourier filtering [59]. However, the spatial Fourier spectrum of oceanic turbulence is broadband [75]; the lack of scale separation between different flow components renders the Fourier filtering highly inefficient in that there is no clear choice for the cut-off wavenumber. To this end, the problem of efficient ZELTs extraction becomes equivocal: both time-averaging and spatial Fourier filtering can produce spurious flow patterns. A double-spectral approach seems to be an appealing remedy to this problem [54]. The approach consists of dividing a dataset into several frequency bands by means of wavelet transform to apply spatial Fourier decomposition within each of the bands. [55] find that zonally-elongated patterns have a dominant period greater than 2.5 years. While such an approach certainly helps to identify the time scale, as mentioned above, spatial Fourier decomposition precludes unambiguous detection of a particular flow pattern.

Many environmental parameters can influence the variability of oceanic turbulence and its associated anisotropy, including Earth’s rotation, stratification and dissipative processes. Simulations of geostrophic turbulence suggest that varying bottom friction causes a tenfold change in the magnitude of the stationary zonal jets; a less pronounced effect is observed due to variations in lateral friction [12]. The anisotropy
of the time-variable flow associated with ZELTs, on the other hand, can respond differently to variations in friction. The impact of Earth’s rotation ($\beta$-effect) and stratification on the anisotropy of ocean variability also remains unknown.

We perform simulations in a two-layer quasi-geostrophic model. This model is conceptually simple yet represents the essential dynamics well; furthermore, its computational efficiency allows an efficient exploration of the model’s parameter space. A more complex and costly model would be impractical computationally because of the long integration times required for each numerical simulation. The details of the numerical model are provided in section 2. By applying Fourier and EOF decomposition to simulated isotropic and anisotropic turbulent flows, we demonstrate that EOF decomposition is a more appropriate technique to identify ZELTs. We then quantify the propagation of ZELTs by means of the Extended EOF decomposition. We compare the variance spectrum, the leading EOF and autocorrelation function (ACF) of the corresponding Principal Component (PC) for several values of $\beta$ and bottom drag coefficient. Finally, we quantify the sensitivity of the anisotropy associated with the leading EOF to simultaneous variations in $\beta$ and bottom drag coefficient by constructing a response surface within $\beta$-bottom drag parameter space.

4.2 Phenomenology

4.2.1 Separating ZELTs from the background field

In this section we perform simulations of anisotropic ($\gamma = 3 \times 10^{-7} \text{ s}^{-1}$, $\beta = 2.15 \times 10^{-11} \text{ m}^{-1}\text{s}^{-1}$) and isotropic ($\beta = 0 \times 10^{-11} \text{ m}^{-1}\text{s}^{-1}$) turbulent flows and utilize them as a testbed for the efficiency of EOF versus Fourier decompositions in extracting
ZELTs from the background flow. The simulations are performed on a square domain for the ease of comparison. To unambiguously distinguish ZELTs from previously analyzed zonal jets ([12] and all references therein; see also [110]), we remove zonal and time-mean component of the total flow.

The Fourier spatial spectra of the velocity streamfunction for the isotropic and anisotropic turbulence simulations are shown in Fig. 4.1. Both spectra are symmetric with respect to the origin due to the doubly-periodic boundary conditions. The spectrum of the anisotropic turbulent flow reveals two power peaks located at short and long zonal wavelengths, respectively (Fig. 4.1a). While the origin of the former spectral peak traces back to the linear instability theory and can be explained as a contribution from the most unstable mode [8], the spectral peak located at long zonal wavelengths, as indicated by squares, hints at the presence of ZELTs in the turbulent flow. In contrast, the power of isotropic turbulence spectrum is distributed almost uniformly among all wavenumbers within a spectral circle (Fig. 4.1b). The squares indicate exactly the same spots as in the spectrum of anisotropic turbulent flow. Applying spatial filtering within squares, we extract similar zonally-elongated patterns in both cases (the nondimensional wavenumbers are 1 and 3 along \( k \)-axis, and 8 and 12 along \( l \)-axis) (Fig. 4.2). This suggests that spatial Fourier filtering can artificially produce ZELTs from purely isotropic flow, in which these structures cannot be expected to exist. We therefore use the alternative approach based on EOF decomposition (see [21]). The technical details of this approach are presented in the Appendix B.

Comparing the amount of the explained variance by each of the EOF modes, one can notice that the variance spectrum of the anisotropic turbulence is steeper than the
one of isotropic turbulence (Fig. 4.3). While the EOF modes of the isotropic turbulent flow are not well-separated, the first four EOF modes of anisotropic turbulent flow are clearly distinct from the rest of the spectrum. This is further confirmed by the spatial structure of the first four EOFs displayed in Fig. 4.4a. Unlike higher order EOFs that do not exhibit long zonal lengthscales (not shown), these four leading modes have the form of multiple zonally-elongated alternating patterns with several eddies superimposed on them. This spatial structure is also in sharp contrast to EOFs of the isotropic turbulent flow, which display randomly distributed eddies (Fig. 4.4b).

The autocorrelation function (ACF) of the corresponding Principal Components (PC) oscillates and decays slowly for all four leading EOFs of the anisotropic turbulent flow (Fig. 4.5a). This goes at odds with four leading EOFs of isotropic turbulent flow, for which the ACF of the corresponding PC decays quickly and fluctuates around zero.

In the rest of the study, we associate ZELTs with zonally-elongated modes of flow variability. A convenient measure to characterize anisotropic properties of the flow field is the anisotropic ratio defined as \[ \alpha = \frac{\langle u'^2 \rangle - \langle v'^2 \rangle}{\langle u'^2 \rangle + \langle v'^2 \rangle} \], where \( \langle \rangle \) denotes spatial averaging, \( u' \) and \( v' \) are zonal and meridional eddy velocities, respectively. Its values ranges from \(-1\), for which the velocities are purely meridional, to \(1\), for which velocities are purely zonal. The value of \( \alpha = 0 \) indicates that the flow field is perfectly isotropic. We can use this parameter to define the number of EOFs that represent ZELTs as those with anisotropic ratio \( \alpha \geq 0.6 \). In the case discussed above only 4 EOF modes have anisotropic ratio \( \alpha \geq 0.6 \). Finally, the EOF decomposition is more efficient than the Fourier decomposition in that it does not produce spurious flow patterns. The disadvantage of the EOF decomposition is that the resulting ZELTs
correspond to several zonal lengthscales, representing both zonally extended patterns and eddies straddling them (see also [55]).

4.2.2 Propagation of ZELTs

One important property of a distinct flow structure is its ability to propagate across the medium. By construction, each separate EOF mode can be interpreted as a stationary oscillation, so the propagation of the flow structure it represents cannot be tracked. Unlike regular EOF decomposition, which makes use only of spatial correlation of the dataset, Extended EOF decomposition also takes into account the fact that geophysical data are usually highly correlated in time. This allows us to track the propagation of ZELTs and estimate their speed.

Fig. 4.6 shows the zonal-time and meridional-time Hovmöller diagrams of the Extended EOF1. The spatial structure of these extended EOFs closely matches the regular EOFs discussed earlier and is not shown here. Due to enormous computational expenses required to compute Extended EOFs we use only three time lags; 500, 1000 and 1500 days. As indicated by slanted contours in the zonal-time diagram, the signal propagates westward with the speed of around 1.15 cms$^{-1}$. Nearly vertical contours in the meridional-time diagram suggest that ZELTs have negligible meridional speed. The zonal propagation speed can be compared to the zonal phase speed of linear modes for the Phillips model, which gives 4.75 - 5 cms$^{-1}$ for small zonal wavenumbers (see formula 7.11.6 in [81]). Higher order Extended EOFs, which are involved in representing ZELTs, show similar slopes in the zonal-time and meridional-time Hovmöller diagrams.
The discrepancy between the values of propagation speed suggests that ZELTs are nonlinear phenomenon and can not be explained by linear dynamics. However, [8, 9] demonstrate that accounting for modulations of the mean flow by stationary zonal jets can alter the propagation of linear modes and result in close match between nonlinear and linear dynamics.

4.3 Sensitivity Analysis

4.3.1 Sensitivity to a single parameter

In this section we examine the sensitivity of ZELTs properties to two model parameters, planetary vorticity gradient $\beta$ and bottom friction $\gamma$. Sensitivity to other model parameters, such as vertical stratification and lateral friction, is found to be weak and is not discussed in this section. We first study the sensitivity to each of these parameters separately, by fixing one parameter and varying the other. The corresponding values are given in Table 4.1.

Variance spectra are shown in Fig. 4.7 (varying $\gamma$, fixed $\beta$) and Fig. 4.8 (varying $\beta$, fixed $\gamma$). The variance spectrum of isotropic turbulence also shown in these plots is identical to the one in Fig. 4.3. The general property of the spectrum is that the larger values of $\gamma$ and the smaller values of $\beta$, the flatter the spectrum tends to be. For $\gamma = 9 \times 10^{-7}$ s$^{-1}$ and $\beta = 1.14 \times 10^{-11}$ m$^{-1}$s$^{-1}$ the spectra of the isotropic and anisotropic turbulence are nearly identical. The regime at low values of bottom friction is unusual. First, for the case of fixed $\beta = 2.15 \times 10^{-11}$ m$^{-1}$s$^{-1}$ and $\gamma = 1 \times 10^{-7}$ s$^{-1}$ only two EOF modes are strongly separated from the rest of the spectrum (Fig. 4.6a). Second,
there is a substantial decrease in the amount of the explained variance by the leading EOFs between the cases of $\gamma = 1 \times 10^{-7}$ s$^{-1}$ and $\gamma = 2 \times 10^{-7}$ s$^{-1}$ (Fig. 4.6a,b).

We next discuss the sensitivity of spatial structure of ZELTs. Even though ZELTs are represented as several leading EOFs (Table 4.1), the changes in ZELT structure can be derived from analyzing only the first mode (EOF1). For some parameters there are no EOFs with anisotropic ratio $\alpha \geq 0.6$ (Table 4.1). However, this value was chosen rather arbitrarily and, as we will see below, some leading EOFs are still characterized by rather noticeable degree of anisotropy. The changes in EOF1 confirms the expectation from the analysis of the variance spectrum: EOF1 becomes more isotropic with increasing bottom drag $\gamma$ (Fig. 4.9) and decreasing $\beta$ (Fig. 4.10). EOF1 remains anisotropic even for very large values of $\gamma$ ($\gamma = 9 \times 10^{-7}$ s$^{-1}$), whereas no signature of zonal anisotropy is observed for small values of $\beta$ ($\beta = 1.14 \times 10^{-11}$ m$^{-1}$s$^{-1}$). The sharp transition in the shape of the variance spectra between $\gamma = 1 \times 10^{-7}$ s$^{-1}$ and $\gamma = 2 \times 10^{-7}$ s$^{-1}$ is also reflected in the spatial structure of EOF1, which changes from strictly structured, nearly zonal patterns to meandering bands with eddies embedded in them.

The impact of environmental parameters on temporal variability of ZELTs can be deduced from changes in ACF of PC1 structure. The ACF of PC1 for the cases with fixed $\beta = 2.15 \times 10^{-11}$ m$^{-1}$s$^{-1}$ shows that the leading mode of variability exhibits perfect oscillations for low values of bottom friction ($\gamma = 1 \times 10^{-7}$ s$^{-1}$), damped oscillations for the moderate values ($\gamma = 2 \times 10^{-7}$ s$^{-1}$, $3 \times 10^{-7}$ s$^{-1}$, $4 \times 10^{-7}$ s$^{-1}$) and a noisy decay modulated by wiggles for the high values ($\gamma = 7 \times 10^{-7}$ s$^{-1}$, $9 \times 10^{-7}$ s$^{-1}$) (Fig. 4.11). For the cases of fixed $\gamma$, the ACF of PC1 is characterized only by decaying behavior, and the decay occurs faster with decreasing $\beta$ (Fig. 4.12). This is evidenced
by decreasing decorrelation time scale, defined here and elsewhere as a time lag of first zero crossing of ACF. When $\beta$ reaches its minimum value $1.14 \times 10^{-11}$ m$^{-1}$s$^{-1}$, the decorrelation time scale becomes very short and the ACF of PC1 bears a lot of resemblance to the one of isotropic turbulence simulation.

Finally, following [104] we attempt to identify the regime of geostrophic turbulence in which ZELTs exist. Zonostrophic regime is defined by the requirement that the zonostrophy index $R_{\beta} = 0.7 \left( \frac{\epsilon \gamma}{\gamma^2} \right)^{\frac{1}{2}} \geq 2.5$, where $\epsilon$ is an energy input rate into the fluid system. This regime is also characterized by prominent stationary zonal jets as well as zonons, the nonlinear waves excited by resonant interactions between Rossby waves [105]. The turbulent regime with $1.5 \leq R_{\beta} < 2.5$ is called transitional; cascading energy in this regime just starts to undergo anisotropization and zonal flows are very erratic. The values $R_{\beta} \leq 1.5$ correspond to friction-dominated regime with classical isotropic turbulence energy spectrum. This index was, however, introduced in the studies on barotropic turbulence [39,104] and its relevance to a baroclinic flow remains to be explored. In this study we estimate the zonostrophy index for the barotropic mode taking $\epsilon$ as the energy input rate into the barotropic mode by external forcing:

$$
\epsilon = US_1 \theta_{11} \psi_{BT} \frac{\partial \psi_{BC}}{\partial x} + U \theta_{11} \psi_{BT} \frac{\partial q_{BT}}{\partial x} + U \theta_{11} \theta_{12} \psi_{BT} \frac{\partial q_{BC}}{\partial x}
$$

$$
q_{BT} = \nabla^2 \frac{H_1 \psi_1 + H_2 \psi_2}{H_1 + H_2} = \nabla^2 \psi_{BT}
$$

$$
q_{BC} = \nabla^2 (\psi_1 - \psi_2) - (S_1 + S_2)(\psi_1 - \psi_2) = \nabla^2 \psi_{BC} - (S_1 + S_2) \psi_{BC}
$$

$$
\theta_{11} = \frac{H_1}{H_1 + H_2}
$$
\[
\theta_{12} = \frac{H_2}{H_1 + H_2}
\]

\[
\theta_{21} = 1
\]

\[
\theta_{22} = -1
\]

The overbar denotes both time and spatial average. The values of the computed parameters \( R_\beta \) are listed in Table 4.1. All values of barotropic zonostrophy index, \( R_\beta \), indicate that the barotropic mode is either in the transitional or friction dominated regime, and as a consequence, ZELTs can not be associated with zonons, despite the fact that both correspond to a sharp peak in the energy spectrum at long zonal wavelengths \([24, 105]\)

\begin{table}[h!]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\((\beta; \gamma)\) & \(R_\beta\) & \# of EOF modes with \(\alpha \geq 0.6\) & \((\beta; \gamma)\) & \(R_\beta\) & \# of EOF modes with \(\alpha \geq 0.6\) \\
\hline
\( (2.15; 1) \) & 1.88 & 3 & \( (1.14; 5) \) & 1.29 & 0 \\
\hline
\( (2.15; 2) \) & 1.59 & 0 & \( (1.47; 5) \) & 1.29 & 0 \\
\hline
\( (2.15; 3) \) & 1.45 & 9 & \( (1.75; 5) \) & 1.29 & 0 \\
\hline
\( (2.15; 4) \) & 1.35 & 11 & \( (1.98; 5) \) & 1.28 & 0 \\
\hline
\( (2.15; 7) \) & 1.18 & 0 & \( (2.15; 5) \) & 1.28 & 5 \\
\hline
\( (2.15; 9) \) & 1.10 & 0 & \( (2.25; 5) \) & 1.27 & 7 \\
\hline
\end{tabular}
\caption{Values of different parameters for 1-D sensitivity analysis. The units of \( \beta \) and \( \gamma \) are \( \text{m}^{-1}\text{s}^{-1} \) and \( \text{s}^{-1} \), respectively.}
\end{table}

4.3.2 Joint sensitivity analysis

In the previous subsection we showed that for some parameter values the EOF1 apparently lacks zonal anisotropy. In this section we investigate anisotropy of EOF1 by constructing the response surface, which shows the changes in anisotropic ratio \( \alpha \) due to simultaneous variations in \( \gamma \) and \( \beta \). Being a non-quadratic quantity, the
anisotropic ratio cannot be easily split into contributions from several EOFs. We
construct the response surface for EOF1 only and associate the changes in it with the
changes in ZELTs. The response surfaces of several higher order leading EOFs do
not provide any additional information. The technical details of the response surface
construction are presented in Appendix C.

The anisotropic ratio is $< 0.2$ in the upper-left corner of the response surface,
indicating that the leading EOF is nearly isotropic and ZELTs do not exist in that
region of the parameter space Fig. 4.14b. Increasing from the upper-left to the
lower-right corner along the diagonal, the anisotropic ratio goes through intermediate
values ($0.3$-$0.7$) and reaches high values ($\sim 0.8$). We associate the values of $\alpha \geq 0.6$
with ZELTs being present in the flow field and identify the swath of the $\alpha \sim 0.8$ as
the region, in which ZELTs are the most pronounced. Another swath with $\alpha \sim 0.9$
indicates that EOF1 is strongly anisotropic analogous to the one seen in Fig. 4.9a.
These two swathes are separated by the tongue of moderate values of $\alpha \simeq 0.5$, the
origin of which remains unclear. We were not able to run the numerical model for
$\gamma < 1 \times 10^{-7}$ and small values of $\beta$. This limits the considered parameter space.

Finally, we compare the sensitivity of anisotropic ratio $\alpha$ in our model with geo-
ographical distribution of $\alpha$ estimated from observations in [92]. The highest values of
anisotropic ratio $\alpha$ in Figure 4 of [92] are 0.5-0.6 and are found at around 20°S-30°S,
decreasing poleward up to 0.3-0.4 between 30°S-40°S and reaching the minimum val-
ues of $\sim 0.1$ at 30°S-60°S, although some spots of large values of $\alpha$ are also observed.
The range of the values of bottom drag $\gamma$ in the real ocean is largely unknown. How-
ever, taking this value to be equal to $5 \times 10^{-7}$ we readily observe the consistency
between the prediction of idealized model and observations. We also note that the
values of anisotropic ratio computed from altimetry data never reach the values higher than 0.6. This can be explained by the fact that the intensity of signal associated with ZELTs in the real ocean can be weakened by other types of mesoscale variability not included in our idealized model.
Figure 4.2: Low-passed Fourier filtered velocity streamfunction $\times 10^3$ of a) anisotropic and b) isotropic turbulent flows.

Figure 4.3: Amount of explained variance by 30 leading EOF modes of simulated anisotropic (red curve) and isotropic (black curve) turbulent flows.
Figure 4.4: The first four leading EOFs×10⁻³. Left column: anisotropic turbulent flow. Right column: isotropic turbulent flow. a) and b) EOF1, c) and d) EOF2, e) and f) EOF3, g) and h) EOF4.
Figure 4.5: ACF of PCs corresponding to several leading EOF modes. Left column: anisotropic turbulent flow. Right column: isotropic turbulent flow. a) and b) PC1, c) and d) PC2, e) and f) PC3, g) and h) PC4.
Figure 4.6: Extended EOF1 $\times 10^{-3}$ along a) zonal direction and b) meridional direction as a function of time lag.
Figure 4.7: Amount of explained variance by 12 leading EOF modes of simulated anisotropic turbulent flow for $\beta = 2.15 \times 10^{-11} \text{ m}^{-1}\text{s}^{-1}$ (red curves) and a) $\gamma = 1 \times 10^{-7} \text{ s}^{-1}$, b) $\gamma = 2 \times 10^{-7} \text{ s}^{-1}$, c) $\gamma = 3 \times 10^{-7} \text{ s}^{-1}$, d) $\gamma = 4 \times 10^{-7} \text{ s}^{-1}$, e) $\gamma = 7 \times 10^{-7} \text{ s}^{-1}$, f) $\gamma = 9 \times 10^{-7} \text{ s}^{-1}$. The variance spectrum of simulated isotropic turbulent flow (black curves) with the same parameters as in fig.4.3 is plotted for comparison.
Figure 4.8: Amount of explained variance by 12 leading EOF modes of simulated anisotropic turbulent flow (red curves) for $\gamma = 5 \times 10^{-7}$ s$^{-1}$ and a) $\beta = 1.14 \times 10^{-11}$ m$^{-1}$s$^{-1}$, b) $\beta = 1.47 \times 10^{-11}$ m$^{-1}$s$^{-1}$, c) $\beta = 1.75 \times 10^{-11}$ m$^{-1}$s$^{-1}$, d) $\beta = 1.98 \times 10^{-11}$ m$^{-1}$s$^{-1}$, e) $\beta = 2.15 \times 10^{-11}$ m$^{-1}$s$^{-1}$, f) $\beta = 2.25 \times 10^{-11}$ m$^{-1}$s$^{-1}$. The variance spectrum of simulated isotropic turbulent flow (black curves) with the same parameters as in fig.4.3 is plotted for comparison.
Figure 4.9: EOF1\times 10^{-3} of simulated anisotropic turbulent flow (red curves) for $\beta = 2.15 \times 10^{-11} \text{ m}^{-1} \text{s}^{-1}$ (red curves) and\n\begin{itemize}
\item a) $\gamma = 1 \times 10^{-7} \text{ s}^{-1}$,
\item b) $\gamma = 2 \times 10^{-7} \text{ s}^{-1}$,
\item c) $\gamma = 3 \times 10^{-7} \text{ s}^{-1}$,
\item d) $\gamma = 4 \times 10^{-7} \text{ s}^{-1}$,
\item e) $\gamma = 7 \times 10^{-7} \text{ s}^{-1}$,
\item f) $\gamma = 9 \times 10^{-7} \text{ s}^{-1}$.
\end{itemize}
Figure 4.10: EOF1×10^{-3} of simulated anisotropic turbulent flow for γ = 5 \times 10^{-7} \text{s}^{-1} and a) \beta = 1.14 \times 10^{-11} \text{m}^{-1}\text{s}^{-1}, b) \beta = 1.47 \times 10^{-11} \text{m}^{-1}\text{s}^{-1}, c) \beta = 1.75 \times 10^{-11} \text{m}^{-1}\text{s}^{-1}, d) \beta = 1.98 \times 10^{-11} \text{m}^{-1}\text{s}^{-1}, e) \beta = 2.15 \times 10^{-11} \text{m}^{-1}\text{s}^{-1}, f) \beta = 2.25 \times 10^{-11} \text{m}^{-1}\text{s}^{-1}. 
Figure 4.11: ACF of PC1 of simulated anisotropic turbulent flow for $\beta = 2.15 \times 10^{-11} \text{ m}^{-1}\text{s}^{-1}$ (red curves) and a) $\gamma = 1 \times 10^{-7} \text{s}^{-1}$, b) $\gamma = 2 \times 10^{-7} \text{s}^{-1}$, c) $\gamma = 3 \times 10^{-7} \text{s}^{-1}$, d) $\gamma = 4 \times 10^{-7} \text{s}^{-1}$, e) $\gamma = 7 \times 10^{-7} \text{s}^{-1}$, f) $\gamma = 9 \times 10^{-7} \text{s}^{-1}$. 
Figure 4.12: ACF of PC1 of simulated anisotropic turbulent flow for $\gamma = 5 \times 10^{-7}$ s$^{-1}$ and a) $\beta = 1.14 \times 10^{-11}$ m$^{-1}$s$^{-1}$, b) $\beta = 1.47 \times 10^{-11}$ m$^{-1}$s$^{-1}$ c) $\beta = 1.75 \times 10^{-11}$ m$^{-1}$s$^{-1}$, d) $\beta = 1.98 \times 10^{-11}$ m$^{-1}$s$^{-1}$, e) $\beta = 2.15 \times 10^{-11}$ m$^{-1}$s$^{-1}$, f) $\beta = 2.25 \times 10^{-11}$ m$^{-1}$s$^{-1}$. 
Figure 4.13: Interpolation points in $\beta - \gamma$ parameter space (black *); the blue and green circles show the parameter values for which the $1 - D$ sensitivity analysis was presented; diamonds (○) are additional interpolation points; red * are the points at which the response surface is validated.
Figure 4.14: Response surface of anisotropic ratio derived from the EOF1 for smoothing parameter $p = 0.5$. The surfaces obtained a) without and b) with additional interpolation points.

Figure 4.15: Validation error of the response surface for smoothing parameter $p = 0.5$. The error computed for response surfaces a) without and b) with additional interpolation points.
CHAPTER 5

Zonally-elongated transient flows: dynamics

5.1 Background

Observed in satellite data [55, 72] and numerical model simulations [58], low-frequency, zonally-elongated structures (ZELTs) populate almost every basin of the World Ocean. Their spatiotemporal variability is highly sensitive to environmental parameters such as bottom drag and meridional gradient of planetary vorticity (chapter 2). These flow structures play a crucial role in regulating material transport [59, 67, 87] and statistical characteristics of oceanic turbulence [48, 92, 102]. This chapter is concerned with identifying the underpinning mechanisms of ZELT formation and their role in ocean energetics.

Two distinct mechanisms have been proposed to explain the emergence of zonal flows in geophysical turbulence. The first mechanism solely relies on inverse cascade of energy: the energy being injected at small scales propagates to ever larger scales until the energy cascade undergoes anisotropization owing to the inherent anisotropy imposed by the \( \beta \)-effect. The energy transport itself occurs via either spectrally local or non-local nonlinear interactions between eddies. The local energy cascade is
a step-by-step process, in which small-scale/small-scale eddy interactions results in
deposition of energy to small scales. In barotropic freely decaying turbulence, the
scale at which the local energy cascade is slowed down is defined by the balance
of nonlinear and the $\beta$-terms [84]. Forced-dissipative fluid system possesses a new
scaling associated for energy cascade anisotropization; the scale, $k_\beta$, is defined as the
cross-over between Rossby waves spectrum and Kolmogorov-Kraichman spectrum of
isotropic turbulence [109]. In fact, if the range between the Rhines scale and $k_\beta$ is wide
enough, barotropic turbulence develops an inertial range with prominent zonal flows
[104]. Additionally, this regime of geostrophic turbulence contains zonons - nonlinear
waves with sharp peak of energy spectrum at off-zonal modes [105]. However, as
argued in Chapter 4, oceanic mesoscale turbulence is in so-called transitional regime,
in which the flow field just starts to undergo anisotropization and ZELTs can not
be associated with zonons. Nonlocal energy cascade occurs via Reynolds and form
stresses, which involve interactions of eddies to produce direct transfer of energy
to small zonal wavenumbers. This mechanism has been shown to maintain jets in
both barotropic [49, 71] and baroclinic turbulence simulations [11, 58, 80], though its
relevance to the formation of ZELTs has yet to be established.

Linear instability is the second mechanism associated with the development of
zonal flows. Meridional eastern boundary currents are subject to radiating instability,
which produces zonally-elongated modes [46]. Although these modes are weakly-
damped, they can be considerably energized by the resonant triad interactions with
the most unstable mode [111]. Baroclinic instability also provides a framework for
the organization of the flow into multiple alternating zonal jets: the most unstable
mode with $k = 0$ emerges as a result of instability of a combined zonally-uniform
baroclinic shear and meridional modes \((l = 0)\) (“noodle” modes \([81]\)). This most unstable mode equilibrates at a finite amplitude in fully non-linear simulations. The meridional length scale of the most unstable mode sets the meridional length scale of jets \([10]\). Furthermore, zonally-elongated flow patterns can emerge as a result of instability of a reduced-dynamics system. A single mode with \(l = 0\) is prone to modulational instability, which is a 4-mode truncation of the fully non-linear system \([23]\). Modulational instability leads to the emergence of either zonal or off-zonal most unstable modes depending on the degree of nonlinearity of the truncated system. However, since the degree of nonlinearity in the real ocean is uncertain, modulational instability can not be viewed as the only mechanism of ZELTs generation. Finally, the instability of the dynamical system with removed eddy-eddy interactions also results in the emergence of the most unstable mode with \(k = 0\) followed by its equilibration \((\[31\], \[32\], \[101]\)). Such a reduced-dynamics system basically generalizes 4-mode truncated system by allowing the basic state to interact with more modes; this system also precludes both local energy and enstrophy cascades, and as such, can serve as a good tool for identifying different mechanisms of ZELTs formation.

We perform a series of simulations in reduced-dynamics models with fully or partially precluded energy and enstrophy transfers to find the mechanism of ZELTs formation. We utilize Fourier and Empirical Orthogonal Functions decomposition to compare simulations of the reduced-dynamics versus nonlinear model.
5.2 Dynamics and energetics of ZELTs

5.2.1 Reduced-dynamics models

To study the dynamics of ZELTs we decompose the flow field into “mean” (zonally-averaged) and “eddy” (deviations from zonal average) components, the governing equations for which are derived in Appendix D. In this formulation, ZELTs is a part of eddy field. Although this decomposition is not unique and can be dynamically not consistent [13], our choice is driven by its transparency and the ease of implementation. The nonlinear forcing entering the right-hand side of the equations for the evolution of eddy components can be divided into two parts: the first part describes the interactions between eddies and the mean flow (EME interactions: \[ J(\overline{p}_p, q'_p) + J(\overline{p}_c, q'_c) \], \[ J(\overline{c}_p, q'_c) + J(\overline{c}_c, q'_p) \], \[ J(\overline{c}_c, q'_0c) + J(\overline{c}_0c, q'_c) \], \[ J(\overline{c}_0c, q'_0p) + J(\overline{c}_p, q'_c) \], \[ J(\overline{c}_p, q'_0p) + J(\overline{c}_0p, q'_c) \], \[ J(\overline{c}_0p, q'_0c) + J(\overline{c}_c, q'_p) \], \[ J(\overline{c}_c, q'_0c) + J(\overline{c}_0c, q'_c) \] \]), while the second part is due to interactions between eddies only (EEE interactions: \[ J(\psi'_p, q'_p) ' + J(\psi'_c, q'_c) ' \], \[ J(\psi'_c, q'_c) ' + J(\psi'_c, q'_p) ' \], \[ J(\psi'_c, q'_0c) ' + J(\psi'_0c, q'_c) ' \], \[ J(\psi'_c, q'_0p) ' + J(\psi'_0p, q'_c) ' \]). Our goal is to identify which type of interactions lead to the formation of ZELTs. To accomplish this we perform several simulations with fully removed or partially modified EEE forcing. Truncated models are conveniently described in Table 5.1. The acronyms follow the convention: QL is a quasi-linear model with removed \[ J(\psi'_p, q'_p) ' \], \[ J(\psi'_c, q'_c) ' \], \[ J(\psi'_c, q'_p) ' \] and \[ J(\psi'_c, q'_c) ' \] terms, QL_T is a model with no \[ J(\psi'_p, q'_p) ' \] term, QL_CE stands for the model without \[ J(\psi'_c, q'_p) ' \] \[ J(\psi'_c, q'_c) ' \] terms, QL_MXE model does not have \[ J(\psi'_p, q'_c) ' \] \[ J(\psi'_c, q'_p) ' \] terms, in QL_CQ model the terms \[ J(\psi'_p, q'_c) ' \] \[ J(\psi'_c, q'_c) ' \] are absent, QL_MXQ has no \[ J(\psi'_c, q'_p) ' \] \[ J(\psi'_c, q'_c) ' \] terms and NL is a fully non-linear model. For the reasons unknown to us, the truncated model QL_CQ does not converge.
We first compare spatial Fourier spectra and EOF1 for simulations with moderate \( \gamma = 3 \times 10^{-7} \) (Fig. 5.1 and 5.2) and high \( \gamma = 9 \times 10^{-7} \) (Fig. 5.3 and 5.4) values of bottom drag coefficient. The choice for these parameter values was motivated by sensitivity analysis of the previous chapter, which shows that ZELTs are pronounced in simulations with \( \gamma = 3 \times 10^{-7} \) and less easily detectable in simulations with \( \gamma = 9 \times 10^{-7} \) (see also [12] for explanation of the role of bottom drag in causing latency of zonal jets). For unambiguous comparison we have removed the Fourier mode with zonal wavenumber \( k = 0 \). Spatial Fourier spectra of QL simulations show no power at long zonal wavelengths hinting at the absence of ZELTs in the turbulent flow (Figs. 5.1a and 5.2a). Spatial structure of EOF1 confirms these expectations: alternating meridionally-elongated patterns populate the entire domain (Figs. 5.3a and 5.4a). This result is in a good agreement with previous studies on barotropic turbulence [70, 101] and it also emphasizes a crucial role of EEE interactions for ZELTS emergence. Similar to the full model, QL_T simulations with precluded both energy and enstrophy cascades in barotropic mode, show strong spectral peak at long zonal wavelengths (Fig. 5.1b and 5.2b). However, the spatial structure of EOF1 highlights the differences between two cases: the EOF1 for simulation with \( \gamma = 9 \times 10^{-7} \) looks similar to the one of fully nonlinear simulations, the EOF1 for simulation with \( \tau = 3 \times 10^{-7} \) demonstrates rather irregular trains of eddies (Fig. 5.3b and 5.4b). This again confirms the inefficiency of Fourier filtering to extract ZELTs (see chapter 3). Simulations with modified baroclinic mode, QL_CE, QL_MXE and QL_MXQ, reveal a more complex picture. Contrary to the fully nonlinear model, spatial Fourier spectra for simulations with no cascade of energy in baroclinic mode (QL_CE) has almost no power at long zonal wavelengths (Fig. 5.1c and 5.2c). Same
true for EOF1: a highly structured array of small-scale vortices is observed in $\gamma = 3 \times 10^{-7}$ simulation (Fig. 5.3c) and an array of eddies with slight meridional elongation appears in $\gamma = 9 \times 10^{-7}$ simulation (Fig. 5.4c). If energy exchanges between barotropic and baroclinic modes are suppressed (QL\_MXE model), spatial Fourier spectra does contain some power at long zonal wavelengths, although the peak is relatively weak compared to the other parts of the spectra (Fig. 5.1d and 5.2d). The spatial structure EOF1 shows either small-scale eddies with a slight meridional elongation (Fig.5.3d) or tilted meridionally-elongated patterns (Fig. 5.4d). These simulations indicate the first order relevance of mixed-mode interactions, and as a consequence, transport of energy in baroclinic mode, for the formation of ZELTs. Finally, in simulations with no cross-modal enstrophy exchanges (QL\_MXQ model), the spatial Fourier spectra exhibit a well-defined spectral peak at long zonal wavelengths (Fig. 5.1d and 5.2d), yet EOF1 shows trains of eddies in simulations with both values of $\gamma$ (Fig. 5.3d and 5.4d). The EOF1 in QL\_MXQ and QL\_T simulations with $\gamma = 9 \times 10^{-7}$ bear a lot of resemblance.

We next discuss temporal variability of the reduced dynamics models by analyzing ACF of PC1. The ACF of PC1 for simulations with $\gamma = 3 \times 10^{-7}$ and $\gamma = 9 \times 10^{-7}$ are shown in Fig. 5.5 and 5.6, respectively. ACF for QL (Fig. 5.5a and 5.6a) and QL\_MXE (Fig. 5.5d and 5.6d) exhibit high-frequency oscillations with almost negligibly small decorrelation time scale indicating that ZELTs are absent in the flow field. In contrast, the ACF for QL\_CE simulations shows high-frequency variability for the case with $\gamma = 9 \times 10^{-7}$ (Fig. 5.5c), while for the case with $\gamma = 3 \times 10^{-7}$ several oscillations are observed (Fig. 5.5d); decorrelation time scales for $\tau = 3 \times 10^{-7}$ case are comparable in QL\_CE and NL simulations. Such a behavior of ACF for $\gamma = 3 \times 10^{-7}$
Abbreviation | Model | Physical Meaning |
---|---|---|
QL | $J(\psi_p', q_p') \to 0 \quad J(\psi_c', q_c') \to 0$ | No energy\enstrophy cascade |
QL_T | $J(\psi_p', q_p') \to 0$ | No energy\enstrophy cascade in barotropic mode |
QL_CE | $J(\psi_c', q_c') \to 0 \quad J(\psi_p', q_p') \to 0$ | No energy cascade in baroclinic mode. Enstrophy is not preserved |
QL_MXE | $J(\psi_p', q_p') \to 0 \quad J(\psi_c', q_c') \to 0$ | No eddy energy exchanges between barotropic and baroclinic modes. Enstrophy is not preserved |
QL_CQ | $J(\psi_p', q_p') \to 0 \quad J(\psi_c', q_c') \to 0$ | No enstrophy cascade in baroclinic mode. Energy is not preserved |
QL_MXQ | $J(\psi_c', q_c') \to 0 \quad J(\psi_p', q_p') \to 0$ | No eddy enstrophy exchanges between barotropic and baroclinic modes. Energy is not preserved |
NL | $J(\psi_c', q_c') \to 0 \quad J(\psi_p', q_p') \to 0$ | Full nonlinear dynamics |

Table 5.1: Reduced dynamics models

case does not conform our expectations and eludes any explanations. Comparing ACF in QL_T and NL simulations, one can conclude that both decorrelation time scale and a subsequent behavior are similar for $\tau = 9 \times 10^{-7}$ case, and rather different for $\gamma = 3 \times 10^{-7}$ case (Fig. 5.5b and 5.6b). This suggests that flow patterns extracted from QL_T and NL simulations with $\gamma = 9 \times 10^{-7}$ (i.e. ZELTs) evolve on the same time scale, while the same flow patterns in simulations with $\gamma = 3 \times 10^{-7}$ exhibit dissimilar evolution. The ACF in simulations with no barotropic/baroclinic enstrophy exchanges (QL_MXQ model) shows rapid decay with subsequent fluctuations around zero (Fig. 5.5e and 5.6e), apparently different behavior from ACF in NL simulations.

Two major points are worth mentioning from the analysis presented above. First, since ZELTs do not show up in the QL simulations, the mechanism based on energy cascade arguments is more relevant to the formation of ZELTs. Second, contrary to barotropic mode, the energy and enstrophy transfers in baroclinic mode play a crucial role in regulating the dynamics of ZELTs.
Figure 5.1: Spatial Fourier spectrum of streamfunction for simulations with $\beta = 2.15 \times 10^{-11} \text{ m}^{-1} \text{s}^{-1}$ and $\gamma = 3 \times 10^{-7} \text{ s}^{-1}$. a) QL, b) QL_T, c) QL_CE, d) QL_MXE, e) QL_MXQ, f) NL
Figure 5.2: Spatial Fourier spectrum of streamfunction for simulations with $\beta = 2.15 \times 10^{-11}$ m$^{-1}$s$^{-1}$ and $\gamma = 9 \times 10^{-7}$ s$^{-1}$. a) QL, b) QL$_T$, c) QL$_{CE}$, d) QL$_{MXE}$, e) QL$_{MXQ}$, f) NL.
Figure 5.3: EOF 1×10^{-3} for simulations with $\beta = 2.15 \times 10^{-11} \text{ m}^{-1} \text{s}^{-1}$ and $\gamma = 3 \times 10^{-7} \text{ s}^{-1}$. a) QL, b) QL_T, c) QL_CE, d) QL_MXE, e) QL_MXQ, f) NL.
Figure 5.4: EOF1×10^{-3} for simulations with \( \beta = 2.15 \times 10^{-11} \text{ m}^{-1}\text{s}^{-1} \) and \( \gamma = 9 \times 10^{-7} \text{ s}^{-1} \). a) QL, b) QL_T, c) QL_CE, d) QL_MXE, e) QL_MXQ, f) NL.
Figure 5.5: ACF of PC1 for simulations with $\beta = 2.15 \times 10^{-11}$ m$^{-1}$s$^{-1}$ and $\gamma = 3 \times 10^{-7}$ s$^{-1}$. a) QL, b) QL_T, c) QL_CE, d) QL_MXE, e) QL_MXQ, f) NL
Figure 5.6: ACF of PC1 for simulations with $\beta = 2.15 \times 10^{-11} \text{ m}^{-1} \text{s}^{-1}$ and $\gamma = 9 \times 10^{-7} \text{ s}^{-1}$. a) QL, b) QL_T, c) QL_CE, d) QL_MXE, e) QL_MXQ, f) NL
CHAPTER 6

Conclusions

Anomalous structures of oceanic turbulence, as exemplified by vortices and zonally-elongated transient patterns, strongly affect the statistical characteristics of mesoscale turbulence and regulate material transport throughout the ocean interior. Motivated by their significance for ocean circulation, we have studied their statistical, dynamical and transport properties.

In the first chapter, we have studied the stability and transport of baroclinic vortices on the $\beta$-plane. Globally integrated and local energy balance allows the examination of energy exchanges between the basic-state flow (“vortex”) and perturbations (“waves”). We find that the time evolution of globally integrated energy exhibits two significant transformations, one associated with the deformation of the vortex and another with its fragmentation. At the time of the first minimum of energy change, the distorted vortex is able to regain its shape without spawning further eddies; the breakdown of the vortex occurs during the second significant energy loss and, unlike on the $f$-plane, both baroclinic and barotropic instabilities can be the cause. Analyzing the energy balance locally, we found that the maximum barotropic energy gain occurs at the vortex centre and the baroclinic energy loss takes place either uniformly within the core when the vortex deforms or decreases rapidly just before $R_{\text{max}}$ while
disintegrating. The former explains the vortex breakdown as a fissure of the vortex core. It is also interesting that the results of local energy balance can give different insight into the problem of instability than globally integrated energy balance.

We also compare mixing induced by stable and unstable vortical flows. Utilizing FTLE maps, we have shown that the most active mixing in the upper layer occurs within the ring region, which can be either continuous or volute depending on the strength of the wave radiation. During the vortex breakdown, the ring region is destroyed and several FTLE filaments connect the main vortex with pinched off smaller vortices. This indicates the exchange of water within and outside of the vortex core as a result of instability. As revealed by wave-like FTLE patterns, a clear signature of wave induced mixing is observed in the far-field. We also found that the lower layer FTLE patterns for the cases of stable vortices have a wavy form, while the unstable cases show irregular filaments of FTLE.

Finally, both stable and unstable vortices affect the large-scale distribution of a tracer. We claim that these impacts are both due to the vortex propagation and wave radiation, and, in the case of unstable vortices, to the vortex fragmentation. The upper-layer transport is dominated by coherent structure with the contribution of waves being less important. The instability of the vortices results in the reduction of particle concentration at some latitudes. In the lower layer, the transport is entirely produced by waves and wave-like eddies. Although the change in the large-scale tracer gradient is weaker than in the upper layer, this effect is still substantial and indicates an overall importance of the surface-intensified vortices for the material transport in the deep ocean.
The findings of the first chapter emphasize a significant role of the \( \beta \)-effect and stratification in modifying stability as well as transport properties of oceanic vortices compared to the \( f \)-plane and barotropic studies.

In the second chapter, we test the performance of two filtering techniques to efficiently separate ZELTs from the background flow and quantify the sensitivity of ZELTs properties to changes in bottom drag coefficient and meridional gradient of planetary vorticity. Low pass Fourier filtered isotropic and anisotropic turbulent flow fields reveal a series of similar zonally-elongated patterns, thereby suggesting the inadequacy of this technique to identify ZELTs. Alternatively, the EOF decomposition provides a framework for efficient representation of ZELTs. The leading EOF modes derived from the anisotropic turbulent flow field reveal a series of zonally-elongated patterns with eddies embedded in them, while each EOF mode of an isotropic turbulent flow field consists of randomly distributed eddies. The variance spectrum of anisotropic turbulent flow is steeper and several leading EOFs are well separated. Analyzing the ACF of the corresponding PCs, we found that it oscillates with a slow decay for the anisotropic turbulent flow and fluctuates around zero followed by a fast decay for the isotropic turbulent flow. We also investigated the propagation of ZELTs by means of Extended EOF decomposition. As revealed by Hovmöller diagrams of the leading Extended EOF, the patterns propagates in zonal direction.

The spatiotemporal variability of ZELTs is strongly affected by bottom drag \((\gamma)\) and meridional gradient of planetary vorticity \((\beta)\). Decreasing \(\beta\) or increasing \(\gamma\) each leads to flattening of the variance spectra, isotropization of the EOF1 and faster decay of ACF of PC1. Our estimates of the zonostrophy index for different values of \(\beta\) and \(\gamma\) suggest that ZELTs exist in a transitional or friction-dominated regime of
geostrophic turbulence, that is not dominated by stationary zonal jets. The response surface of the anisotropic ratio reveals the region of relatively high values ($\sim 0.8$) in the $\beta - \gamma$ parameter space; ZELTs are the most pronounced in this region.

The second chapter suggests that the mesoscale variability at low- and mid-latitudes can be dominated by ZELTs, and, as a consequence, any attempts to parameterize the eddy-induced stirring by the isotropic diffusivity will lead to biases in tracer distributions [59].

The third chapter elucidates the underpinning mechanisms of ZELTs formation in oceanic turbulence. We test the performance of the reduced dynamics models, in which the nonlinear eddy-eddy forcing is either completely or partially absent, to reproduce ZELTs as in fully nonlinear simulations. Guided by the sensitivity analysis of anisotropy associated with EOF1, we perform simulations with moderate and high values of bottom drag coefficient. Computing spatial Fourier spectra, EOF1 and the ACF of PC1 we argue that eddy-eddy interactions are of vital importance for ZELTs emergence in both moderate- and high-drag simulations. As such, ZELTs is highly non-linear phenomenon owing its existence to the anisotropic upscale energy cascade. The major contribution to this energy transport comes from the baroclinic-baroclinic and mixed-mode eddy interactions, while barotropic-barotropic interactions are of less (moderate-drag simulations) or no importance (high-drag simulations) for ZELTs dynamics.

6.1 Most significant findings and future work

In this concluding section we list the most significant findings of this work and provide some ideas for future studies.
1. Depth-compensated vortices are subject to both barotropic and baroclinic instabilities.

2. Instability of vortices entails significant local and large-scale transport.

3. ZELTs can be conveniently identified as several leading modes of mesoscale variability.

4. Anisotropy associated with ZELTs varies geographically and due to bottom friction.

5. The emergence of ZELTs can be attributed to upscale energy cascade; non-uniform stratification has a significant impact on this energy transport.

In the first chapter we consider the problem of vortex stability by examining the energy exchanges between the vortex and waves. Such an approach was adopted because the undisturbed vortex is not a stationary solution of the governing equations. However, the flow field can be linearized around time-evolving vortex with subsequent computation of singular vectors of the linearized system [30]. Such an approach can give a more direct way to contrast the stability properties of baroclinic vortices on the $f$ and $\beta$-planes by comparing the most unstable eigenvectors of the $f$-plane problem and singular vectors of $\beta$-plane problem.

Since the dynamics of ZELTs is primarily governed by eddy energy cascade, it would be interesting to explicitly examine the spectral structure of energy fluxes and quantify the directions of energy cascade. The impact of higher baroclinic modes on both dynamics of ZELTs and energy transport (see [96]) should also be addressed.

Finally, the dynamics of vortices and ZELTs should be examined in full-physics models on sufficiently fine resolution to capture submesoscale eddies. This can par-
tially contribute to the open problem of energy routes to dissipation and improve parameterization schemes in climate models.
APPENDIX A

Energy balance

We first derive dynamical balance for barotropic and baroclinic modes. Following [83], we switch to the system of reference moving with the vortex center $\tilde{x} = x - X(t); \tilde{y} = y - Y(t)$, where $\sim$ denotes the new system of reference, $X(t)$ and $Y(t)$ is a center of vortex defined as an extremum of the baroclinic potential vorticity field. The modified equations for barotropic and baroclinic modes in the new reference frame are as follows:

\begin{align*}
\frac{\partial q_{BT}}{\partial t} &+ J(q_{BT}, q_{BT}) + \alpha_T J(q_{BC}, q_{BC}) + \beta \frac{\partial q_{BT}}{\partial x} - U(t) \frac{\partial q_{BT}}{\partial x} - V(t) \frac{\partial q_{BT}}{\partial y} = D_{BT} \\
\frac{\partial q_{BC}}{\partial t} &+ J(q_{BC}, q_{BC}) + J(q_{BC}, q_{BT}) + \alpha_C J(q_{BC}, q_{BC}) + \beta \frac{\partial q_{BC}}{\partial x} - U(t) \frac{\partial q_{BC}}{\partial x} - V(t) \frac{\partial q_{BC}}{\partial y} = D_{BC}
\end{align*}

(1)

where

\begin{align*}
q_{BT} &= \nabla^2 \frac{H_1 \psi_1 + H_2 \psi_2}{H_1 + H_2} = \nabla^2 \psi_{BT} \\
q_{BC} &= \nabla^2 (\psi_1 - \psi_2) - (S_1 + S_2)(\psi_1 - \psi_2) = \nabla^2 \psi_{BC} - (S_1 + S_2) \psi_{BC} = \xi_{BC} - (S_1 + S_2) \psi_{BC} \\
\alpha_T &= \frac{H_1 H_2}{(H_1 + H_2)^2} \\
\alpha_C &= \frac{H_2^2 - H_1^2}{(H_1 + H_2)^2} \\
U(t) &= \frac{dX(t)}{dt} \\
V(t) &= \frac{dY(t)}{dt}
\end{align*}
and the tilde is omitted for convenience. To describe how waves change the structure of the vortex, we switch to the polar coordinates with the origin in the vortex center and decompose each variable $a$ into an azimuthally averaged part representing the axisymmetric vortex,

$$<a> = \frac{1}{2\pi} \int_0^{2\pi} a(r, \theta) d\theta$$

and an asymmetric part $a'$ representing waves and defined as deviation from the azimuthal mean. Applying this decomposition to each variable and then taking an azimuthal average of the left- and right-hand side of equation 20 yields:

$$\frac{\partial <q_{BT}>}{\partial t} = - <J(\psi'_{BT}, q'_{BT})> - \alpha_T <J(\psi'_{BC}, q'_{BC})> - \beta \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) <\psi'_{BT} \cos \theta>$$
$$+ U(t) \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) <q'_{BT} \cos \theta> + V(t) \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) <q'_{BT} \sin \theta>$$
$$+ <D_{BT}>$$

Analogously, we get the budget for the baroclinic mode:

$$\frac{\partial <q_{BC}>}{\partial t} = - <J(\psi'_{BT}, q'_{BC})> - <J(\psi'_{BC}, q'_{BT})> - \alpha_C <J(\psi'_{BC}, q'_{BC})>$$
$$- \beta \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) <\psi'_{BC} \cos \theta>$$
$$+ U(t) \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) <q'_{BC} \cos \theta> + V(t) \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) <q'_{BC} \sin \theta>$$
$$+ <D_{BC}>$$

The energy balance is derived by multiplying the equations A3 and A4 by azimuthally averaged batortopic and baroclinic streamfunctions, respectively, and then taking the integral of both right- and left-hand sides over the area of the domain.
\[ \int_0^\infty \frac{\partial < E_{BT} >}{\partial t} r dr = \int_0^\infty \frac{1}{\alpha_T} < \psi_{BT} >= J(\psi'_{BT}, q'_{BT}) + < \psi_{BT} >= J(\psi'_{BC}, q'_{BC}) \\
+ \frac{1}{\alpha_T} < \psi_{BT} >= \beta \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) < \psi'_{BT} \cos \theta > \\
- \frac{1}{\alpha_T} < \psi_{BT} >= \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) (U(t) < q'_{BT} \cos \theta > + V(t) < q'_{BT} \sin \theta >) \\
+ \frac{1}{\alpha_T} < \psi_{BT} >= D_{BT} > r dr \quad (5) \]

\[ \int_0^\infty \frac{\partial < E_{BC} >}{\partial t} r dr = \int_0^\infty < \psi_{BC} >= J(\psi'_{BT}, q'_{BC}) + < \psi_{BC} >= J(\psi'_{BC}, q'_{BT}) \\
- (S_1 + S_2) < \psi_{BC} >= J(\psi'_{BC}, \psi'_{BT}) > \\
+ \alpha_C < \psi_{BC} >= J(\psi'_{BC}, q'_{BC}) > \\
+ < \psi_{BC} >= \beta \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) < \psi'_{BC} \cos \theta > \\
- < \psi_{BC} >= \left( \frac{1}{r} + \frac{\partial}{\partial r} \right) (U(t) < q'_{BC} \cos \theta > + V(t) < q'_{BC} \sin \theta >) \\
+ < \psi_{BC} >= D_{BC} > r dr \quad (6) \]

where

\[ \frac{\partial < E_{BT} >}{\partial t} = \frac{1}{\alpha_T} < \psi_{BT} > \frac{\partial < q_{BT} >}{\partial t} \quad (7) \]

\[ \frac{\partial < E_{BC} >}{\partial t} = < \psi_{BC} > \frac{\partial < q_{BC} >}{\partial t} \quad (8) \]
APPENDIX B

Computational aspects of EOF/EEOF

We discuss some of the key properties of Empirical Orthogonal Functions (EOF) and Extended Empirical Orthogonal Functions (EEOF), and present the technical details of their computation pertinent to our model configuration. For general discussion of EOF application to turbulent flows the reader can consult [2, 7, 45, 97, 98].

The output of the model is represented as a two dimensional matrix $X_{ml}$, where index $m$ corresponds to a grid point and index $l$ denotes sampling time, $m = 1, \ldots, N_{gr}$, $l = 1, \ldots, T$. Here $N_{gr}$ is the number of grid points of the computational domain, $T$ is the number of samples. Each column of this matrix is a state vector at a particular instance, which is assumed to have zero time mean. The matrix is a discrete version of time evolving, continuous state vector $x(t)$. The computation of EOFs boils down to the eigenvalue problem

$$A_{mm}\phi_m = \lambda_m \phi_m$$

(9)

where $\lambda_m$ are eigenvalues and $\phi_m$ are corresponding eigenvectors (EOF). $A_{mm}$ is a covariance matrix defined as

$$A_{mm} = \sum_{l=1}^{T} X_{ml}X_{lm}$$

(10)

Direct computation of eigenvectors and eigenvalues of matrix $A_{mm}$ is impossible due to the enormous size of the matrix. To circumvent this problem, we instead compute the eigenvectors and eigenvalues of matrix $B_{ll} = \sum_{m=1}^{N_{gr}} X_{lm}X_{ml}$, which has the same non-zero eigenvalues as matrix $A_{mm}$ and its eigenvectors related to the eigenvectors $\eta_l$ of matrix $B_{ll}$ through a simple linear transformation [103].
Since the covariance matrix $A_{mm}$ is positive definite, the eigenvectors (EOF) of this matrix form a complete orthonormal basis in the space with appropriately chosen inner product. Examples of the inner product include area integrated energy, enstrophy or squared streamfunction. Since the focus of this study is on large-scale flows, we use the inner product in the form of squared streamfunction [95]. Now, the state vector $x(t)$, that is streamfunction, can be expanded in series over this basis

$$x(t) = \sum_{k=1}^{T} \alpha_k(t) \phi_k$$

where the expansion coefficients $\alpha_k(t)$ are referred to as Principal Components (PC). The basis spanned by EOF suggest that the squared norm generated by the inner product, or equivalently the variance, is minimized. The EOFs are typically arranged according to the fraction of explained variance, which can be expressed as

$$\frac{\text{var}(x_k)}{\sum_{i=1}^{T} \text{var}x_i} = \frac{\lambda_k}{\sum_{i=1}^{T} \lambda_i}$$

where $\text{var}x_k$ is variance of the $k$-th EOF mode.

For Extended EOFs the data matrix takes into account the correlation in time, so that index $m$ runs as $m = 1, \ldots, M \times N_{gr}$, and index $l$ runs as $l = 1, \ldots, T - M + 1$. Here $M$ is a number of time lags. For the rest, Extended and regular EOFs are computed in a similar fashion.

$$\phi_m = \sum_{l=1}^{T} \frac{X_{ml}}{\det(X_{ml})} \eta_l$$
APPENDIX C

Response surface construction

We introduce the basic idea of the response surface construction [53]. For simplicity, consider anisotropic ratio $\alpha$, which depends only on a single parameter $\gamma$. The $\tilde{\alpha}(\gamma)$ approximates the true anisotropic ratio $\alpha(\gamma)$ computed at several points in the parameter space and can be represented as a series expansion over some basis:

$$\alpha(\gamma) = \tilde{\alpha}(\gamma) + \epsilon(\gamma) = \sum_{i=1}^{M+3} c_i \phi_i(\gamma) + \epsilon(\gamma)$$  \hspace{1cm} (13)

where $\epsilon$ is an error committed by the approximation, $\phi_i(\gamma)$ are the basis functions, $c_i$ are the expansion coefficients to be determined. In this study we choose $\phi_i(\gamma)$ in the form of cubic B-splines [15,22,52].

$$\phi_i(\gamma) = \begin{cases} 
0, & \gamma < \xi_{i-4} \\
B_1(\frac{\gamma - \xi_{i-4}}{f}), & \xi_{i-4} < \gamma < \xi_{i-3} \\
B_2(\frac{\gamma - \xi_{i-3}}{f}), & \xi_{i-3} < \gamma < \xi_{i-2} \\
B_3(\frac{\gamma - \xi_{i-2}}{f}), & \xi_{i-2} < \gamma < \xi_{i-1} \\
B_4(\frac{\gamma - \xi_{i-1}}{f}), & \xi_{i-1} < \gamma < \xi_i \\
0, & \xi_i < \gamma.
\end{cases}$$  \hspace{1cm} (14)

where $\xi_{-3} \ldots \xi_0, \xi_1, \ldots, \xi_M \ldots \xi_{M+3}$ are equispaced knots, $f$ is a common knot interval and

$$B_1(r) = \frac{r^3}{6}$$

$$B_2(r) = \frac{-3r^3 + 3r^2 + 3r + 1}{6}$$
\[ B_3(r) = \frac{3r^3 - 6r^2 + 4}{6} \]
\[ B_4(r) = \frac{-r^3 + 3r^2 - 3r + 1}{6} \]

93 The coefficients \( c_i \) are determined by minimizing the error norm. We choose the norm in the following form:

\[ ||\epsilon(\gamma)|| = p \sum_{i=1}^{M+3} \left( \frac{f(\gamma_i) - \tilde{f}(\gamma_i)}{\delta f(\gamma_i)} \right)^2 + (1 - p) \int_{\gamma_1}^{\gamma_M} f^2 d\gamma \]

(15)

Here \( \alpha(\gamma_i) \) and \( \tilde{\alpha}(\gamma_i) \) are the values of anisotropic ratio \( \alpha(\gamma) \) and its approximation \( \tilde{\alpha}(\gamma) \) at the data sites \( \gamma_i \), \( \delta \alpha(\gamma_i) \) is an estimate of the standard deviation in \( \alpha(\gamma_i) \). The smoothing parameter \( p \) ranges from \( p = 0 \), in which case the fit is just a straight line, to \( p = 1 \) - the least-square fit. The generalization for multiple parameters case is straightforward. Now, our anisotropic ratio \( \alpha(\gamma, \beta) \) depends on two parameters \( \gamma \) and \( \beta \), and is approximated by \( \tilde{\alpha}(\gamma, \beta) \) with committed error \( \epsilon(\gamma, \beta) \)

\[ \alpha(\gamma, \beta) = \tilde{\alpha}(\gamma, \beta) + \epsilon(\gamma, \beta) = \sum_{i=1}^{M+3} \sum_{j=1}^{N+3} c_{ij} \phi_i(\gamma) \phi_j(\beta) + \epsilon(\gamma, \beta) \]

(16)

and the coefficients \( c_{ij} \) are found by minimizing the error norm of the form

\[ ||\epsilon(\gamma, \beta)|| = p \sum_{i=1}^{M+3} \sum_{j=1}^{N+3} \left( \frac{\alpha(\gamma_i, \beta_j) - \tilde{\alpha}(\gamma_i, \beta_j)}{\delta \alpha(\gamma_i, \beta_j)} \right)^2 + (1 - p) \int_{\gamma_1}^{\gamma_M} \int_{\beta_1}^{\beta_N} (\alpha_{\gamma})^2 + (\alpha_{\beta})^2 d\gamma d\beta \]

(17)

The values of data sites in the parameter space are shown in fig.4.13 with black stars. The validation error is computed as

\[ ||\epsilon(\gamma_v, \beta_v)|| = |\alpha(\gamma_v, \beta_v) - \tilde{\alpha}(\gamma_v, \beta_v)| \]

(18)

The index \( v \) refers to the location of validation points in the parameter space marked by red stars in fig.4.13.
Fig. 4.14a displays the response surface obtained with interpolation points marked by black stars in fig. 4.13. Even though the value 0.5 of smoothing parameter gives the best approximation, the validation error shown in fig. 4.15a indicates that the surface approximates the anisotropic ratio poorly in the lower right corner. After adding several interpolation points marked by diamonds as shown in fig. 4.13, the new response surface resolves more finer structures in the lower right corner (fig. 4.14b) and the validation error is significantly decreased in that area (fig. 4.15b).
APPENDIX D

Dynamical balance

The governing equations of quasi-geostrophic dynamics can be projected onto a set of vertical modes, which are the eigenfunctions of Sturm-Liouville operator [81]:

\[
\frac{d}{dz} \left( f^2 \frac{d\phi}{dz} \right) = \lambda^2 \phi z \in (-H, 0) \quad (19)
\]

In the case of the two-layer system, only the barotropic and the first baroclinic modes are retained. The projection of the governing equations onto these two modes gives

\[
\frac{\partial q_p}{\partial t} + J(\psi_p, q_p) + \theta_{11} \theta_{12} J(\psi_c, q_c) + \beta \frac{\partial \psi_p}{\partial x} + U S_1 \theta_{11} \frac{\partial \psi_c}{\partial x} = -U \theta_{11} \frac{\partial q_p}{\partial x} - U \theta_{12} \frac{\partial q_c}{\partial x} - \gamma \theta_{12} q_p + \gamma \theta_{11} \theta_{12} \xi_c + \nu \nabla^2 q_p \quad (20)
\]

\[
\frac{\partial q_c}{\partial t} + J(\psi_p, q_c) + J(\psi_c, q_p) + (\theta_{11}^2 - \theta_{12}^2) J(\psi_c, q_c) + (\beta + U(S_1 \theta_{12} - S_2 \theta_{11})) \frac{\partial \psi_c}{\partial x} + U(S_1 + S_2) \frac{\partial \psi_c}{\partial x} = -U \frac{\partial q_c}{\partial x} - U \theta_{12} \frac{\partial q_c}{\partial x} - \gamma q_p + \gamma \theta_{11} \xi_c + \nu \nabla^2 \xi_c \quad (21)
\]

where

\[
\theta_{11} = \frac{H_1}{H_1 + H_2} = \frac{H}{H + 1} = \frac{\delta}{\delta + 1}
\]

\[
\theta_{12} = \frac{H_2}{H_1 + H_2} = \frac{1}{H + 1} = \frac{1}{\delta + 1}
\]

\[
q_p = \nabla^2 \frac{H_1 \psi_1 + H_2 \psi_2}{H_1 + H_2} = \nabla^2 \psi_p
\]
\[ q_c = \nabla^2 (\psi_1 - \psi_2) - (S_1 + S_2)(\psi_1 - \psi_2) = \nabla^2 \psi_c - (S_1 + S_2)\psi_c = \xi_c - (S_1 + S_2)\psi_c \]

Decomposing the flow field into “mean” (zonally-averaged) and “eddy” (deviation from zonally-averaged) components leads to a set of four equations, which is a framework for constructing reduced-dynamics models.

\[
\frac{\partial \bar{q}_p}{\partial t} = -J(\bar{\psi}_p', q'_p) - \theta_{11}^1 \theta_{12} \bar{J}(\psi'_c, q'_c) - \gamma \theta_{12} \xi_p + \gamma \theta_{11} \theta_{12} \xi_c + \nu \nabla^2 \xi_p \tag{22}
\]

\[
\frac{\partial q'_p}{\partial t} = -J(\bar{\psi}_p', q'_p) - J(\psi'_p, q'_p) - J(\psi'_p, q'_p)' - \theta_{11} \theta_{12} [J(\bar{\psi}_c, q'_c) + J(\psi'_c, q'_c)] \\
- \beta \frac{\partial \psi'_p}{\partial x} - US_1 \theta_{11} \frac{\partial \psi'_c}{\partial x} - U \theta_{11} \frac{\partial q'_p}{\partial x} - U \theta_{11} \theta_{12} \frac{\partial q'_c}{\partial x} \\
- \gamma \theta_{12} \xi'_p + \gamma \theta_{11} \theta_{12} \xi'_c + \nu \nabla^2 \xi'_p \tag{23}
\]

\[
\frac{\partial \bar{q}_c}{\partial t} = -J(\bar{\psi}_p', q'_c) - J(\psi'_c, q'_c) - (\theta_{11}^2 - \theta_{12}^2) \bar{J}(\psi'_c, q'_c) - \gamma \xi_p + \gamma \theta_{11} \xi_c + \nu \nabla^2 \xi_c \tag{24}
\]

\[
\frac{\partial q'_c}{\partial t} = -J(\bar{\psi}_p, q_c) - J(\psi'_p, q_c) - J(\psi'_p, q_c)' - J(\bar{\psi}_c, q'_c) - J(\psi'_c, q'_c)' \\
- (\theta_{11}^2 - \theta_{12}^2) [J(\bar{\psi}_c, q'_c) + J(\psi'_c, q'_c)] \\
- (\beta + U(S_1 \theta_{12} - S_2 \theta_{11})) \frac{\partial \psi'_c}{\partial x} - U(S_1 + S_2) \frac{\partial \psi'_c}{\partial x} - U \frac{\partial q'_p}{\partial x} - U \theta_{12} \frac{\partial q'_c}{\partial x} \\
- \gamma \xi'_c + \gamma \theta_{11} \xi'_c + \nu \nabla^2 \xi'_c \tag{25}
\]
Bibliography


