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Non-Equilibrium Fluctuation Limit of a Mean Field AIMD Model

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NON-EQUILIBRIUM FLUCTUATION LIMIT OF A MEAN FIELD AIMD MODEL

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In this paper we study the fluctuation limit of a particle system in non-equilibrium. Each individual among n particles with current position $x(t)$ moves on the positive axis according to a Poisson clock. With probability $1-p$, depending on the average position of the particle configuration, it moves to $x(t)+1$ and with probability p to $\gamma\, x(t)$, $\gamma \in (0,1)$. This is the *Additive Increase Multiplicative Decrease* (AIMD) internet traffic protocol, where $x(t)$ is the data transmission rate of a given user. Under proper scaling, when $n \to \infty$, the system has a deterministic *fluid limit* described as the solution of an ordinary differential equation. We are looking at the functional second approximation $\xi(t)$, i.e. departures from this limit, on a Central Limit Theorem scale. The random field $\xi(t)$ is identified by its action on special test functions ϕ, where $t \to \langle \xi(t), \phi \rangle$ are formally identified as diffusions. For polynomial test functions, the central limit theorem fluctuation field is tight and we identify its limit explicitly. Labeling the random field $Z(k,t)$ for each monomial $\phi(x) = x^k$ of degree $k \geq 1$ we obtain a hierarchical system of diffusions, in the following sense: the vector $(Z(1,t), \ldots, Z(m,t))$ is a linear diffusion with time dependent explicit coefficients and the system for $m' > m$ is consistent with the system for m in that the matrix is sub-diagonal. When the initial data is Gaussian, the infinite-dimensional process, indexed by $\phi(x)$ polynomials, is a Gaussian process. The abstract random field limit is formulated as a generalized Ornstein-Uhlenbeck process and we discuss some open problems.

Dedication

I dedicate my dissertation work to my family and many friends, for their help for many years.

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Chapter 1

Introduction

1.1 The general interacting particle model

In the following, we study a random system of n particles

$$\mathbf{x}(n,t) = (x_1(t), x_2(t), ..., x_n(t)) \in (0,\infty)^n\,, \qquad t \geq 0 \tag{1.1}$$

where the components $x_i(t)$, $1 \leq i \leq n$, denoting their positions at time t, undergo a random motion on the positive real axis guided by a variant of the *Additive Increase Multiplicative Decrease* traffic control protocol (TCP). This is well known in the engineering literature as the AIMD internet traffic protocol and this model had been analyzed in [3, 12, 27, 10, 25, 9, 11].

We begin by describing a more general model and then specialize to our case. The dynamics will be a pure jump Markov process in continuous time $t \geq 0$. By the law of the movement of particles, as will be given below, the state space is $S = (0,\infty)^n$, with elements $\mathbf{x} = (x_1, \ldots, x_n)$. For every $1 \leq i \leq n$, we denote the initial position $x_i(0) > 0$.

Using the standard construction of a pure jump process, for example in [24], as well as [7], each particle carries a Poisson clock governing the jump times. All clocks are independently distributed with parameter $\lambda = 1$. When x_i, $1 \leq i \leq n$ (without loss of generality) jumps at time τ, the particle has a probability p of moving backward and a probability $1 - p$ of moving forward. The position of the particle x_i when it jumps at time τ will be described precisely in the following.

Attached to every particle x_i are the deterministic measurable functions

$$\zeta_i : S \to [0, 1] \,, \qquad 1 \leq i \leq n \tag{1.2}$$

denoting jump probabilities $p = \zeta_i(\mathbf{x}(n, t))$ whenever a jump occurs at time t (we note that in the mean field model, the functions $\zeta_i(\mathbf{x}) = p(\bar{x})$, where $\bar{x} = (x_1 + \ldots x_n)/n$, p continuous). In addition, a constant $0 < \gamma < 1$ will be set for all particles and all times. The AIMD model implies that

$$\begin{aligned} x_i(\tau) &= \gamma x_i(\tau-) \quad \text{with probability} \quad \zeta_i(\mathbf{x}(n, \tau-)) \\ x_i(\tau) &= x_i(\tau-) + 1 \quad \text{with probability} \quad 1 - \zeta_i(\mathbf{x}(n, \tau-)) \,. \end{aligned} \tag{1.3}$$

General models of this kind have a detailed explanation in [2]. The first line in (1.3) is the multiplicative decrease, and the second line the additive increase. The step size for increase can be scaled (not equal to one unit) and the process speed $\lambda > 0$, can be scaled as well, leading to other versions of the dynamics, studied in the literature [2, 3].

The moving law of the particle implies that for every $1 \leq i \leq n$, at any moment $t \geq 0$, $x_i(t) > 0$ almost surely, unless either a particle tends to zero or escapes to infinity in finite time with positive probability. These scenarios of explosion can be removed by setting

$$0 < p_0 \le \zeta_i(\mathbf{x}) \le p_0' < 1\,. \tag{1.4}$$

These bounds are very strong, and in general can be relaxed significantly (see [15] and others) when studying questions like irreducibility, ergodicity, and so on. In this work they are needed because we are interested in the second order approximation of the empirical measure, the *fluctuation limit in non-equilibrium*, that raises its own difficulties and requires tighter bounds on the coefficients. They are formally stated in Assumption 1.2, Part (**1**).

1.2 Generator and Martingales

A detailed explanation for a general pure jump process can be seen in [4].

For a point in the state space $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n) \in S = (0, \infty)^n$ we introduce the two shift notation

$$L_i\mathbf{x} = (x_1, \ldots, \gamma\, x_i, \ldots, x_n)$$

meaning that the i-th particle jumps to the left (the multiplicative decrease part), and

$$R_i\mathbf{x} = (x_1, \ldots, x_i + 1, \ldots, x_n)$$

meaning that the i-th particle moves to the right by one step (the additive increase).

For a test function $f \in C_b([0, \infty) \times S)$-the space of continuous bounded functions in $(t, \mathbf{x})$, the generator is

$$\mathcal{A}_{t,n} f(t, \mathbf{x}) = \lambda\Big[\sum_{i=1}^{n} \Big((1 - \zeta_i(\mathbf{x}))\,(f(t, R_i\mathbf{x}) - f(t, \mathbf{x})) + \zeta_i(\mathbf{x})(f(t, L_i\mathbf{x}) - f(t, \mathbf{x}))\Big)\Big] \tag{1.5}$$

Note that the jump probabilities (1.2) are naturally bounded above by one and cannot vanish at the same time. Then we can state the following general result defining the process.

Definition 1.1. *The generator (1.5) defines a pure jump Markov process* $\boldsymbol{x}(n,t,\omega)$*,* $t \geq 0$*, adapted to a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$*. It can be assumed that it satisfies* the usual conditions*, i.e. the paths are right-continuous with left limits (RCLL) and the filtration* $(\mathcal{F}_t)_{t\geq 0}$ *is right-continuous. We shall not write the notation* ω *and use the simplified* $\boldsymbol{x}(n,t)$*,* $t \geq 0$ *instead. In this work, the process speed is not scaled and* $\lambda = 1$.

Then, if $M_f(n,t)$ denotes the expression

$$\begin{aligned} M_f(n,t) = f(t, \mathbf{x}(n,t)) - f(0, \mathbf{x}(n,0)) \\ - \int_0^t \{\partial_s f(s, \mathbf{x}(n,s-)) + \mathcal{A}_{s,n} f(s, \mathbf{x}(n,s-))\} ds \,, \end{aligned} \tag{1.6}$$

then the differential formula (1.6) defines the analogue of Ito's lemma for jump processes in the sense that $M_f(n,t)$ is a $(\mathcal{F}_t)_{t\geq 0}$ martingale with quadratic variation given in the formula

$$\langle M_f(n,t)\rangle = \int_0^t [\mathcal{A}_{s,n} f^2(s, \mathbf{x}(n,s-)) - 2f(s, \mathbf{x}(n,s-))\mathcal{A}_{s.n} f(s, \mathbf{x}(n,s-))] ds \,. \tag{1.7}$$

The last equation can be written explicitly in this case

$$\begin{aligned} \langle M_f(n,t)\rangle = \int_0^t \sum_{i=1}^n \Big((1-\zeta_i(\mathbf{x}(n,s-)))\,(f(t, R_i\mathbf{x}(n,s-)) - f(t, \mathbf{x}(n,s-)))^2 + \\ \zeta_i(\mathbf{x}(n,s-))(f(t, L_i\mathbf{x}(n,s-)) - f(t, \mathbf{x}(n,s-)))^2\Big) ds \,. \end{aligned} \tag{1.8}$$

Remark. In both equations (1.6) and (1.7) the left-limit $s-$ can be dropped since the integration takes place against the continuous Lebesgue measure on the positive axis ds. We discuss the construction and the martingales (1.6) and (1.7) in more generality in following chapters.

1.3 The mean field model

If for any pair $1 \leq i, j \leq n$, $\zeta_i(\mathbf{x}(n,t)) = \zeta_j(\mathbf{x}(n,t))$, we simply denote the common value by $\zeta(\mathbf{x}(n,t))$. Further, let

$$\bar{x}_n(t) = \frac{1}{n}\sum_{i=1}^{n} x_i(t) \tag{1.9}$$

be the empirical average of the particle position at time $t \geq 0$. Assume there exists a continuous real valued function $p : S \to \mathbb{R}$ such that

$$\zeta(\mathbf{x}) = p(\bar{x}_n), \qquad 0 \leq p(x) \leq 1. \tag{1.10}$$

In this case the dynamics is said to be *mean field model*.

Assumption 1.2. *Regularity of the jump rate* $p(x)$*.*

1. *There exist constants* p_0 *and* p_0', $0 < p_0 \leq p_0' < 1$, *such that for any* $x \in (0, +\infty)$,

$$0 < p_0 \leq p(x) \leq p_0' < 1. \tag{1.11}$$

2. $p \in C_b^2((0,+\infty))$, *i.e.* p *is twice continuously differentiable with bounded derivatives* p' *and* p''*.*

3. p *is non-decreasing.*

Under (1.10), and even more so under Assumption 1.2, the process $\mathbf{x}(n,t,\omega)$, $t \geq 0$ is well defined for all times. Denote its empirical measures

$$\mu_n(t, dy) = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i(t)}(dy) \in M_1((0,\infty)), \quad t \geq 0, \tag{1.12}$$

where $M_1(X)$, respectively $\mathcal{M}(X)$ is the space of probability measures, signed measures on the polish space X. In this paper, X is $(0,\infty)$ equipped with the Euclidean norm, and thus (1.12) is a probability measure on the Borel σ-algebra.

Definition 1.3. *Applied to test functions $\phi \in C((0,\infty), \mathbb{R})$, we write*

$$\langle \nu, \phi \rangle = \int_{(0,\infty)} \phi(y)\nu(dy)$$

for a signed measure $\nu \in \mathcal{M}((0,\infty))$. We shall also use the notation

$$\langle \mu_n(t, dy), \phi \rangle = \bar{x}_n(\phi, t), \quad \textit{respectively} \quad \langle \mu(t, dy), \phi \rangle = \bar{x}(\phi, t) \tag{1.13}$$

if $\mu_n(t, dy)$ has a limit $\mu(t, dy)$ as $n \to \infty$. In the special case $\phi(x) = x^k$, $k \geq 0$, we use the shorthand

$$\langle \mu_n(t, dy), \phi \rangle = \bar{x}_n(k, t), \quad \textit{respectively} \quad \langle \mu(t, dy), \phi \rangle = \bar{x}(k, t) \tag{1.14}$$

for the limit, whenever it exists.

The differential formulas (1.6) and (1.7) can be applied to a larger test function space. We follow [15] to define the space of functions with positive exponential moment $q > 0$.

1.4 Test functions

Definition 1.4. *Given $q > 0$, we say that $\phi \in C^2_q((0,+\infty))$ if it is a function in $C^2((0,+\infty))$ with all derivatives up to second order having an exponential moment up to q as $x \to +\infty$ and all negative moments as $x \to 0$, which means that there exists $k > 0$ and a positive constant $K_\phi(k)$ such that*

$$\sup_{0\leq b\leq 2} \sup_{x\in(0,+\infty)} x^k e^{-qx} \left|\phi^{(b)}(x)\right| = K_\phi(k) < \infty \,.$$

Remark 1.5. *Due to the bound away from zero of the jump probabilities, the definition can be simplified by removing the condition as $x \to 0$.*

Remark 1.6. *The main purpose of defining these test functions is to be able to include not only continuous, bounded functions as in (1.5), but also polynomials. More generally, any $\phi(x) = P(x)e^{q'x}$, $q' < q$ and P polynomial, is in the C^2_q - class.*

We need a technical result to guarantee a larger set of test functions, including polynomials.

Proposition 1.7. *The differential formulas apply to functions ϕ of the C^2_q - class where q is determined by the bounds p_0, p'_0 in eq. (1.11), Assumption 1.2.*

Chapter 2

Scaling limits

2.1 Fluid limit - the Law of Large Numbers

Theorem 2 in [15], here labeled Theorem 2.3 proves that the *empirical measures* (2.25) have a deterministic limit. This scaling limit is a Law of Large Numbers for correlated processes $(x_i(t))$. Scaling limits of different models had also been analyzed in [14].

We can start with the particular case of the average $\bar{x}_n(t)$. This deterministic limit $\bar{x}(t)$ is given as the solution of an ordinary differential equation, i.e. a deterministic trajectory with $t \geq 0$. Two other assumptions on the initial configuration for the process are necessary.

Assumption 2.1. *There exists a constant $\eta_0 > 0$ such that*

$$\limsup_{n\to+\infty} E[\langle e^{\eta_0 x}, \mu_n(0,dx)\rangle] < +\infty \tag{2.1}$$

Assumption 2.2. *At $t = 0$, for every test function ϕ, as $n \to +\infty$,*

$$\mu_n(0,dy) \;\to\; \mu(0,dy) \in M_1((0,\infty)) \quad \textit{a deterministic probability measure .}$$

One can consider $\phi(x)$ in the theorem simply a polynomial, but the test functions are more general. Please refer to Definition 1.4 for the notation $\phi \in C^2_\eta((0, +\infty))$, which will be used in the statement of Theorem 2.3.

Theorem 2.3 (Theorem 2 in [15]). *Under Assumptions 1.2, 2.1 and 2.2, the average $\bar{x}_n(\phi, t)$ is tight in the Skorohod space $D([0, \infty), (0, \infty))$ and any limit point $\bar{x}(t)$ is the unique deterministic solution of the ordinary differential equation*

$$\frac{dy_1}{dt} = 1 - p(y_1) - (1 - \gamma)p(y_1)y_1 \tag{2.2}$$

with $y_1(0) = \lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^n x_i(0)$. The empirical measure process (1.12) *is tight in the Skorohod space of time indexed measure -valued paths $D([0, \infty), M_1((0, \infty)))$ and any limit point is the unique solution that verifies the equation*

$$\begin{aligned} &\langle \mu(t, dx), \phi(x)\rangle - \langle \mu(0, dx), \phi(x)\rangle \\ &\qquad - \int_0^t \langle \mu(s, dx), (1 - p(\bar{x}(s)))(\phi(x + 1) - \phi(x)) + p(\bar{x}(s))(\phi(\gamma x) - \phi(x))\rangle ds = 0 \end{aligned} \tag{2.3}$$

for any $\phi \in C^2_\eta((0, +\infty))$, with $\mu(0, dx)$ defined in Assumption 2.2.

Remark 2.4. *In [15], assumptions for the jump probability function and the initial distribution are given with less requirements. In Assumption 1.2 we require that the jump rate function have a bounded second derivative.*

Remark 2.5. *The fact that the jump rate function is non-decreasing is motivated by the application inspiring our model. Mathematically speaking, it is not necessary to establish our results.*

2.2 Bounds for polynomials

Theorem 2.3 shows that for any arbitrary test function $\phi \in C^2_\eta((0,\infty))$, the average $\bar{x}_n(\phi,t)$ converges in distribution to the deterministic process $\bar{x}(\phi,t)$, with the notations of Definition 1.3, eq. (1.13)-(1.14), when the test function is $\phi(x) = x^k$, with $k \in \mathbb{N}$, we conclude that $\bar{x}(k,t)$ is the solution to an ordinary differential equation.

When $k = 0$, for each n, $\bar{x}_n(0,t) = 1$ and $\bar{x}(0,t) = 1$. When $k = 1$, we can show that $\bar{x}(1,t) = \bar{x}(t)$ (for the simple average we drop the index $k = 1$) is the unique deterministic solution to the ordinary differential equation (2.2). We have

$$\begin{aligned} M_n(k,t) = \bar{x}_n(k,t) - \bar{x}_n(k,0) - \frac{1}{n}\sum_{i=1}^{n}\int_0^t \Big\{(1-p(\bar{x}_n(s))) \\ \times\Big[\sum_{j=1}^{k}\binom{k}{j}x_i^{k-j}(s)\Big] - (1-\gamma^k)p(\bar{x}_n(s))x_i^k(s)\Big\}ds \end{aligned} \tag{2.4}$$

$$\begin{aligned} \langle M_n(k,t)\rangle = \int_0^t \Big\{(1-p(\bar{x}_n(s)))\frac{1}{n^2}\sum_{i=1}^{n}\Big[\binom{k}{1}^2 x_i^{2k-2}(s) \\ + 2\binom{k}{1}\binom{k}{2}x_i^{2k-3}(s) + \cdots + 1\Big] \\ + p(\bar{x}_n(s))(1-\gamma^k)^2\frac{1}{n^2}\sum_{i=1}^{n}x_i^{2k}(s)\Big\}ds \end{aligned} \tag{2.5}$$

Denote $\langle M_n(k,t)\rangle$ as the quadratic variation of $M_n(k,t)$, denote $\phi_k(x) = x^k$. Theorem 2.3 is based on the fact that $\langle M_n(k,t)\rangle$ vanishes uniformly with respect to t in a bounded interval as $n \to +\infty$. Then, $\bar{x}_n(k,t)$ converges in distribution to a deterministic process $\bar{x}(k,t)$ and $\bar{x}(k,t)$ is the unique deterministic solution to the ordinary differential equation

$$\frac{dy_k}{dt} = (1-p(y_1))\sum_{j=1}^{k}\binom{k}{j}y_{k-j} - (1-\gamma^k)p(y_1)y_k \tag{2.6}$$

with initial condition

$$y_k(0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i^k(0)\,.$$

In here the initial value is non-random and $y_0(0) = 1$.

We need to show that the non-random $\bar{x}(k,t)$ and $\mathrm{E}[\bar{x}_n(k,t)]$ are bounded for every k and n when $t \in [0,T]$, $T > 0$ arbitrary but fixed. This is our Lemma 2.6 and Lemma 2.7. These two lemmas are very important in our following proof, as they provide some boundedness property of motions of our particle system.

Lemma 2.6. *Let $\gamma \in (0,1)$ be the multiplicative decrease rate in (1.5) and $p_0 > 0$ the lower bound of the probability $p(x)$ in (1.11). For every $k \in \mathbb{N}$, there exists $C_0(k) > 0$ depending on k such that*

$$\bar{x}(k,t) \le (\bar{x}(k,0) - \frac{C_0(k)}{(1-\gamma^k)p_0}) \exp(-(1-\gamma^k)p_0 t) + \frac{C_0(k)}{(1-\gamma^k)p_0} \tag{2.7}$$

Proof. by (2.2) and Assumption 1.2, we have

$$\frac{dy_1}{dt} \le (1-p_0) - (1-\gamma)p_0 y_1 \tag{2.8}$$

Replacing

$$y_1(t) - \frac{1-p_0}{(1-\gamma)p_0}$$

by $z_1(t)$, (2.8) will be

$$\frac{dz_1}{dt} \le -(1-\gamma)p_0 z_1\,. \tag{2.9}$$

Consider $z_2(t) = e^{-(1-\gamma)p_0 t}$ with the property that $z_2'(t) = -(1-\gamma)p_0 z_2(t)$ and the derivative of $z_1(t)/z_2(t)$,

$$
\begin{aligned}
[\frac{z_1(t)}{z_2(t)}]' &= \frac{z_1'(t)z_2(t) - z_1(t)z_2'(t)}{z_2^2(t)} \\
&\leq \frac{-(1-\gamma)p_0 z_1(t)z_2(t) + (1-\gamma)p_0 z_1(t)z_2(t)}{z^2(t)} \leq 0\,.
\end{aligned}
\tag{2.10}
$$

According to (2.10), for every $t \geq 0$,

$$
\frac{z_1(t)}{z_2(t)} \leq \frac{z_1(0)}{z_2(0)}\,,
$$

thus

$$
y_1(t) - \frac{1-p_0}{(1-\gamma)p_0} \leq \Big(y_1(0) - \frac{1-p_0}{(1-\gamma)p_0}\Big)e^{-(1-\gamma)p_0 t} \tag{2.11}
$$

By arranging, we have

$$
y_1(t) \leq \Big(y_1(0) - \frac{1-p_0}{(1-\gamma)p_0}\Big)e^{-(1-\gamma)p_0 t} + \frac{1-p_0}{(1-\gamma)p_0}\,. \tag{2.12}
$$

Seen from (2.12), $y_1(t)$ is bounded. Using induction, if $y_1(t)$, $y_2(t)$,...,$y_{k-1}(t)$ are bounded, then for $y_k(t)$, by (2.6) and Assumption 1.2, there exists a constant $C_0(k) > 0$, $C_0(k)$ depends on k, such that

$$
\frac{dy_k(t)}{dt} \leq C_0(k) - (1-\gamma^k)p_0 y_k\,. \tag{2.13}
$$

Using the similar argument as for the $k = 1$ case,

$$
y_k(t) \leq \Big(y_k(0) - \frac{C_0(k)}{(1-\gamma^k)p_0}\Big)\exp(-(1-\gamma^k)p_0 t) + \frac{C_0(k)}{(1-\gamma^k)p_0}\,. \tag{2.14}
$$

Thus $y_k(t)$ is also bounded above by a constant depending on k, which completes our proof.

$\square$

Using a similar method as what we did in the proof of Lemma 2.6, we can extend our conclusion, that for every n, $\mathrm{E}[\bar{x}_n(k,t)]$ is bounded, which is our Lemma 2.7.

Lemma 2.7. *For every $k \in \mathbb{N}$, $n \in \mathbb{N}$, there exists $C_1(k) > 0$ depending on k such that*

$$E[\bar{x}_n(k,t)] \leq \Big(E[\bar{x}_n(k,0)] - \frac{C_1(k)}{(1-\gamma^k)p_0}\Big)\exp(-(1-\gamma^k)p_0 t) + \frac{C_1(k)}{(1-\gamma^k)p_0} \tag{2.15}$$

Proof. By (2.4), when $k = 1$, since $\mathrm{E}[M_n(t)] = 0$, so we have

$$\mathrm{E}[\bar{x}_n(t)] = \mathrm{E}[\bar{x}_n(0)] + \mathrm{E}[\int_0^t (1 - p(\bar{x}_n(s))) - (1-\gamma)p(\bar{x}_n(s))\bar{x}_n(s)ds] \tag{2.16}$$

followed by

$$\frac{d\mathrm{E}[\bar{x}_n(t)]}{dt} = \mathrm{E}[(1 - p(\bar{x}_n(t))) - (1-\gamma)p(\bar{x}_n(t))\bar{x}_n(t)]\,. \tag{2.17}$$

By our assumption, we have that

$$\frac{d\mathrm{E}[\bar{x}_n(t)]}{dt} \leq (1-p_0) - (1-\gamma)p_0\mathrm{E}[\bar{x}_n(t)] \tag{2.18}$$

Referring to the proof of our Lemma 2.6, we conclude that

$$E[\bar{x}_n(t)] \leq \Big(\mathrm{E}[\bar{x}_n(0)] - \frac{1-p_0}{(1-\gamma)p_0}\Big)e^{-(1-\gamma)p_0 t} + \frac{1-p_0}{(1-\gamma)p_0} \tag{2.19}$$

Also by induction, if for every $1 \leq j \leq k-1$, $E[\bar{x}_n(j,t)]$ is bounded, since

$$\begin{aligned} E[\bar{x}_n(k,t)] = E[\bar{x}_n(k,0)] + E\Big[\int_0^t \Big\{(1 - p(\bar{x}_n(s))) \\ \times \sum_{j=1}^{k} \binom{k}{j} \bar{x}_n(k-j,s) - (1-\gamma^k)p(\bar{x}_n(s))\bar{x}_n(k,s)\Big\}ds\Big] \end{aligned} \tag{2.20}$$

which is

$$\frac{dE[\bar{x}_n(k,t)]}{dt} \leq (1-p_0) \times \sum_{j=1}^{k} \binom{k}{j} E[\bar{x}_n(k-j,t)] - (1-\gamma^k)p_0 E[\bar{x}_n(k,t)] \tag{2.21}$$

By the boundedness of $E[\bar{x}_n(j,t)]$, $1 \leq j \leq k-1$, there exists a positive constant $C_1(k)$, which depends on k, such that

$$\frac{dE[\bar{x}_n(k,t)]}{dt} \leq C_1(k) - (1-\gamma^k)p_0 E[\bar{x}_n(k,t)] \tag{2.22}$$

Also referring to the proof of our Lemma 2.6, we conclude that

$$E[\bar{x}_n(k,t)] \leq \Big(E[\bar{x}_n(k,0)] - \frac{C_1(k)}{(1-\gamma^k)p_0}\Big) \exp(-(1-\gamma^k)p_0 t) + \frac{C_1(k)}{(1-\gamma^k)p_0}, \tag{2.23}$$

which shows that for every k, n, the value $\mathrm{E}[\bar{x}_n(k,t)]$ is bounded above by a constant depending on k.

□

2.3 Fluctuation limit - Central Limit Theorem

In this work, we pursue the second order scaling limit, namely, fluctuation limits. It can be regarded as a generalization of the Central Limit Theorem at two levels. One, it is at the level of empirical distributions for a time-indexed process, as in (2.26)-(2.27). The other level is determined by the interaction among the particles. The jump probabilities depend on the position of *all other particles in the system.*

Definition 1.4, which is cited from [15] establishes that we apply the martingale formulas (1.5)-(1.6) to a larger class of test functions than just the class of continuous and bounded functions, for instance including all polynomials.

For the moment, let us assume that we can formally work with polynomials ϕ. Starting with the simplest case, corresponding to the test function equal to the identity $\phi(x) = x$, let

$$Z_n(t) = \sqrt{n}[\bar{x}_n(t) - \bar{x}(t)] , \qquad t \geq 0 . \tag{2.24}$$

The sequence of stochastic processes $\{Z_n(t)\}_{t\geq 0}$ is indexed by $n \geq 1$, the number of particles. Under some assumptions (Assumption 1.2) we will show that $Z_n(t)$ converges in distribution to $Z(t)$, which is a diffusion of affine type (7.4), ([22], Section 5.6 and also [18]).

To establish the limit of $Z_n(t)$, one needs to show

(i) tightness, more precisely C-tightness, of the sequence of the stochastic processes $Z_n(t)$, showing the existence of at least one limiting process with continuous paths;

(ii) the limiting process of $Z_n(t)$, is a specific Itô process with the appropriate coefficients.

Moving to the next level, we look at the empirical measures (1.12).

$$\langle \mu_n(t, dy), \phi \rangle = \int \phi(y)\, \mu_n(t, dy) = \frac{1}{n} \sum_{i=1}^{n} \phi(x_i(t)) , \qquad t \geq 0 . \tag{2.25}$$

Theorem 2.3 shows that

$$\mu_n(\cdot, dy) \quad \to \quad \text{(in probability)} \quad \mu(\cdot, dy) \quad \text{(a deterministic trajectory)}$$

in the sense that, for any fixed ϕ, the stochastic process $\langle \mu_n(t, dy), \phi \rangle$ converges weakly to a deterministic process, which we denote as $\langle \mu(t, dy), \phi \rangle$. Since this limit is a Law of Large Numbers (LLN), we look at the corresponding Central Limit Theorem (CLT) associated.

To describe the fluctuation field we set

$$\xi_n(t, dy) = \sqrt{n}[\mu_n(t, dy) - \mu(t, dy)], \qquad t \geq 0. \tag{2.26}$$

With a Gaussian (or deterministic) initial condition, for instance, if the particles are i.i.d. at time $t = 0$, the limit will be Gaussian.

Definition 2.8. *The abstract random field (2.26) is defined against test functions. Some notations will be used throughout the paper.*

(i) For $\phi \in C((0, \infty))$

$$Z_n(\phi, t) = \int \phi(y)\, \xi_n(t, dy). \qquad t \geq 0. \tag{2.27}$$

(ii) For convenience, we will use $Z_n(k, t)$ to represent the case when the test function is $\phi(x) = x^k$, $k \in \mathbb{N}$, more precisely $Z_n(k, t) = \langle \xi_n(t, dx), x^k \rangle$.

(iii) When $k = 1$, we just denote $Z_n(t) = \langle \xi_n(t, dx), x \rangle$ as in (2.24).

The main result is Theorem 3.4 in Chapter 3. It will establish the exact form of the limit $\xi(t, dy)$

$$\langle \phi, \xi_n(t, dy) \rangle \Rightarrow \langle \phi, \xi(t, dy) \rangle \tag{2.28}$$

when ϕ is a polynomial test function. The theorem can be stated in a more general form, with $\langle \phi, \xi(t, dy) \rangle$ being the solution to a stochastic differential equation.

When the test function ϕ is a polynomial function, $\langle \phi, \xi(t, dy) \rangle$, which is the weak limit of $Z_n(\phi, t)$, is a Gaussian process. In fact, we prove more. Our result shows that the family of processes indexed by polynomials ϕ

$$\phi \to \langle \phi, \xi(t, dy) \rangle \tag{2.29}$$

are jointly Gaussian with explicit mean and variance as functions of t and ϕ. Definitions of Gaussian random field and its special case Gaussian process will be given in Appendix.

Similar results exist, mostly involving diffusions, in [8], [13], [28] and [29]. In [8], [13], fluctuation limits of different models had been analyzed, even though models are quite different in nature, fluctuation limits of those models are also diffusions. In spite of the simplicity of the model presented here, the fluctuation limit is obtained *in non-equilibrium*, which is hard to capture in explicit form.

2.4 The polynomial case

Part of the problem is to define a suitable function space to work with by indexing the random field. Ideally, we could consider

$$C = \left\{ \phi \in C((0,\infty), \mathbb{R}) \,\middle|\, \quad Z_n(\phi, \cdot) \quad \text{exists and converges in distribution as } n \to \infty \right\}. \tag{2.30}$$

Definition 1.4 shows that in the differential formulas (1.6)-(1.7) we can use test functions with finite exponential moments. This includes polynomials.

Theorem 3.4 and 3.6 show that we can identify the limits $Z(\phi, t)$ of $Z_n(\phi, t)$ from (2.27) when $\phi(x) = x^k$, $k \in \mathbb{N}$. The limits are linear diffusions with coefficients depending on the smaller powers. In this sense we call the limit hierarchical. The linearity is a consequence of the fact that $\xi_n(t, dy)$ applies to ϕ as an integral against a signed measure.

When

$$\phi(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0 \,,$$

then

$$Z_n(\phi, t) = \sum_{j=0}^{k} a_j Z_n(j, t)\,.$$

By linearity, it would be suffice in the following chapters to prove our main results for the monomial case, meaning we prove the inclusion

$$C \supseteq \mathbb{R}[x] \qquad \text{(polynomials over the field of reals)}\,.$$

We begin with a discussion of the simplest case, when the test function $\phi(x) = x$. By showing the C - tightness of the sequence of stochastic processes $Z_n(t)$, we are allowed to pick a limit point, here a process with continuous paths. The limit process must satisfy certain properties that identify it as unique. In this case it will be an Ito process (diffusion). In more detail, the weak limit, which we denote as $Z(t)$, is a linear one dimensional diffusion and Gaussian process (if the initial configuration is Gaussian). The covariance formula is explicit and will be given in Theorem 6.3 and eq. (6.10).

Using induction, for higher power monomials, $Z_n(k, t)$ is also tight and the limiting process is again a solution to a stochastic differential equation. For each $k \in \mathbb{N}$, the limit process $Z(k, t)$ contains terms with lower order $Z(j, t)$, with $1 \leq j \leq k-1$. We shall see that $Z(k, \cdot)$ depends only on $\{Z(j, \cdot)\}_{1 \leq j \leq k-1}$, which casts the problem, as far as polynomials are concerned, into a finite closable equation. Moreover, it turns out that for any given $k \in \mathbb{N}$, the system of processes

$$(Z_n(t), Z_n(2, t), ..., Z_n(k, t))$$

converges in distribution to a multidimensional linear process

$$(Z(t), Z(2, t), ..., Z(k, t))\,.$$

This multidimensional linear process is a k-dimensional linear diffusion process, and is Gaussian when its initial condition is Gaussian. We shall also give a formula for the joint distribution. Given different polynomial test functions ϕ_1, ϕ_2, the cross variation between the martingale parts of $Z(\phi_1, t)$ and $Z(\phi_2, t)$ will be obtained explicitly in Theorem 6.3 as well.

Chapter 3

Main result

3.1 The problem

The limit of the empirical measures in Theorem 2.3 is a generalization of the Law of Large Numbers. We are interested in the next order of magnitude of the limit, which is corresponding to a Central Limit Theorem for an interacting particle system. The following question is the motivation of this work.

Theorem (Fluctuation field). *Under the same assumptions as in Theorem 2.3, consider the limit $\mu(t, dx)$ and denote for $t \geq 0$*

$$\xi_n(t, dx) = \sqrt{n}[\mu_n(t, dx) - \mu(t, dx)] . \tag{3.1}$$

Determine the conditions such that the random field ξ_n converges in distribution and, if that is the case, characterize its limit ξ.

The limit is a *random field* because it is random, as in the CLT, and is a field because it is defined weakly, in the general sense of being an object in a dual space, i.e. integrated against a test function ϕ. In that case, $\langle \xi_n(t, dx), \phi \rangle$ is a family of processes in time $t \geq 0$, indexed by the test functions ϕ. We shall see that if the test functions are polynomials, each such process can be characterized as a diffusion. The joint distribution can be determined because for monomials $\phi(x) = x^k$, $k \geq 0$, the processes are jointly a linear diffusion.

3.2 Plan of the proof

Theorems 3.4 and 3.6 answer the questions posed here. The key derivation, even though only formal, is to obtain equation (3.11). Theorem 3.4 gives a rigorous formulation of (3.11), characterizing the limit ξ as a random field of the generalized Ornstein-Uhlenbeck type, given in (3.11). We note that the drift term is linear with $\mathcal{L}_s^\circ$ from (3.10) containing $\mathcal{L}_s$ from the fluid limit (Theorem 2.3), plus a non-equilibrium component (3.9).

In fact the equation can be written, purely formally, as

$$d\xi(t) = (\mathcal{L}_s^\circ)^* \xi(t)\, dt + dW(t)\,, \tag{3.2}$$

where the star stands for the formal adjoint and dW is a Gaussian zero mean noise defined by (3.15). In particle systems, such limits are introduced in not only Holley - Stroock [19, 18] but also Chang and Yau [8]. Theorem 3.6 translates the more abstract Theorem 3.4 by breaking down polynomials into monomials x^k, ordering them by powers k and studying the joint distribution of the corresponding diffusions.

- Theorem 6.3 (Chapter 6) implies Theorem 3.6 which implies Theorem 3.4.

- Theorem 6.3 is a statement on the degree k of the polynomial ϕ. It is proven by induction, in the two steps below.

- Theorems 4.4 and 4.6 (Chapter 4) answer the simpler question for $\phi(x) = x$, which is also the verification step $k = 1$ of the induction over k.
- Theorem 5.1 (Chapter 5) is the induction step, showing the fact that the fluctuation field for $\phi(x) = x^k$ is a diffusion with coefficients depending on the fluctuation field applied to $\phi(x) = 1, x, x^2, \dots, x^k$.

3.3 Formal equation for the fluctuation field

For the process $\mathbf{x}(n, t) = (x_1(t), x_2(t), ...x_n(t))$ with average $\bar{x}_n(t)$ and a continous function f, denote

$$\mathcal{L}_{t,n} f(x) = (1 - p(\bar{x}_n(t)))(f(x+1) - f(x)) + p(\bar{x}_n(t))(f(\gamma x) - f(x)). \tag{3.3}$$

After $n \to \infty$, we want to denote

$$\mathcal{L}_t f(x) = (1 - p(\bar{x}(t)))(f(x+1) - f(x)) + p(\bar{x}(t))(f(\gamma x) - f(x))\,. \tag{3.4}$$

The differential formulas applied to the test function $\langle \mu_n(t, dx), \phi(x) \rangle$ show that

$$M_n(\phi, t) = \langle \mu_n(t, dx), \phi(x) \rangle - \langle \mu_n(0, dx), \phi(x) \rangle - \int_0^t \langle \mu_n(s, dx), \mathcal{L}_{s,n} \phi(x) \rangle ds \tag{3.5}$$

is a martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ of the process.

According to Theorem 2.3, in the limit

$$0 = \langle \mu(t, dx), \phi(x) \rangle - \langle \mu(0, dx), \phi(x) \rangle - \int_0^t \langle \mu(s, dx), \mathcal{L}_s \phi(x) \rangle ds\,. \tag{3.6}$$

Subtracting and multiplying by the Central Limit Theorem scaling factor $\sqrt{n}$ we obtain

$$\begin{aligned}\sqrt{n}M_n(\phi,t) &= \langle \xi_n(t,dx),\phi(x)\rangle - \langle \xi_n(0,dx),\phi(x)\rangle \\ \text{Part I} \qquad & - \int_0^t \langle \xi_n(s,dx), \mathcal{L}_s\phi(x)\rangle ds \\ \text{Part II} \qquad & - \int_0^t \langle \mu_n(s,dx),\ \sqrt{n}(\mathcal{L}_{s,n}\phi(x) - \mathcal{L}_s\phi(x))\rangle ds\,.\end{aligned} \tag{3.7}$$

Throughout this paper we use the notations

$$Z_n(\phi,t) = \langle \xi_n(t,dx),\phi(x)\rangle = \sqrt{n}[\bar{x}_n(\phi,t) - \bar{x}(\phi,t)]. \tag{3.8}$$

Since in the expression of $Z_n(\phi,t) = \langle \sqrt{n}(\mu_n(t,dx) - \mu(t,dx)),\phi(x)\rangle$ the element $\mu(t,dx)$ is deterministic, the randomness is only shown in the martingale part that is equal to $\sqrt{n}M_n(\phi,t)$.

For any $s \geq 0$, we define the mapping between test functions

$$\phi \to \mathcal{G}_s\phi\,, \quad \text{where} \quad \mathcal{G}_s\phi(x) = x\left[p'(\bar{x}(s))\langle \mu(s,dx), -\phi(x+1) + \phi(\gamma x)\rangle\right]. \tag{3.9}$$

Finally denote

$$\phi \to \mathcal{L}_s^\circ\phi\,, \quad \text{where} \quad \mathcal{L}_s^\circ\phi(x) = (\mathcal{L}_s + \mathcal{G}_s)\phi(x)\,. \tag{3.10}$$

Part I in (3.7) corresponds to $\mathcal{L}_s$ and Part II corresponds to $\mathcal{G}_s$. The second term is discussed below.

Interpreting (3.7) we see that if the martingales converge in distribution to a Gaussian random field $W(\phi,t) = \lim_{n\to\infty} \sqrt{n}M_n(\phi,t)$, then the right hand side is simply the Ito formula for a process $\xi = \lim_{n\to\infty} \xi_n$. Putting everything together, formally, the limit ξ should satisfy

$$W(\phi, t) = \langle \xi(t, dx), \phi(x)\rangle - \langle \xi(0, dx), \phi(x)\rangle - \int_0^t \langle \xi(s, dx), \mathcal{L}^{\circ}_{s,n}\phi(x)\rangle ds\,. \tag{3.11}$$

3.4 The non-equilibrium component

There are two integral terms in (3.7). The first contains $\mathcal{L}_s\phi(x)$ and can be thought of as the *equilibrium* part of the equation. The second is characteristic to *non-equilibrium*.

In Chapters 4 and 5 we show that for ϕ a polynomial, as $n \to +\infty$, $(Z_n(\phi, t))_{t\geq 0}$ converges in distribution to a diffusion. Let $Z_n(t)$, respectively $Z(t)$ be the fluctuation field before and after scaling corresponding to $\phi(x) = x$, i.e.

$$Z(t) = \langle \xi(t, dx), x\rangle = \lim_{n\to\infty} \langle \xi_n(t, dx), x\rangle = \lim_{n\to\infty} \sqrt{n}(\bar{x}_n(t) - \bar{x}(t))\,.$$

As $n \to +\infty$, we will show that, in distribution,

$$\lim_{n\to\infty} \int_0^t \langle \mu_n(s, dx),\ \sqrt{n}(\mathcal{L}_{s,n}\phi - \mathcal{L}_s\phi)\rangle ds = \int_0^t \langle \xi(s, dx), \mathcal{G}_s\phi\rangle\, ds\,.$$

To see that, consider

$$\sqrt{n} \int_0^t \langle \mu_n(s, dx), \mathcal{L}_{s,n}\phi - \mathcal{L}_s\phi\rangle ds,$$

which is given as

$$\begin{aligned} &\sqrt{n}\langle \mu_n(s, dx), \mathcal{L}_{s,n}\phi - \mathcal{L}_s\phi\rangle = \\ &\quad \sqrt{n}(p(\bar{x}_n(s)) - p(\bar{x}(s))) \cdot [-\frac{1}{n}\sum_{i=1}^n \phi(x_i(s) + 1) + \frac{1}{n}\sum_{i=1}^n \phi(\gamma x_i(s))]\,. \end{aligned} \tag{3.12}$$

We have that

$$\sqrt{n}(p(\bar{x}_n(s)) - p(\bar{x}(s))) = \quad p'(\bar{x}(s))Z_n(s) + o(\frac{1}{\sqrt{n}}) \qquad \text{from (4.42)}\,. \tag{3.13}$$

and as we shall show later in (4.45), we have that as $n \to +\infty$, (3.12) converges in distribution to

$$\int_0^t Z(s)p'(\bar{x}(s))\langle\mu(s,dx), -\phi(x+1) + \phi(\gamma x)\rangle ds = \int_0^t \langle\xi(s,dx), \mathcal{G}_s\phi\rangle\, ds\,.$$

3.5 The Gaussian random field

The notion of Gaussian random filed is briefly introduced in the Appendix, Section 7.1. Technically we work in finite setting, as the main theroem is proved for monomial, and then polynomials of degrees $k \in \mathbb{N}$. Denote $\mathbb{R}[x]$ the ring of polynomials with real coefficients

$$\mathbb{R}[x] = \left\{\phi(x) \,\middle|\, \phi(x) = \sum_{j=0}^{k} a_j x^j \,:\, a_j \in \mathbb{R},\ k \in \mathbb{N}\right\}.$$

Recall that

$$Z_n(\phi, t) = \langle\xi_n(t, dy), \phi\rangle \quad \text{and} \quad Z_n(k, t) = Z_n(x^k, t)\,.$$

Define the quadratic form

$$\mathcal{D}_s\phi = \mathcal{L}_s\phi^2 - 2\phi\mathcal{L}_s\phi\,. \tag{3.14}$$

Definition 3.1. *Define the centered (mean zero) Gaussian random field* $(\phi, t) \to W(\phi, t)$*,*

$t \geq 0$, $\phi \in \mathbb{R}[x]$ with covariance

$$\langle W(\phi_1, t_1), W(\phi_2, t_2) \rangle = \frac{1}{4} \int_0^{t_1 \wedge t_2} \langle \mu(s, dx), \mathcal{D}_s(\phi_1 + \phi_2) - \mathcal{D}_s(\phi_1 - \phi_2) \rangle ds \, . \tag{3.15}$$

Proposition 3.2. *In here, there exists an extension $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ of the filtration of the process $(\mathcal{F}_t)_{t \geq 0}$ from Definition 1.1 such that $W(\phi, t)$, $t \geq 0$ are continuous, square integrable martingales. In particular*

$$Cov(W(\phi, t_1), W(\phi, t_2)) = \int_0^{t_1 \wedge t_2} \langle \mu(s, dx), \mathcal{D}_s \phi \rangle ds \, . \tag{3.16}$$

Proof. The doubly indexed Gaussian random field is well defined as long as the covariance is bilinear and positive definite. Bilinearity is a consequence of the polarization formula for $\mathcal{D}_s$. To show it is positive definite, we set $\phi_1 = \phi_2 = \phi$ in Definition 3.1 and obtain (3.16). For $t_1 = t_2 = t$, and based on Definition (3.14) we see that

$$\mathcal{D}_s \phi(x) = (1 - p(\bar{x}(s)))(\phi^2(x+1) - \phi^2(x)) + p(\bar{x}(s))(\phi^2(\gamma x) - \phi^2(x))$$

$$-2\phi(x)[(1 - p(\bar{x}(s)))(\phi(x+1) - \phi(x)) + p(\bar{x}(s))(\phi(\gamma x) - \phi(x))]$$

$$= (1 - p(\bar{x}(s)))(\phi(x+1) - \phi(x))^2 + p(\bar{x}(s))(\phi(\gamma x) - \phi(x))^2 \geq 0 \, .$$

□

Theorem 3.4 is our main result and we will devote our next several chapters proving this result step by step, and we state here. We recall Assumption 2.2 on the existence of the initial measure

$$\mu_n(0, dy) \;\rightarrow\; \mu(0, dy) \in M_1((0, \infty)) \, .$$

We need a similar assumption on the initial fluctuation variable.

Assumption 3.3. *At $t = 0$, for every test function ϕ, as $n \to +\infty$,*

$$Z_n(\phi, 0) = \sqrt{n}[\langle \mu_n(0, dy), \phi \rangle - \langle \mu(0, dy), \phi \rangle]$$

converges in distribution to $Z(\phi, 0)$, where $Z(\phi, 0)$ is normally distributed.

Theorem 3.4. *Under the assumptions 1.2, 2.1 and 2.2 of Theorem 2.3, together with assumption 3.3, for any polynomial $\phi(x)$, the fluctuation process, indexed by ϕ*

$$Z_n(\phi, t) = \langle \xi_n(t, dx), \phi \rangle \,, \qquad t \geq 0$$

converges in distribution, as $n \to \infty$, to a diffusion process $Z(\phi, t)$ satisfying

$$Z(\phi, t) - Z(\phi, 0) - \int_0^t \langle \xi(s, dx), \mathcal{L}_s^\circ \phi \rangle ds = W(\phi, t) \,, \tag{3.17}$$

where $\mathcal{L}_s^\circ$ is defined in (3.10) and the martingale $(W(\phi, t))$ is the centered Gaussian random field defined in (3.15).

The next result is *the sub-diagonal lemma*. While simply based on an observation, it is essential because it allows the sub-diagonality of the infinite system $Z(\phi, \cdot)$ for $\phi(x) = 1, x^1, x^2, \ldots$ (monomials), and in general to prove that the joint process of the first k monomial is a linear diffusion with explicit form of the coefficients (Theorem 6.3), $k \in \mathbb{N}$.

Lemma 3.5. *For a fixed non-constant $\phi \in \mathbb{R}[x]$,*

$$deg(\mathcal{L}_s^\circ \phi) \leq deg(\phi) \,.$$

Proof. In (3.4) and (3.10), the coefficients (jump probabilities) are functions of t only. In x, the expressions $\phi(\gamma x) - \phi(x)$ (for the first) and $\phi(\gamma x) - \phi(x+1)$ (for the second) have the same degree as ϕ. $\square$

Remark. Any subsystem of polynomials $\phi_k(x)$, $k \in \mathbb{N}$, with $deg(\phi_k(x)) = k$, will be *hierarchical*, in the sense that $Z(\phi_k, t)$ is a fully determined Itô process.

To illustrate the hierarchy, start writing the martingales $M_n(k, t)$ explicitly. In case $\phi(x) = x$,

$$M_n(t) = \bar{x}_n(t) - \bar{x}_n(0) - \int_0^t (1 - p(\bar{x}_n(s))) - (1-\gamma)p(\bar{x}_n(s))\bar{x}_n(s)ds\,. \tag{3.18}$$

In case $\phi(x) = x^2$,

$$\langle \mu_n(t, dx), \phi_2(x) \rangle = \frac{1}{n}\sum_{i=1}^{n} x_i^2(t) = \bar{x}_n(2, t) \tag{3.19}$$

and the corresponding martingale is

$$M_n(2, t) = \bar{x}_n(2, t) - \bar{x}_n(2, 0) - \int_0^t (1 - p(\bar{x}_n(s)))(2\bar{x}_n(s) + 1) - (1-\gamma^2)p(\bar{x}_n(s))\bar{x}_n(2, s)ds \tag{3.20}$$

and continue with $\phi_k(x) = x^k$, $k = 3, 4, \ldots$ thereafter.

These considerations are contained in the following theorem.

Theorem 3.6. *Under the assumptions 1.2, 2.1 and 2.2 of Theorem 2.3, together with assumption 3.3, let $k \geq 1$ and $Z_n(j, t) = Z_n(x^j, t)$, i.e. the fluctuation process applied to the test function $\phi(x) = x^j$, for all $1 \leq j \leq k$. Then, the joint k - dimensional process converges in distribution to a linear diffusion with explicit coefficients obtained from (3.17) when applied to the polynomials x^j, $1 \leq j \leq k$. As a consequence, the fluctuation random field indexed by polynomials is jointly Gaussian.*

In the next several chapters, we will complete the proof step by step.

3.6 Martingales and the Gaussian random field

The first important thing is that in the limit $\sqrt{n}M_n(\phi, t)$ converges in distribution to a Brownian motion with time dependent variance. The second important feature is that for polynomials the formula is hierarchic, meaning that the terms appearing in the drift of $Z_n(\phi, t)$ have degrees less or equal than $deg(\phi)$. This allows us to obtain an explicit formula for polynomials, given in Theorem 6.3.

First we look at the quadratic variation of the martingale.

Proposition 3.7. *From (1.8), the quadratic variation of $M_n(\phi, t)$ is given by*

$$\langle M_n(\phi, t)\rangle = \frac{1}{n}\int_0^t \langle \mu_n(s, dx), \mathcal{D}_{s,n}\phi\rangle ds\,, \quad \mathcal{D}_{s,n}\phi = \mathcal{L}_{s,n}\phi^2 - 2\phi\mathcal{L}_{s,n}\phi\,. \tag{3.21}$$

It follows that

$$\lim_{n\to\infty}\langle \sqrt{n}M_n(\phi, t)\rangle = \lim_{n\to\infty}\int_0^t \langle \mu_n(s, dx), \mathcal{D}_s\phi\rangle ds\,, \quad \mathcal{D}_s\phi = \mathcal{L}_s\phi^2 - 2\phi\mathcal{L}_s\phi\,. \tag{3.22}$$

Proof. The quadratic variation formula (1.7) is given in general form. When applied to the generator (3.3) we see that by collecting all jump terms we obtain exactly the expression (3.21). It is integrated against the empirical measure $\mu_n(s, dx)$ because the same jump occurs for all particles i and is summed up.

For $Z_n(\phi, t) = \langle \xi_n(t, dx), \phi(x)\rangle$, which has already been emphasized as our central topic, since

$$\langle \mu_n(t,dx),\phi\rangle = \langle \mu_n(0,dx),\phi\rangle + \int_0^t \langle \mu_n(s,dx),\mathcal{L}_{s,n}\phi\rangle ds + M_n(\phi,t) \tag{3.23}$$

$$\langle \mu(t,dx),\phi\rangle = \langle \mu(0,dx),\phi\rangle + \int_0^t \langle \mu(s,dx),\mathcal{L}_{s}\phi\rangle ds\,, \tag{3.24}$$

then

$$\begin{aligned}\sqrt{n}M_n(\phi,t) = \langle \xi_n(t,dx),\phi\rangle - \langle \xi_n(0,dx),\phi\rangle - \int_0^t \langle \xi_n(s,dx),\mathcal{L}_s\phi\rangle ds \\ - \sqrt{n}\int_0^t \langle \mu_n(s,dx),\mathcal{L}_{s,n}\phi - \mathcal{L}_s\phi\rangle ds\end{aligned} \tag{3.25}$$

is a martingale with quadratic variation

$$\begin{aligned}\langle \sqrt{n}M_n(\phi,t)\rangle = \frac{1}{n}\sum_{i=1}^{n}\int_0^t (1-p(\bar{x}_n(s)))(\phi(x_i(s)+1)-\phi(x_i(s)))^2 \\ +p(\bar{x}_n(s))(\phi(\gamma x_i(s))-\phi(x_i(s)))^2 ds\,.\end{aligned} \tag{3.26}$$

□

Chapter 4

Fluctuation limit, $\phi(x) = x^k$, $k = 1$, verification step

After the preliminary work on the scaling limit and formulating the problem and explaining the outline of the proof, we begin the proof on the fluctuation limit with the case of $\phi(x) = x$, when the power of the monomial is $k = 1$, the simplest case. Of course, the trivial $k = 0$ gives null fluctuations.

In the central limit theorem, if we have a sequence of independent identical distributed random variables, denoted as $\{x_n\}$, with finite variance σ^2 and expected value μ for each x_i, once we denote $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$, then $\sqrt{n}(\bar{x}_n - \mu)$ converges in distribution to a normal $N(0, \sigma^2)$.

In Chapter 2, we have already defined by eq. (2.27) a sequence of stochastic processes $Z_n(\phi, t)$, ϕ a test function in the space $C_q^2((0, \infty))$ defined in Definition 1.4. In this chapter, $\phi(x) = x^k$, with $k = 1$ for the verification step.

Recall that the fluctuation field is $\langle \xi_n(t, dx), \phi \rangle$. For $\phi(x) = x^k$ it is denoted by $Z_n(k, t)$.

In the following we simplify the notation by writing

$$Z_n(1,t) := Z_n(t)\,.$$

Step by step, we first prove that $Z_n(t)$ is tight. By the tightness of $Z_n(t)$, we denote the limit point of $Z_n(t)$ as $Z(t)$, we then prove that $Z(t)$ is a diffusion process and the solution to a stochastic differential equation. The property that $Z(t)$ is a Gaussian process will be proved at the end of this chapter.

4.1 Tightness of $Z_n(t)$

Tightness, in fact C-tightness, proves that a sequence of probability measures (or laws of processes) on the Skorohod space of right-continuous with left-limits paths is pre-compact in the $J-1$ metric. Moreover, the criterion we give in Proposition 7.8 (Appendix) guarantees that any limit point is concentrated on the class of continuous paths.

We need to apply Gronwall's inequality in Proposition 4.3. The lemma is stated in many textbooks, see [26] (Exercise 5.17) assuming continuity of the function under consideration. In the following lemma we generalize to cover the case when the function is only bounded (the same proof can be used for a locally integrable function). In all our applications the function is the expected value of a function of a RCLL (Skorohod space) path.

Lemma 4.1 (Gronwall's Inequality)**.** *If C_1, C_2 are positive constants and $y(t)$ is a nonnegative function.*

(i) If $y(t)$ is continuous, then from

$$y(t) \leq C_1 + C_2 \int_0^t y(s)ds \quad for \quad 0 \leq t \leq T \tag{4.1}$$

it follows that

$$y(t) \leq C_1 \exp(C_2 t) \quad for \quad 0 \leq t \leq T. \tag{4.2}$$

(ii) Assume $y(t)$ is bounded and inequality (4.1) is satisfied. Then there exists a constant $C(T)$ depending on T only such that

$$y(t) \leq C(T) \exp(C_2 t) \quad for \quad 0 \leq t \leq T. \tag{4.3}$$

Proof. Part (i):

Denote $F(t) = \int_0^t y(s)ds$, our (4.1) is

$$F'(t) \leq C_2(F(t) + \frac{C_1}{C_2}) \tag{4.4}$$

Consider $F_1(t) = (F(t) + \frac{C_1}{C_2})e^{-C_2 t}$, by the above inequality, $F_1'(t) \leq 0$, so

$$(F(t) + \frac{C_1}{C_2})e^{-C_2 t} = F_1(t) \leq F_1(0) = \frac{C_1}{C_2},$$

and $y(t) \leq C_2(F(t) + \frac{C_1}{C_2}) \leq C_1 \exp(C_2 t)$, for $0 \leq t \leq T$.

Part (ii):

Denote $G(t) = \int_0^t y(s)ds$, then $G(t)$ is continuous. By (4.1), we have

$$\int_0^t y(s)ds \leq C_1 t + C_2 \int_0^t G(s)ds \leq C_1 T + C_2 \int_0^t G(s)ds \tag{4.5}$$

By the continuity property of $G(t)$ and (i), we deduce that

$$G(t) \leq C_1 T \exp(C_2 t).$$

It follows that

$$y(t) \leq C_1(1 + C_2 T \exp(C_2 t)) \tag{4.6}$$

Choose $C(T) = C_1(1 + C_2 T)$, by (4.6), there exists a constant $C(T)$ depending on T only such that

$$y(t) \leq C(T) \exp(C_2 t) \quad \text{for} \quad 0 \leq t \leq T. \tag{4.7}$$

□

We also need another lemma in the proof of Proposition 4.3, in here, we present without proof, for a detailed explanation, please refer to [4].

Lemma 4.2. *Due to the boundedness of all moments, $\{M_n(t)\}$ is a sequence of square integrable $\mathcal{F}_t$-martingales. Then, by Doob-Meyer's decomposition theorem,*

$$\forall n \geq 1 \qquad [M_n(t)]^2 - \langle M_n(t) \rangle, \quad \textit{respectively} \quad [\sqrt{n} M_n(t)]^2 - n \langle M_n(t) \rangle$$

are martingales.

Proposition 4.3. *For the sequence of stochastic processes defined as $Z_n(t) = \langle \xi_n(t, dx), x \rangle$, $\{Z_n(t)\}$ is tight.*

Proof. Now let us consider $Z_n(t) = \langle \xi_n(t, dx), x \rangle = \sqrt{n}[\bar{x}_n(t) - \bar{x}(t)]$. There exists $T > 0$, such that $0 \leq t \leq T$. By (3.18),

$$\bar{x}_n(t) = \bar{x}_n(0) + \int_0^t (1 - p(\bar{x}_n(s))) - (1 - \gamma) p(\bar{x}_n(s)) \bar{x}_n(s) ds + M_n(t) \tag{4.8}$$

with $\langle M_n(t)\rangle$ vanishes uniformly in t as $n \to +\infty$, so we also have that

$$\bar{x}(t) = \bar{x}(0) + \int_0^t (1 - p(\bar{x}(s))) - (1-\gamma)p(\bar{x}(s))\bar{x}(s)ds \tag{4.9}$$

so we have

$$\begin{aligned} Z_n(t) = Z_n(0) + \int_0^t -\sqrt{n}(p(\bar{x}_n(s)) - p(\bar{x}(s))) - (1-\gamma)\sqrt{n}[p(\bar{x}_n(s))(\bar{x}_n(s) - \bar{x}(s)) \\ +(p(\bar{x}_n(s)) - p(\bar{x}(s)))\bar{x}(s)]ds + \sqrt{n}M_n(t) \end{aligned} \tag{4.10}$$

For the term $p(\bar{x}_n(s)) - p(\bar{x}(s))$, by the mean value theorem, there exists $v_n(s)$, with $\{v_n(s)\}$ a sequence of stochastic processes, such that for every n, $p(\bar{x}_n(s)) - p(\bar{x}(s)) = p'(v_n(s))(\bar{x}_n(s) - \bar{x}(s))$.

In the following discussion, when we come across such terms, we will adopt same notations.

Denote $q_n(v,s) = -p'(v_n(s)) - (1-\gamma)[p(\bar{x}_n(s)) + p'(v_n(s))\bar{x}(s)]$.

By (4.5), $Z_n(t)$ itself is (4.6) below.

$$Z_n(t) = Z_n(0) + \int_0^t q_n(v,s)Z_n(s)ds + \sqrt{n}M_n(t) \tag{4.11}$$

$$\begin{aligned} \langle \sqrt{n}M_n(t)\rangle &= \int_0^t (1 - p(\bar{x}_n(s))) \sum_{i=1}^{n} [\sqrt{n}\frac{1}{n}((x_i(s)+1) - x_i(s))]^2 \\ &\quad + p(\bar{x}_n(s)) \sum_{i=1}^{n} [\sqrt{n}\frac{1}{n}((\gamma x_i(s)) - x_i(s))]^2 ds \\ &= \int_0^t (1 - p(\bar{x}_n(s))) + (1-\gamma)^2 p(\bar{x}_n(s))\bar{x}_n(2,s)ds \end{aligned} \tag{4.12}$$

By (4.7), we have

$$\mathrm{E}[\sup_{0\le t\le T}\langle\sqrt{n}M_n(t)\rangle] \le T(1+\mathrm{E}[\sup_{0\le t\le T}\bar{x}_n(2,t)]) \tag{4.13}$$

$\mathrm{E}[\sup_{0\le t\le T}\bar{x}_n(2,t)]$ is bounded, thus we know that $\mathrm{E}[\sup_{0\le t\le T}\langle\sqrt{n}M_n(t)\rangle]$ is bounded.

Now let us show the tightness of $\{Z_n(t)\}$ in two parts. If our sequence of stochastic processes satisfy two conditions stated in Definition 7.8, which we will prove below, then this sequence of stochastic processes is tight.

Part 1. Based on our Assumption 1.2 and Lemma 2.6, there exists $M_0 > 0$, such that $|q_n(v,s)| \le M_0$. We have

$$\left|Z_n(t)\right|^2 \le 3\left|Z_n(0)\right|^2 + 3\left|\int_0^t \left|q_n(v,s)\right|\left|Z_n(s)\right|ds\right|^2 + 3\left|\sqrt{n}M_n(t)\right|^2 \tag{4.14}$$

By Cauchy-Schwartz inequality,

$$\left|\int_0^t \left|q_n(v,s)\right|\left|Z_n(s)\right|ds\right|^2 \le \int_0^t \left|q_n(v,s)\right|^2 ds \int_0^t \left|Z_n(s)\right|^2 ds \tag{4.15}$$

Thus we have that

$$\left|Z_n(t)\right|^2 \le 3\left|Z_n(0)\right|^2 + 3\int_0^t \left|q_n(v,s)\right|^2 ds \int_0^t \left|Z_n(s)\right|^2 ds + 3\left|\sqrt{n}M_n(t)\right|^2 \tag{4.16}$$

Combining with the fact that $|q_n(v,s)| \le M_0$ implies

$$|Z_n(t)|^2 \le 3|Z_n(0)|^2 + 3M_0^2T\int_0^t |Z_n(s)|^2\, ds + 3|\sqrt{n}M_n(t)|^2 \tag{4.17}$$

Denote $A_n(\omega) = 3\left|Z_n(0)\right|^2 + 3\sup_{0\le t\le T}\left|\sqrt{n}M_n(t)\right|^2$, by (4.12), for all $0 \le t' \le t$,

$$\left|Z_n(t')\right|^2 \le A_n(\omega) + 3M_0^2T\int_0^{t'}\left|Z_n(s)\right|^2 ds \tag{4.18}$$

$$\left|Z_n(t')\right|^2 \le A_n(\omega) + 3M_0^2T\int_0^{t}\sup_{0\le s'\le s}\left|Z_n(s')\right|^2 ds \tag{4.19}$$

Then

$$\sup_{0\le t'\le t}\left|Z_n(t')\right|^2 \le A_n(\omega) + 3M_0^2T\int_0^{t}\sup_{0\le s'\le s}\left|Z_n(s')\right|^2 ds \tag{4.20}$$

Denote $U_n(s) = \sup_{0\le s'\le s} \mid Z_n(s') \mid^2$. We point out that $\mathrm{E}[U_n(s)]$, $0 \le s \le T$ is bounded. To see this we notice that all particles $x_i(s)$ are bounded above by n independently identically distributed Poisson processes $X_i(t)$ advancing at any Poisson clock by exactly one unit. This shows that in the formula of $U_n(s)$ we only have to write upper bounds for suprema of the form

$$\mathrm{E}[\sup_{0\le s'\le s}|x_i(s')|^k] \le \mathrm{E}[|X_i(s)|^k] < \infty\,, \qquad \text{for } k \ge 2\,. \tag{4.21}$$

Then we write the inequality after taking the expected value on both sides of (4.20)

$$\mathrm{E}[U_n(t)] \le \mathrm{E}[A_n(\omega)] + 3M_0^2T\int_0^t \mathrm{E}[U_n(s)]ds\,. \tag{4.22}$$

We have already shown that

$$\mathrm{E}[\sup_{0\le t\le T}\langle\sqrt{n}M_n(t)\rangle] = \mathrm{E}[\langle\sqrt{n}M_n(T)\rangle]$$

is bounded uniformly in $n \ge 1$.

So there exists $C_1' > 0$, independent of n such that

$$\mathrm{E}[\sup_{0\le t\le T} \left| \sqrt{n} M_n(t) \right|^2] \le C_1' < \infty\,,$$

therefore there exists $C_1 > 0$ independent of n such that $\mathrm{E}[A_n(\omega)] \le C_1$. By Lemma 4.1,

$$\mathrm{E}[U_n(t)] \le C(T) \exp(3M_0^2 T t) \tag{4.23}$$

Since $0 \le t \le T$, $\mathrm{E}[\sup_{0\le t\le T} \left| Z_n(t) \right|^2]$ is bounded, thus Chebyshev's inequality implies

$$\lim_{K\to+\infty} \limsup_{n\to+\infty} P(\sup_{0\le t\le T} \left| Z_n(t) \right| > K) = 0\,, \tag{4.24}$$

which completes the proof of the part 1.

Part 2. Also consider $\left| Z_n(t) - Z_n(r) \right|^2$, with $0 \le r < t \le T$.

$$\left| Z_n(t) - Z_n(r) \right|^2 \le 2 \int_r^t \left| q_n(v,s) \right|^2 ds \int_r^t \left| Z_n(s) \right|^2 ds + 2 \left| \sqrt{n}(M_n(t) - M_n(r)) \right|^2 \tag{4.25}$$

$$\left| Z_n(t) - Z_n(r) \right|^2 \le 2M_0^2 (t-r) \int_r^t \left| Z_n(s) \right|^2 ds + 2 \left| \sqrt{n}(M_n(t) - M_n(r)) \right|^2 \tag{4.26}$$

$$\begin{aligned} \sup_{0\le r<t\le T, 0<t-r<\delta} \left| Z_n(t) - Z_n(r) \right|^2 &\le 2M_0^2 \delta \int_r^t \left| Z_n(s) \right|^2 ds \\ &+2 \sup_{0\le r<t\le T, 0<t-r<\delta} \left| \sqrt{n}(M_n(t) - M_n(r)) \right|^2 \end{aligned} \tag{4.27}$$

Consider $\mathrm{E}[\sup_{0\le r<t\le T, 0<t-r<\delta} \left| \sqrt{n}(M_n(t) - M_n(r)) \right|^2]$, which we have that

$$\mathrm{E}[\sup_{0\le r<t\le T, 0<t-r<\delta} \left| \sqrt{n}(M_n(t) - M_n(r)) \right|^2] \le \delta(1 + \mathrm{E}[\sup_{0\le t\le T} \bar{x}_n(2,t)]) \tag{4.28}$$

Combining (4.21) and (4.22), we conclude that

$$\begin{aligned}\mathrm{E}[\sup_{0\le r<t\le T, 0<t-r<\delta} \left|Z_n(t) - Z_n(r)\right|^2] \le & 2\delta\{M_0^2 \mathrm{E}[\int_r^t \sup_{0\le s'\le s} \left|Z_n(s')\right|^2 ds] \\ & + 1 + \mathrm{E}[\sup_{0\le t\le T} \bar{x}_n(2,t)]\}\end{aligned} \tag{4.29}$$

By (4.23) and above statements, it can be verified that as $n \to +\infty$, for any $\epsilon > 0$, there exists $\delta' > 0$, such that if $0 < t - r < \delta'$, we have

$$\limsup_{n\to+\infty} \mathrm{E}[\sup_{0\le r<t\le T, 0<t-r<\delta'} \left|Z_n(t) - Z_n(r)\right|^2] \le \epsilon \tag{4.30}$$

and so we conclude that for any positive number $\epsilon_0 > 0$, we have

$$\lim_{\lambda\to 0} \limsup_{n\to+\infty} P(\sup_{0\le r<t\le T, 0<t-r<\lambda} \left|Z_n(t) - Z_n(r)\right| > \epsilon_0) = 0 \tag{4.31}$$

The sequence of stochastic processes $\{Z_n(t)\}_{n\ge 1}$ is tight. □

By Proposition 4.3, since $\{Z_n(t)\}$ is tight, just denote a limit point of $\{Z_n(t)\}$ as $Z(t)$.

In Theorem 4.4, we will show that a limit point $Z(t)$ is the solution to a stochastic differential equation and $Z(t)$ is a diffusion process.

4.2 Proof that $Z(t)$ is a diffusion process

For the convenience of our statement, we will adopt the notation that

$$\begin{aligned} q_n(s) &= -(p'(\bar{x}(s)) + (1-\gamma)[p(\bar{x}_n(s)) + \bar{x}(s)p'(\bar{x}(s))]) \\ r_n(s) &= (1 - p(\bar{x}_n(s))) + (1-\gamma)^2 p(\bar{x}_n(s))\bar{x}_n(2, s) \end{aligned} \tag{4.32}$$

Moreover,

$$q(s) = -\Big(p'(\bar{x}(s)) + (1-\gamma)\Big[p(\bar{x}(s)) + \bar{x}(s)p'(\bar{x}(s))\Big]\Big) \tag{4.33}$$
$$r(s) = (1 - p(\bar{x}(s))) + (1-\gamma)^2 p(\bar{x}(s))\bar{x}(2, s)\,.$$

In here, $q(s)$ to represent for the limit of $q_n(s)$ and $r(s)$ to represent for the limit of $r_n(s)$, both with respect to n. By (4.33), we conclude that $q(s) < 0$, and $r(s) > 0$.

Theorem 4.4. *Let $Z(t)$ be a limit point of the tight sequence of stochastic processes $\{Z_n(t)\}$, in other words there exists a subsequence converging in distribution to $Z(t)$. Then for any function $\varphi \in C^3(\mathbb{R})$, with $|\varphi'|$, $|\varphi''|$ and $|\varphi^{(3)}|$bounded,*

$$M(\varphi, Z(t), t) = \varphi(Z(t)) - \varphi(Z(0)) - \int_0^t [q(s)Z(s)\varphi'(Z(s)) + \frac{1}{2}r(s)\varphi''(Z(s))]ds\,, \tag{4.34}$$

where $M(\varphi, Z(t), t)$ is a $\mathcal{F}_t$- martingale.

Remark. The space or $C_c^\infty(\mathbb{R})$ with all bounded derivatives trivially satisfies this condition for the test function φ.

Definition 4.5. *By* (1.6) *and* (1.7)*, for the pure jump processes $\{Z_n(t)\}$, $0 \leq t \leq T$,*

$$\varphi(Z_n(t)) = \varphi(Z_n(0)) + \int_0^t -\sqrt{n}\varphi'(Z_n(s))\bar{x}'(s) + n\{(1 - p(\bar{x}_n(s)))[\varphi(Z_n(s) + \frac{1}{\sqrt{n}})$$
$$-\varphi(Z_n(s))] - p(\bar{x}_n(s))\varphi(Z_n(s))\} + p(\bar{x}_n(s))\sum_{i=1}^{n}\varphi(Z_n(s) - \frac{(1-\gamma)}{\sqrt{n}}x_i(s))ds + M_n(\varphi, Z(t), t) \tag{4.35}$$

This equation serves as definition of the martingales $M_n(\varphi, Z(t), t)$.

Proof. (Theorem 4.4)

By Taylor's formula with mean value form of remainder, there exists $\{\epsilon_i(n,s)\}$(possibly random), $0 \le i \le n$, with the property that $0 < \epsilon_i(n,s) < 1$, such that

$$\varphi(Z_n(s) + \frac{1}{\sqrt{n}}) - \varphi(Z_n(s)) = \frac{1}{\sqrt{n}}\varphi'(Z_n(s)) + \frac{1}{2n}\varphi''(Z_n(s)) + \frac{1}{6n^{\frac{3}{2}}}\varphi^{(3)}(Z_n(s) + \frac{\epsilon_0(n,s)}{\sqrt{n}}) \tag{4.36}$$

and

$$\begin{aligned}\varphi(Z_n(s) - \frac{(1-\gamma)}{\sqrt{n}}x_i(s)) - \varphi(Z_n(s)) = -\frac{(1-\gamma)}{\sqrt{n}}x_i(s)\varphi'(Z_n(s))\\ +\frac{(1-\gamma)^2}{2n}x_i^2(s)\varphi''(Z_n(s)) - \frac{(1-\gamma)^3}{6n^{\frac{3}{2}}}x_i^3(s)\varphi^{(3)}(Z_n(s) - \frac{(1-\gamma)\epsilon_i(n,s)}{\sqrt{n}}x_i(s))\end{aligned} \tag{4.37}$$

$$\bar{x}'(s) = (1 - p(\bar{x}(s))) - (1-\gamma)p(\bar{x}(s))\bar{x}(s) \tag{4.38}$$

So our (4.35) is

$$\begin{aligned}\varphi(Z_n(t)) = \varphi(Z_n(0)) + \int_0^t \sqrt{n}\{[(1 - p(\bar{x}_n(s))) - (1-\gamma)p(\bar{x}_n(s))\bar{x}_n(s)] - [(1 - p(\bar{x}(s)))\\ - (1-\gamma)p(\bar{x}(s))\bar{x}(s)]\}\varphi'(Z_n(s)) + \frac{1}{2}[(1 - p(\bar{x}_n(s)))\\ + (1-\gamma)^2 p(\bar{x}_n(s))\bar{x}_n(2,s)]\varphi''(Z_n(s))ds + R_n(\varphi,t) + M_n(\varphi, Z(t), t)\end{aligned} \tag{4.39}$$

In here,

$$\begin{aligned}R_n(\varphi,t) = \int_0^t \frac{1}{6\sqrt{n}}[(1 - p(\bar{x}_n(s)))\varphi^{(3)}(Z_n(s) + \frac{\epsilon_0(n,s)}{\sqrt{n}}) -\\ (1-\gamma)^3 p(\bar{x}_n(s))\frac{1}{n}\sum_{i=1}^{n} x_i^3(s)\varphi^{(3)}(Z_n(s) - \frac{(1-\gamma)\epsilon_i(n,s)}{\sqrt{n}}x_i(s))]ds\end{aligned} \tag{4.40}$$

with the existence of a positive constant $C_{R_n(\varphi,t)}$, such that

$$\mathrm{E}[\left|R_n(\varphi,t)\right|^2] \le \frac{C_{R_n(\varphi,t)}}{n} \tag{4.41}$$

As $n \to +\infty$, $\mathrm{E}[\left|R_n(\varphi, t)\right|^2]$ vanishes uniformly in t.

Also by applying Taylor's formula, there exists $0 \leq \eta_n(s) \leq 1$ such that

$$p(\bar{x}_n(s)) = p(\bar{x}(s)) + p'(\bar{x}(s))(\bar{x}_n(s) - \bar{x}(s)) + \frac{1}{2}p''(\bar{x}(s) + \eta_n(s)(\bar{x}_n(s) - \bar{x}(s)))(\bar{x}_n(s) - \bar{x}(s))^2 \tag{4.42}$$

and

$$\begin{aligned} &\sqrt{n}\{[(1 - p(\bar{x}_n(s))) - (1 - \gamma)p(\bar{x}_n(s))\bar{x}_n(s)] - [(1 - p(\bar{x}(s))) - (1 - \gamma)p(\bar{x}(s))\bar{x}(s)]\} = \\ &- p'(\bar{x}(s))Z_n(s) - (1 - \gamma)(p'(\bar{x}(s))\bar{x}(s) + p(\bar{x}_n(s)))Z_n(s) + R_n(2, s) \end{aligned} \tag{4.43}$$

In here

$$R_n(2, s) = -\frac{1}{2}(1 + (1 - \gamma)\bar{x}(s))p''(\bar{x}(s) + \eta_n(s)(\bar{x}_n(s) - \bar{x}(s)))(\bar{x}_n(s) - \bar{x}(s))Z_n(s)\,. \tag{4.44}$$

One step further, since $\bar{x}(s)$, $p''(\bar{x}(s)+\eta_n(s)(\bar{x}_n(s)-\bar{x}(s)))$ are bounded, there exists a constant $C_{R_n(2,s)}$, such that

$$\mathrm{E}[R_n(2, s)] \leq C_{R_n(2,s)}\mathrm{E}^{\frac{1}{2}}[|\bar{x}_n(s) - \bar{x}(s)|^2]\mathrm{E}^{\frac{1}{2}}[|Z_n(s)|^2]\,. \tag{4.45}$$

As we have already shown in Proposition 4.3, since $E[|Z_n(s)|^2]$ is bounded, as $n \to +\infty$, $\mathrm{E}[R_n(2, s)]$ vanishes uniformly in s.

We have

$$\begin{aligned} \varphi(Z_n(t)) = \varphi(Z_n(0)) &+ \int_0^t q_n(s)Z_n(s)\varphi'(Z_n(s)) + \frac{1}{2}r_n(s)\varphi''(Z_n(s))ds \\ &+ M_n(\varphi, Z(t), t) + \int_0^t R_n(2, s)\varphi'(Z_n(s))ds + R_n(\varphi, t) \end{aligned} \tag{4.46}$$

By (4.46), the sequence of martingales $\{M_n(\varphi, Z(t), t)\}$ is given by

$$\begin{aligned} M_n(\varphi, Z(t), t) = \varphi(Z_n(t)) - \varphi(Z_n(0)) - \int_0^t q_n(s) Z_n(s) \varphi'(Z_n(s)) + \frac{1}{2} r_n(s) \varphi''(Z_n(s)) ds \\ - \int_0^t R_n(2, s) \varphi'(Z_n(s)) ds - R_n(\varphi, t) \end{aligned} \tag{4.47}$$

Let's arbitrarily choose a bounded random variable $H(\omega) \in \mathcal{F}_{t'}$ and $0 \leq t' < t \leq T$. Define the functional

$$V(t) \mapsto [\varphi(V(t)) - \varphi(V(t')) - \int_{t'}^t q(s) V(s) \varphi'(V(s)) + \frac{1}{2} r(s) \varphi''(V(s)) ds] H(\omega) \tag{4.48}$$

Denote this functional as ρ, by the continuous property of φ, φ' and φ'', for every $V(t) \in \mathrm{D}([t', T], \mathbb{R})$, $\rho(V(t))$ is a continuous functional and we have that $\mathrm{E}[\rho(Z_n(t))]$ converges to $\mathrm{E}[\rho(Z(t))]$.

Just denote $f(n, \varphi, t, t') = \int_{t'}^t (q(s) - q_n(s)) Z_n(s) \varphi'(Z_n(s)) + \frac{1}{2}(r(s) - r_n(s)) \varphi''(Z_n(s)) - R_n(2, s) \varphi'(Z_n(s)) ds - (R_n(\varphi, t) - R_n(\varphi, t'))$, using a similar argument as we did to $\mathrm{E}[R_n(2, s)]$, we can show that $\mathrm{E}[f(n, \varphi, t, t')]$ vanishes as $n \to \infty$.

By (4.47), when $0 \leq t' \leq t \leq T$, for every n,

$$\begin{aligned} \mathrm{E}\{[M_n(\varphi, Z(t), t) - M_n(\varphi, Z(t'), t')] H(\omega)\} = \mathrm{E}\{[\varphi(Z_n(t)) - \varphi(Z_n(t')) - \\ \int_{t'}^t q_n(s) Z_n(s) \varphi'(Z_n(s)) + \frac{1}{2} r_n(s) \varphi''(Z_n(s)) ds - \int_{t'}^t R_n(2, s) \varphi'(Z_n(s)) ds - \\ (R_n(\varphi, t) - R_n(\varphi, t'))] H(\omega)\} = \mathrm{E}[\rho(Z_n(t))] + \mathrm{E}[f(n, \varphi, t, t') H(\omega)] = 0 \end{aligned} \tag{4.49}$$

Since $\lim_{n\to+\infty} \mathrm{E}[f(n,\varphi,t,t')H(\omega)] = 0$, as $n \to +\infty$, we have

$$\lim_{n\to+\infty} \mathrm{E}[\rho(Z_n(t))] = \mathrm{E}\{[\varphi(Z(t)) - \varphi(Z(t')) \\ - \int_{t'}^{t} q(s)Z(s)\varphi'(Z(s)) + \frac{1}{2}r(s)\varphi''(Z(s))ds]H(\omega)\} = 0\,. \tag{4.50}$$

We conclude that $\varphi(Z(t)) - \varphi(Z(0)) - \int_0^t q(s)Z(s)\varphi'(Z(s)) + \frac{1}{2}r(s)\varphi''(Z(s))ds$ is a $\mathcal{F}_t$-martingale. We just denote this martingale as $M(\varphi, Z(t), t)$.

Thus we have

$$\varphi(Z(t)) = \varphi(Z(0)) + \int_0^t q(s)Z(s)\varphi'(Z(s)) + \frac{1}{2}r(s)\varphi''(Z(s))ds + M(\varphi, Z(t), t) \tag{4.51}$$

When $\varphi(x) = x$, we have that

$$Z(t) = Z(0) + \int_0^t q(s)Z(s)ds + M(\varphi, Z(t), t)\,, \qquad \varphi(z) = z \tag{4.52}$$

This concludes the proof. □

4.3 $Z(t)$ is a one dimensional linear diffusion process and its explicit formula

In our next Theorem 4.6, we will show that $Z_n(t)$ converges in distribution to a one dimensional linear diffusion process and will give its explicit formula.

Theorem 4.6. *$Z_n(t)$ converges in distribution to a linear diffusion process $Z(t)$. $Z(t)$ have a* *same distribution to a solution of the below stochastic differential equation*

$$X(t) = X(0) + \int_0^t q(s)X(s)ds + \int_0^t r^{\frac{1}{2}}(s)dW(s), X(0) = Z(0) \tag{4.53}$$

with coefficients given in (4.33). $\{W(t)\}_{t\geq 0}$ *in here is a one-dimensional Brownian motion.*

Proof. By Theorem 4.4, we have already given the explicit formula for $\varphi(Z_n(t))$, which is given in (4.46). For the martingale part, which is $M_n(\varphi, Z(t), t)$, its quadratic variation is given as

$$\begin{aligned}\langle M_n(\varphi, Z(t), t)\rangle = \int_0^t &\{n(1 - p(\bar{x}_n(s)))[\varphi(Z_n(s) + \frac{1}{\sqrt{n}}) - \varphi(Z_n(s))]^2 \\ &+ p(\bar{x}_n(s)) \sum_{i=1}^n [\varphi(Z_n(s) - \frac{(1-\gamma)}{\sqrt{n}} x_i(s)) - \varphi(Z_n(s))]^2\} ds \qquad (4.54) \\ = \int_0^t [(1 - p(\bar{x}_n(s))) &+ (1-\gamma)^2 p(\bar{x}_n(s))\bar{x}_n(2, s)](\varphi'(Z_n(s)))^2 ds + R_n(3, \varphi, t)\end{aligned}$$

with

$$
\begin{aligned}
R_n(3,\varphi,t) = \int_0^t \frac{1}{4n}(\varphi''(Z_n(s)))^2[(1-p(\bar{x}_n(s))) + (1-\gamma)^4 p(\bar{x}_n(s))\bar{x}_n(4,s)] \\
+\frac{1}{36n^2}[(1-p(\bar{x}_n(s)))(\varphi^{(3)}(Z_n(s)+\frac{\epsilon_0(n,s)}{\sqrt{n}}))^2+ \\
(1-\gamma)^6 p(\bar{x}_n(s))\frac{1}{n}\sum_{i=1}^{n} x_i^6(s)(\varphi^{(3)}(Z_n(s)-\frac{(1-\gamma)\epsilon_i(n,s)}{\sqrt{n}}x_i(s)))^2]+ \\
\frac{1}{\sqrt{n}}\varphi'(Z_n(s))\varphi''(Z_n(s))[(1-p(\bar{x}_n(s))) - (1-\gamma)^3 p(\bar{x}_n(s))\bar{x}_n(3,s)]+ \\
\frac{1}{3n}\varphi'(Z_n(s))[(1-p(\bar{x}_n(s)))\varphi^{(3)}(Z_n(s)+\frac{\epsilon_0(n,s)}{\sqrt{n}})+ \\
(1-\gamma)^4 p(\bar{x}_n(s))\frac{1}{n}\sum_{i=1}^{n} x_i^4(s)\varphi^{(3)}(Z_n(s)-\frac{(1-\gamma)\epsilon_i(n,s)}{\sqrt{n}}x_i(s))]+ \\
\frac{1}{6n^{\frac{3}{2}}}\varphi''(Z_n(s))[(1-p(\bar{x}_n(s)))\varphi^{(3)}(Z_n(s)+\frac{\epsilon_0(n,s)}{\sqrt{n}})- \\
(1-\gamma)^5 p(\bar{x}_n(s))\frac{1}{n}\sum_{i=1}^{n} x_i^5(s)\varphi^{(3)}(Z_n(s)-\frac{(1-\gamma)\epsilon_i(n,s)}{\sqrt{n}}x_i(s))]ds
\end{aligned}
\tag{4.55}
$$

and there exists a positive constant $C_{R_n(3,\varphi,t)}$, such that

$$
\mathrm{E}[\left|R_n(3,\varphi,t)\right|^2] \le \frac{C_{R_n(3,\varphi,t)}}{n} \tag{4.56}
$$

As $n \to +\infty$, $\mathrm{E}[\left|R_n(3,\varphi,t)\right|^2]$ vanishes uniformly in t.

Which gives $M_n(\varphi, Z(t), t)$ converges to $M(\varphi, Z(t), t)$ as $n \to +\infty$ with

$$
\langle M(\varphi, Z(t), t)\rangle = \int_0^t [(1-p(\bar{x}(s))) + (1-\gamma)^2 p(\bar{x}(s))\bar{x}(2,s)](\varphi'(Z(s)))^2 ds \tag{4.57}
$$

As $n \to +\infty$, taking limits with respect to n on both sides of (4.46), we have

$$
\varphi(Z(t)) = \varphi(Z(0)) + \int_0^t q(s)Z(s)\varphi'(Z(s)) + \frac{1}{2}r(s)\varphi''(Z(s))ds + M(\varphi, Z(t), t) \tag{4.58}
$$

and by (4.57), there exists a one dimensional Brownian motion $\{W(t)\}_{t\geq 0}$, such that

$$\varphi(Z(t)) = \varphi(Z(0)) + \int_0^t q(s)Z(s)\varphi'(Z(s)) + \frac{1}{2}r(s)\varphi''(Z(s))ds + \int_0^t r^{\frac{1}{2}}(s)\varphi'(Z(s))dW(s) \tag{4.59}$$

choose $\varphi(x) = x$, $Z(t)$ is an Ito process and (4.59) is

$$\begin{aligned} Z(t) = Z(0) + \int_0^t \{-p'(\bar{x}(s)) - (1-\gamma)[p(\bar{x}(s)) + \bar{x}(s)p'(\bar{x}(s))]\}Z(s)ds \\ + \int_0^t [(1 - p(\bar{x}(s))) + (1-\gamma)^2 p(\bar{x}(s))\bar{x}(2,s)]^{\frac{1}{2}} dW(s) \end{aligned} \tag{4.60}$$

$Z(t)$ has the same distribution to $X(t)$, with $X(t) = X(0) + \int_0^t q(s)X(s)ds + \int_0^t r^{\frac{1}{2}}(s)dW(s)$.

Coefficients are given as $q(s) = -p'(\bar{x}(s)) - (1-\gamma)[p(\bar{x}(s)) + \bar{x}(s)p'(\bar{x}(s))]$, $r(s) = (1 - p(\bar{x}(s))) + (1-\gamma)^2 p(\bar{x}(s))\bar{x}(2,s)$. $Z_n(t)$ converges to a linear diffusion process given by (4.53).

□

4.4 Proof that $Z(t)$ is a Gaussian process

The computation process largely depends on the result and method of [22], Sec 5.6. It can also be found in [20]. Define the linear process $(U(t))$ as the solution of the linear stochastic differential equation

$$dU(t) = q(t)U(t)dt + r^{\frac{1}{2}}(t)dW(t) \tag{4.61}$$

with coefficients (r, q) defined in (4.33).

Proposition 4.7. *The limiting process* $(Z(t))_{t\geq 0}$ *is the unique solution of the one-dimensional linear stochastic differential equation (4.61). Assuming that* $\sqrt{n}(\bar{x}_n(0) - \bar{x}(0))$ *converges in* *distribution to normal random variable $Z(0)$, then $(Z(t))$ is Gaussian.*

Proof. By (4.60), we have that $Z(t)$ satisfies (4.61). Consider the deterministic equation

$$d\Phi(t) = q(t)\Phi(t)dt \tag{4.62}$$

with $\Phi(0) = 1$. The solution to (4.62) is

$$\Phi(t) = \exp\Big(\int_0^t q(s)ds\Big).$$

Define

$$Z(t) = \Phi(t)Z(0) + \Phi(t)\int_0^t \Phi^{-1}(s)r^{\frac{1}{2}}(s)dW(s).$$

By Ito's rule, we have that

$$dZ(t) = \Phi'(t)Z(0) + \Phi'(t)\int_0^t \Phi^{-1}(s)r^{\frac{1}{2}}(s)dW(s) + \Phi(t)\Phi^{-1}(t)r^{\frac{1}{2}}(t)dW(t)$$

$$= q(t)Z(t)dt + r^{\frac{1}{2}}(t)dW(t).$$

So $Z(t)$ here is exactly the solution to (4.61).

Uniqueness of $Z(t)$ is verified immediately due to the fact that $r(t)$, $q(t)$ are smooth and not depending on the space variable and the drift coefficient is linear, hence Lipschitz.

The above computation thus implies that

$$Z(t) = \Phi(t)Z(0) + \Phi(t)\int_0^t \Phi^{-1}(s)r^{\frac{1}{2}}(s)dW(s)$$

$$= e^{\int_0^t q(s)ds}Z(0) + e^{\int_0^t q(s)ds}\int_0^t e^{-\int_0^s q(u)du}r^{\frac{1}{2}}(s)dW(s).$$

Provided $Z(0)$ is Gaussian, we see that the stochastic integral is Gaussian as the limit in L^2 of sums

$$\sum_{1\leq j\leq m} \tilde{r}(s_{j-1})(W(s_j) - W(s_{j-1})) \qquad |s_j - s_{j-1}| = t/m \qquad m \to \infty\,,$$

with $\tilde{r}(s) = \Phi^{-1}(s)r^{\frac{1}{2}}(s)$. This concludes the proof. □

4.5 The covariance of $Z(t)$

In [22], there is a direct conclusion indicating that for such stochastic process in the form of $Z(t)$ is a Gaussian process, and we in here analyze the characteristic function of $Z(t)$ at time $t \geq 0$.

Denote $R(t) = \frac{Z(t)}{\Phi(t)}$, and so

$$R(t) = Z(0) + \int_0^t \Phi^{-1}(s)r^{\frac{1}{2}}(s)dW(s) \tag{4.63}$$

with

$$dR(t) = \Phi^{-1}(t)r^{\frac{1}{2}}(t)dW(t) \tag{4.64}$$

Consider $e^{i\xi R(t)}$, by Ito's formula, we have

$$d[e^{i\xi R(t)}] = i\xi e^{i\xi R(t)}\Phi^{-1}(t)r^{\frac{1}{2}}(t)dW(t) - \frac{1}{2}\xi^2 e^{i\xi R(t)}\Phi^{-2}(t)r(t)dt \tag{4.65}$$

and

$$e^{i\xi R(t)} = e^{i\xi R(0)} + i\xi \int_0^t e^{i\xi R(s)}\Phi^{-1}(s)r^{\frac{1}{2}}(s)dW(s) - \frac{1}{2}\xi^2 \int_0^t e^{i\xi R(s)}\Phi^{-2}(s)r(s)ds \tag{4.66}$$

thus

$$\mathrm{E}[e^{i\xi R(t)}] = \mathrm{E}[e^{i\xi R(0)}] - \frac{1}{2}\xi^2 \int_0^t \mathrm{E}[e^{i\xi R(s)}]\Phi^{-2}(s)r(s)ds \tag{4.67}$$

By solving the differential equation (4.67), we conclude that $R(t)$ is normally distributed at any time $t \geq 0$, and so is $Z(t)$.

$Z(t)$ is a Gaussian process given Assumption 2.3, and the covariance form, when $0 \leq t_1 \leq t_2 < \infty$, is given as

$$\begin{aligned} \mathrm{Cov}(Z(t_1), Z(t_2)) &= \exp\Big(\int_0^{t_1} q(s)ds + \int_0^{t_2} q(s)ds\Big) \\ &\quad \times \int_0^{t_1} \exp\Big(-2\int_0^s q(u)du\Big) r(s)ds \\ &= \Phi(t_1)\Phi(t_2)\int_0^{t_1} \Phi^{-2}(s)r(s)ds\,. \end{aligned} \tag{4.68}$$

Chapter 5

Fluctuation limit $\phi(x) = x^k$, $k \geq 2$, induction step

By Proposition 4.3, Theorem 4.4 and Theorem 4.6, we have already shown that $(Z_n(t))$, $n \geq 1$ is tight and the limit point $Z(t)$ is the solution to the linear stochastic differential equation (4.61). Now we introduce the processes $Z_n(k,t)$, $k \geq 2$ and shall show that $Z_n(k,t)$ is also tight. This implies that the joint process

$$\mathbf{Z}_n(k,t) = (Z_n(1,t), Z_n(2,t), \ldots, Z_n(k,t)) \quad t \geq 0\,, \qquad \text{indexed over } n \tag{5.1}$$

is tight. Recall that

$$Z_n(1,t) = Z_n(t)\,, \qquad Z(1,t) = Z(t)\,. \tag{5.2}$$

Consider now a limit point

$$\mathbf{Z}(k,t) = (Z(1,t), Z(2,t), \ldots, Z(k,t)) \quad t \geq 0 \tag{5.3}$$

of this k-dimensional process. Under our assumptions, $(Z(1,t), Z(2,t), ..., Z(k,t))$ is a k-dimensional linear diffusion process and, as a consequence, a Gaussian process, if the initial value is Gaussian.

5.1 The formula of $Z(k,t)$

Using a similar argument as what we did in the proof of Theorem 4.6, in here, notations are a bit different. We denote

$$q_n(k, Z_n(s), Z_n(2,s), ..., Z_n(k,s), s) = \tag{5.4}$$
$$\sum_{j=1}^{k}[(1-p(\bar{x}_n(s)))\binom{k}{j}Z_n(k-j,s) - p'(\bar{x}(s))Z_n(s)\binom{k}{j}\bar{x}(k-j,s)]$$
$$-(1-\gamma^k)[p(\bar{x}_n(s))Z_n(k,s) + p'(\bar{x}(s))Z_n(s)\bar{x}(k,s)]\,,$$

together with

$$r_n(k,s) = (1-p(\bar{x}_n(s)))[\binom{k}{1}^2\bar{x}_n(2k-2,s) + 2\binom{k}{1}\binom{k}{2}\bar{x}_n(2k-3,s) + \cdots + 1]$$
$$+(1-\gamma^k)^2 p(\bar{x}_n(s))\bar{x}_n(2k,s)$$

and again

$$q(k, Z(s), Z(2,s), ..., Z(k,s), s) = \tag{5.5}$$
$$\sum_{j=1}^{k}[(1-p(\bar{x}(s)))\binom{k}{j}Z(k-j,s) - p'(\bar{x}(s))Z(s)\binom{k}{j}\bar{x}(k-j,s)]$$
$$-(1-\gamma^k)[p(\bar{x}(s))Z(k,s) + p'(\bar{x}(s))Z(s)\bar{x}(k,s)]\,,$$

and also

$$r(k,s) = (1-p(\bar{x}(s)))[\binom{k}{1}^2 \bar{x}(2k-2,s) + 2\binom{k}{1}\binom{k}{2}\bar{x}(2k-3,s) + \cdots + 1]$$

$$+(1-\gamma^k)^2 p(\bar{x}(s))\bar{x}(2k,s)\,.$$

Theorem 5.1. *For $k \geq 2$, with the notations from eq. (5.1)-(5.3), assume that $(\mathbf{Z}_n(k-1,t))_{t\geq 0}$ converges in distribution (jointly in all $1 \leq j \leq k-1$) to the process $(\mathbf{Z}(k-1,t))_{t\geq 0}$. Let $(Z(k,t))_{t\geq 0}$ be a limit point of the tight sequence of stochastic processes $\{Z_n(k,t)\}$. Then, for any function $\varphi \in C^3(\mathbb{R})$ with bounded derivatives, the process*

$$\begin{aligned} M(\varphi, Z(k,t), t) &= \varphi(Z(k,t)) - \varphi(Z(k,0)) \\ &\quad - \int_0^t [q(k, Z(s), Z(2,s), ..., Z(k,s), s)\varphi'(Z(k,s)) + \frac{1}{2} r(k,s)\varphi''(Z(k,s))]ds \end{aligned} \tag{5.6}$$

is a $\mathcal{F}_t$- martingale.

Remark. 1) The test functions here are not the same test functions ϕ, including the special case of polynomials, used for the fluctuation random field $\xi_n(t)$, which are defined in Definition 1.4. Here we are simply proving that the limit point of the process $\mathbf{Z}_n(k,t)$ satisfies a certain martingale problem. The test functions $\phi(x) = x^k$ give us a diffusion process $Z(x^k,t)$. Now this process is characterized by its own martingale problem, with test functions $\varphi(z)$. Note the use of the variable z to distinguish the two levels of test functions.

2) We note the set includes smooth functions with compact support. However, we shall need the fact that $\varphi(z) = z$ is a valid test function. Since the derivatives are bounded, ϕ is sub-linear anyway.

Definition 5.2. *When $\varphi(z) = z$ we use the simplified notation*

$$M(\varphi, Z(k,t), t) = M(Z(k,t), t)\,. \tag{5.7}$$

This is consistent with Definition 5.4, eq. (5.7) for $k > 1$ and eq. (4.52) for $k = 1$.

5.2 Differential formula and martingales for $Z_n(k, t)$

First, let us still consider the martingale part involved in this pure jump process, actually large part of the below computations are just repeating what we did before, we repeat the computation process here merely to keep the integrity of our proof.

For a test function ϕ as in Definition 1.4, including polynomials, consider the martingale $M_n(\phi, t)$ given below as

$$\begin{aligned} M_n(\phi, t) = \bar{x}_n(\phi, t) - \bar{x}_n(\phi, 0) - \sum_{i=1}^{n} \int_0^t (1 - p(\bar{x}_n(s)))\frac{1}{n}[\phi(x_i(s) + 1) - \phi(x_i(s))] \\ + p(\bar{x}_n(s))\frac{1}{n}[\phi(\gamma x_i(s)) - \phi(x_i(s))] ds \end{aligned} \tag{5.8}$$

and the quadratic variation is given below as

$$\begin{aligned} \langle M_n(\phi, t)\rangle = \sum_{i=1}^{n} \int_0^t (1 - p(\bar{x}_n(s)))[\frac{1}{n}(\phi(x_i(s) + 1) - \phi(x_i(s)))]^2 \\ + p(\bar{x}_n(s))[\frac{1}{n}(\phi(\gamma x_i(s)) - \phi(x_i(s)))]^2 ds \end{aligned} \tag{5.9}$$

$\langle M_n(k, t)\rangle$ vanishes uniformly in t as $n \to +\infty$. Using same notations, when the test function is $\phi(x) = x^k$, $k \in \mathbb{N}$,

$$\begin{aligned}\bar{x}_n(k,t) = \bar{x}_n(k,0) + \int_0^t (1 - p(\bar{x}_n(s))) \sum_{j=1}^{k} \binom{k}{j} \bar{x}_n(k-j,s) \\ -(1-\gamma^k) p(\bar{x}_n(s)) \bar{x}_n(k,s) ds + M_n(k,t)\end{aligned} \tag{5.10}$$

and as $n \to +\infty$,

$$\bar{x}(k,t) = \bar{x}(k,0) + \int_0^t (1 - p(\bar{x}(s))) \sum_{j=1}^{k} \binom{k}{j} \bar{x}(k-j,s) - (1-\gamma^k) p(\bar{x}(s)) \bar{x}(k,s) ds \tag{5.11}$$

In here, $\bar{x}(k,t)$ is deterministic and still consider their difference,

$$\begin{aligned}\bar{x}_n(k,t) - \bar{x}(k,t) = \bar{x}_n(k,0) - \bar{x}(k,0) + \int_0^t \sum_{j=1}^{k} \binom{k}{j} [(1 - p(\bar{x}_n(s)))(\bar{x}_n(k-j,s) - \bar{x}(k-j,s)) \\ -(p(\bar{x}_n(s)) - p(\bar{x}(s)))\bar{x}(k-j,s)] \\ -(1-\gamma^k)[p(\bar{x}_n(s))(\bar{x}_n(k,s) - \bar{x}(k,s)) + (p(\bar{x}_n(s)) - p(\bar{x}(s)))\bar{x}(k,s)] ds + M_n(k,t)\end{aligned} \tag{5.12}$$

$$\begin{aligned}\sqrt{n}(\bar{x}_n(k,t) - \bar{x}(k,t)) = \sqrt{n}(\bar{x}_n(k,0) - \bar{x}(k,0)) \\ + \int_0^t \sum_{j=1}^{k} \binom{k}{j} [(1 - p(\bar{x}_n(s))) \sqrt{n}(\bar{x}_n(k-j,s) - \bar{x}(k-j,s)) - \sqrt{n}(p(\bar{x}_n(s)) - p(\bar{x}(s)))\bar{x}(k-j,s)] \\ -(1-\gamma^k)[p(\bar{x}_n(s)) \sqrt{n}(\bar{x}_n(k,s) - \bar{x}(k,s)) + \sqrt{n}(p(\bar{x}_n(s)) - p(\bar{x}(s)))\bar{x}(k,s)] ds + \sqrt{n} M_n(k,t)\end{aligned} \tag{5.13}$$

$p(\bar{x}_n(s)) - p(\bar{x}(s))$ is still processed as what we did in Proposition 4.3 of Chapter 4, so for (5.13), we simplify to (5.14), which, there exists $\{v_n(s)\}$, such that

$$Z_n(k,t) = Z_n(k,0) + \int_0^t \sum_{j=1}^{k} [(1 - p(\bar{x}_n(s)))\binom{k}{j} Z_n(k-j,s) - p'(v_n(s))Z_n(s)\binom{k}{j}\bar{x}(k-j,s)]$$
$$-(1-\gamma^k)[p(\bar{x}_n(s))Z_n(k,s) + p'(v_n(s))Z_n(s)\bar{x}(k,s)]ds + \sqrt{n}M_n(k,t) \tag{5.14}$$

This process also has martingale part involved and its quadratic variation is given in (5.15).

$$\langle Z_n(k,t)\rangle = \int_0^t \sum_{i=1}^{n} (1 - p(\bar{x}_n(s)))[\sqrt{n}\frac{1}{n}((x_i(s)+1)^k - x_i^k(s))]^2$$
$$+p(\bar{x}_n(s))[\sqrt{n}\frac{1}{n}((\gamma x_i(s))^k - x_i^k(s))]^2 ds = \int_0^t (1 - p(\bar{x}_n(s)))[\binom{k}{1}^2 \bar{x}_n(2k-2,s) \tag{5.15}$$
$$+2\binom{k}{1}\binom{k}{2}\bar{x}_n(2k-3,s) + \cdots + 1] + (1-\gamma^k)^2 p(\bar{x}_n(s))\bar{x}_n(2k,s)ds$$

We will prove similar conclusions for $\{Z_n(k,t)\}$, with similar methods to the ones we used for $\{Z_n(t)\}$.

5.3 Tightness of $(Z_n(k,t))$, $k \geq 2$

Proposition 5.3. *For the sequence of stochastic processes defined as in (2.27)*

$$Z_n(k,t) = \langle \xi_n(t,dx), x^k \rangle \qquad t \geq 0\,,$$

indexed by $n \geq 1$, is tight as a family of stochastic processes in the Skorokhod space.

Proof. Preliminary considerations. In Proposition 4.3, we have already shown that when $j = 1$, $\{Z_n(j,t)\}$ is tight. Using induction hypothesis, we know that for every $j \leq k-1$,

$0 \leq t \leq T$,

(according to (7.12)) $\int_0^t \mathrm{E}[\sup_{0\leq s'\leq s} \left|Z_n(j,s')\right|^2]ds$ is bounded, and

(according to (7.13)) $\{Z_n(j,t)\}$ satisfies the modulus of continuity bound.

For the tightness of $Z_n(k,t)$, we divide our proof into two parts, as we did for the $k = 1$ case.

Part 1. Equations (5.14) and (5.15) show that there exists $g_j(n,k,s)$, $1 \leq j \leq k$, such that

$$Z_n(k,t) = Z_n(k,0) + \sum_{j=1}^{k} \int_0^t g_j(n,k,s)Z_n(j,s)ds + \sqrt{n}M_n(k,t) \tag{5.16}$$

and

$$\left|Z_n(k,t)\right|^2 \leq 3k\{\left|Z_n(k,0)\right|^2 + \sum_{j=1}^{k} \left|\int_0^t g_j(n,k,s)Z_n(j,s)ds\right|^2 + \left|\sqrt{n}M_n(k,t)\right|^2\}, \tag{5.17}$$

since

$$\left|\int_0^t g_j(n,k,s)Z_n(j,s)ds\right|^2 \leq \int_0^t \left|g_j(n,k,s)\right|^2 ds \int_0^t \left|Z_n(j,s)\right|^2 ds. \tag{5.18}$$

Actually by Lemma 2.6 and (5.14), we already have that for every $n \in \mathbb{N}$, $\left|g_j(n,k,s)\right|$ are bounded by a constant, and in Proposition 4.3 and our assumption, for every $1 \leq j \leq k-1$, at any moment $0 \leq t \leq T$, $\int_0^t \mathrm{E}[\sup_{0\leq s'\leq s} \left|Z_n(j,s')\right|^2]ds$ is bounded, which we still denote this common bound as M_1.

Thus, at any moment $0 \leq t \leq T$, for $1 \leq j \leq k-1$, $\int_0^t \mathrm{E}[\sup_{0\leq s'\leq s} \left|Z_n(j,s')\right|^2]ds < M_1$ and for every $n \in \mathbb{N}$, $\left|g_j(n,k,s)\right| < M_1$. Thus we have that

$$\left|Z_n(k,t)\right|^2 \leq 3k\{\left|Z_n(k,0)\right|^2 + M_1^2 T \sum_{j=1}^{k} \int_0^t \left|Z_n(j,s)\right|^2 ds + \left|\sqrt{n}M_n(k,t)\right|^2\} \tag{5.19}$$

Denote $A_n(k,\omega) = 3k\{\left|Z_n(k,0)\right|^2 + \sup_{0\le t\le T}\left|\sqrt{n}M_n(k,t)\right|^2\}$. Boundedness of $\mathrm{E}[A_n(k,\omega)]$ be proved the same way as for the case when $k = 1$, at any moment $0 \le t' \le t$, we have

$$\left|Z_n(k,t')\right|^2 \le A_n(k,\omega) + 3kM_1^2T\sum_{j=1}^{k-1}\int_0^{t'}\left|Z_n(j,s)\right|^2 ds + 3kM_1^2T\int_0^{t'}\left|Z_n(k,s)\right|^2 ds \tag{5.20}$$

$$\left|Z_n(k,t')\right|^2 \le A_n(k,\omega) + 3kM_1^2T\sum_{j=1}^{k-1}\int_0^{t'}\sup_{0\le s'\le s}\left|Z_n(j,s')\right|^2 ds + 3kM_1^2T\int_0^{t'}\sup_{0\le s'\le s}\left|Z_n(k,s')\right|^2 ds \tag{5.21}$$

then

$$\sup_{0\le t'\le t}\left|Z_n(k,t')\right|^2 \le A_n(k,\omega) + 3kM_1^2T\sum_{j=1}^{k-1}\int_0^{t}\sup_{0\le s'\le s}\left|Z_n(j,s')\right|^2 ds + 3kM_1^2T\int_0^{t}\sup_{0\le s'\le s}\left|Z_n(k,s')\right|^2 ds \tag{5.22}$$

Also denote $U_n(k,s) = \sup_{0\le s'\le s}\left|Z_n(k,s')\right|^2$,

$$\mathrm{E}[U_n(k,t)] \le \mathrm{E}[A_n(k,\omega)] + 3kM_1^2T\sum_{j=1}^{k-1}\int_0^{t}\mathrm{E}[\sup_{0\le s'\le s}\left|Z_n(j,s')\right|^2]ds + 3kM_1^2T\int_0^{t}\mathrm{E}[U_n(k,s)]ds \tag{5.23}$$

$$\mathrm{E}[U_n(k,t)] \le \mathrm{E}[A_n(k,\omega)] + 3k^2M_1^3T^2 + 3kM_1^2T\int_0^{t}\mathrm{E}[U_n(k,s)]ds \tag{5.24}$$

Also by Gronwall inequality (Lemma 4.1),

$$\mathrm{E}[U_n(k,t)] \le (\mathrm{E}[A_n(k,\omega)] + 3k^2M_1^3T^2)\exp(3kM_1^2Tt) \tag{5.25}$$

Thus $\mathrm{E}[\sup_{0\le t\le T}\left|Z_n(k,t)\right|^2]$ is bounded.

We have

$$\lim_{K\to+\infty}\lim_{n\to+\infty}P(\sup_{0\le t\le T}\left|Z_n(k,t)\right| > K) = 0 \tag{5.26}$$

This ends *Part 1*.

Part 2. Also consider $\left|Z_n(k,t) - Z_n(k,r)\right|^2$, with $0 \le r < t \le T$. First,

$$\begin{aligned}\left|Z_n(k,t) - Z_n(k,r)\right|^2 \le 2k\bigg\{ \sum_{j=1}^{k} \int_r^t \left|g_j(n,k,s)\right|^2 ds \int_r^t \left|Z_n(j,s)\right|^2 ds \\ + \left|\sqrt{n}(M_n(k,t) - M_n(k,r))\right|^2 \bigg\}\end{aligned} \tag{5.27}$$

implies

$$\left|Z_n(k,t) - Z_n(k,r)\right|^2 \le 2k\left\{ M_1^2(t-r) \sum_{j=1}^{k} \int_r^t \left|Z_n(j,s)\right|^2 ds + \left|\sqrt{n}(M_n(k,t) - M_n(k,r))\right|^2 \right\}.$$

Notice that

$$\begin{aligned}\sup_{0\le r<t\le T, 0<t-r<\delta} &\left|Z_n(k,t) - Z_n(k,r)\right|^2 \\ &\le 2kM_1^2\delta \sum_{j=1}^{k} \int_r^t \left|Z_n(j,s)\right|^2 ds \\ &+ 2k \sup_{0\le r<t\le T, 0<t-r<\delta} \left|\sqrt{n}(M_n(k,t) - M_n(k,r))\right|^2 .\end{aligned}$$

Considering

$$\mathrm{E}[\sup_{0\le r<t\le T, 0<t-r<\delta} \left|\sqrt{n}(M_n(k,t) - M_n(k,r))\right|^2],$$

there exists a constant C_1, such that

$$\mathrm{E}[\sup_{0\le r<t\le T, 0<t-r<\delta} \left|\sqrt{n}(M_n(k,t) - M_n(k,r))\right|^2] \le C_1\delta .$$

Combining (5.22) and (5.23), we conclude that

$$\mathrm{E}\left[\sup_{0\le r<t\le T, 0<t-r<\delta}\left|Z_n(k,t)-Z_n(k,r)\right|^2\right]$$
$$\le 2\delta k\left\{M_1^2\sum_{j=1}^{k}\mathrm{E}[\int_r^t \sup_{0\le s'\le s}\left|Z_n(j,s')\right|^2 ds] + C_1\right\}.$$

By (5.24) and above statements, it can be verified that for every $\epsilon > 0$, there exists $\delta' > 0$ such that if $0 < t - r < \delta'$, we have

$$\limsup_{n\to+\infty}\mathrm{E}[\sup_{0\le r<t\le T, 0<t-r<\delta'}\left|Z_n(k,t)-Z_n(k,r)\right|^2] \le \epsilon \tag{5.28}$$

so for any positive number $\epsilon_0 > 0$, we have

$$\lim_{\lambda\to 0}\limsup_{n\to+\infty}P(\sup_{0<t-r<\lambda, 0\le r<t\le T}\left|Z_n(k,t)-Z_n(k,r)\right| > \epsilon_0) = 0 \tag{5.29}$$

The sequence of stochastic processes $\{Z_n(k,t)\}_{n\ge 1}$ is tight. □

5.4 Proof of Theorem 5.1

By analyzing our stochastic processes $\{Z_n(k,t)\}$ using the differential formulas (1.6)-(1.7) which is exactly as what we did in our Theorem 4.6, Definition 4.5, we obtain that for a test function $\varphi(x) \in C^3(\mathbb{R})$, with $|\varphi'|$, $|\varphi''|$, and $|\varphi^{(3)}|$ bounded, equation (5.30) holds.

Definition 5.4. *The martingales $M_n(\varphi, Z(k,t), t)$ are defined by the stochastic differential* *formula*

$$\varphi(Z_n(k,t)) = \varphi(Z_n(k,0)) + \int_0^t -\sqrt{n}\varphi'(Z_n(k,s))\bar{x}'(k,s) +$$
$$\sum_{i=1}^{n}\{(1-p(\bar{x}_n(s)))[\varphi(Z_n(k,s)+\frac{1}{\sqrt{n}}(\binom{k}{1}x_i^{k-1}(s)+\binom{k}{2}x_i^{k-2}(s)+\cdots+1))-\varphi(Z_n(k,s))]$$
$$+p(\bar{x}_n(s))[\varphi(Z_n(k,s)-\frac{(1-\gamma^k)}{\sqrt{n}}x_i^k(s))-\varphi(Z_n(k,s))]\}ds + M_n(\varphi, Z(k,t),t) \tag{5.30}$$

In the special case $\varphi(z) = z$ we note that we use the simplified formula (5.7).

By Taylor's formula with mean value form of remainder, there exists $\{\epsilon_i(n,k,s)\}$ (possibly random), $0 \leq i \leq n$, with the property that $0 < \epsilon_i(n,k,s) < 1$, such that

$$\varphi(Z_n(k,s)+\frac{1}{\sqrt{n}}\sum_{j=1}^{k}\binom{k}{j}x_i^{k-j}(s)) - \varphi(Z_n(k,s)) =$$
$$\frac{1}{\sqrt{n}}\sum_{j=1}^{k}\binom{k}{j}x_i^{k-j}(s)\varphi'(Z_n(k,s)) + \frac{1}{2n}(\sum_{j=1}^{k}\binom{k}{j}x_i^{k-j}(s))^2\varphi''(Z_n(k,s)) + \tag{5.31}$$
$$\frac{1}{6n^{\frac{3}{2}}}(\sum_{j=1}^{k}\binom{k}{j}x_i^{k-j}(s))^3\varphi^{(3)}(Z_n(k,s)+\frac{\epsilon_0(n,k,s)}{\sqrt{n}}\sum_{j=1}^{k}\binom{k}{j}x_i^{k-j}(s))$$

and

$$\varphi(Z_n(k,s)-\frac{(1-\gamma^k)}{\sqrt{n}}x_i^k(s)) - \varphi(Z_n(k,s)) =$$
$$-\frac{1}{\sqrt{n}}(1-\gamma^k)x_i^k(s)\varphi'(Z_n(k,s)) + \frac{1}{2n}(1-\gamma^k)^2x_i^{2k}(s)\varphi''(Z_n(k,s)) \tag{5.32}$$
$$-\frac{1}{6n^{\frac{3}{2}}}(1-\gamma^k)^3x_i^{3k}(s)\varphi^{(3)}(Z_n(k,s)-\frac{\epsilon_i(n,k,s)}{\sqrt{n}}(1-\gamma^k)x_i^k(s))$$

and

$$\bar{x}'(k,s) = (1-p(\bar{x}(s)))\sum_{j=1}^{k}\binom{k}{j}\bar{x}(k-j,s) - (1-\gamma^k)p(\bar{x}(s))\bar{x}(k,s) \tag{5.33}$$

So we have

$$\varphi(Z_n(k,t)) = \varphi(Z_n(k,0)) + \int_0^t \sqrt{n}\varphi'(Z_n(k,s))[(1-p(\bar{x}_n(s)))\sum_{j=1}^{k}\binom{k}{j}\bar{x}_n(k-j,s)$$
$$-(1-p(\bar{x}(s)))\sum_{j=1}^{k}\binom{k}{j}\bar{x}(k-j,s) - (1-\gamma^k)(p(\bar{x}_n(s))\bar{x}_n(k,s) - p(\bar{x}(s))\bar{x}(k,s))]+$$
$$\frac{1}{2}\varphi''(Z_n(k,s))[(1-p(\bar{x}_n(s)))\frac{1}{n}\sum_{i=1}^{n}(\sum_{j=1}^{k}\binom{k}{j}x_i^{k-j}(s))^2 + (1-\gamma^k)^2 p(\bar{x}_n(s))\bar{x}_n(2k,s)]ds$$
$$+R_n(\varphi,k,t) + M_n(\varphi, Z(k,t), t) \tag{5.34}$$

$$R_n(\varphi,k,t) = \int_0^t \frac{1}{6\sqrt{n}}\{(1-p(\bar{x}_n(s)))\frac{1}{n}\sum_{i=1}^{n}[(\sum_{j=1}^{k}\binom{k}{j}x_i^{k-j}(s))^3\times$$
$$\varphi^{(3)}(Z_n(k,s) + \frac{\epsilon_0(n,k,s)}{\sqrt{n}}\sum_{j=1}^{k}\binom{k}{j}x_i^{k-j}(s))] - (1-\gamma^k)^3 p(\bar{x}_n(s))\frac{1}{n}\sum_{i=1}^{n}[x_i^{3k}(s)$$
$$\times\varphi^{(3)}(Z_n(k,s) - \frac{\epsilon_i(n,k,s)}{\sqrt{n}}(1-\gamma^k)x_i^k(s))]\}ds \tag{5.35}$$

$$\mathrm{E}[\left|R_n(\varphi,k,t)\right|^2] \le \frac{C_{R_n(\varphi,k,t)}}{n} \tag{5.36}$$

with $C_{R_n(\varphi,k,t)}$ a positive constant.

As $n \to +\infty$, $\mathrm{E}[\left|R_n(\varphi,k,t)\right|^2]$ vanishes uniformly in t.

Since

$$\sqrt{n}[(1-p(\bar{x}_n(s)))\sum_{j=1}^{k}\binom{k}{j}\bar{x}_n(k-j,s) - (1-p(\bar{x}(s)))\sum_{j=1}^{k}\binom{k}{j}\bar{x}(k-j,s)-$$
$$(1-\gamma^k)(p(\bar{x}_n(s))\bar{x}_n(k,s) - p(\bar{x}(s))\bar{x}(k,s))] = \{\sum_{j=1}^{k}\binom{k}{j}[Z_n(k-j,s) - p(\bar{x}_n(s))Z_n(k-j,s)$$
$$-p'(\bar{x}(s))Z_n(s)\bar{x}(k-j,s)] - (1-\gamma^k)(p(\bar{x}_n(s))Z_n(k,s) + p'(\bar{x}(s))Z_n(s)\bar{x}(k,s))\} + R_n(2,k,s) \tag{5.37}$$

with

$$R_n(2,k,s) = -\frac{1}{2}p''(\bar{x}(s)+\eta_n(s)(\bar{x}_n(s)-\bar{x}(s)))[\sum_{j=1}^{k}\binom{k}{j}\bar{x}(k-j,s) \\ +(1-\gamma^k)\bar{x}(k,s)]Z_n(s)(\bar{x}_n(s)-\bar{x}(s)) \tag{5.38}$$

As $n \to +\infty$, $\mathrm{E}[R_n(2,k,s)]$ vanishes uniformly in s.

$$\varphi(Z_n(k,t)) = \varphi(Z_n(k,0)) + \int_0^t \varphi'(Z_n(k,s))\{\sum_{j=1}^{k}[(1-p(\bar{x}_n(s)))\binom{k}{j}Z_n(k-j,s) \\ -p'(\bar{x}(s))Z_n(s)\binom{k}{j}\bar{x}(k-j,s)] - (1-\gamma^k)[p(\bar{x}_n(s))Z_n(k,s) + p'(\bar{x}(s))Z_n(s)\bar{x}(k,s)]\}ds + \\ \frac{1}{2}\int_0^t \varphi''(Z_n(k,s))\{(1-p(\bar{x}_n(s)))[\binom{k}{1}^2\bar{x}_n(2k-2,s) + 2\binom{k}{1}\binom{k}{2}\bar{x}_n(2k-3,s) + \cdots + 1] \\ +(1-\gamma^k)^2 p(\bar{x}_n(s))\bar{x}_n(2k,s)\}ds + \int_0^t R_n(2,k,s)\varphi'(Z_n(k,s))ds + R_n(\varphi,k,t) + M_n(\varphi,Z(k,t),t) \tag{5.39}$$

then for any arbitrarily chosen bounded variable $H(\omega) \in \mathcal{F}_{t'}$,

$$\mathrm{E}\{[\varphi(Z_n(k,t)) - \varphi(Z_n(k,t')) - \int_{t'}^t \varphi'(Z_n(k,s))q_n(k,Z_n(s),Z_n(2,s),...,Z_n(k,s),s) + \\ \frac{1}{2}\varphi''(Z_n(k,s))r_n(k,s)ds - \int_{t'}^t R_n(2,k,s)\varphi'(Z_n(k,s))ds - (R_n(\varphi,k,t) - R_n(\varphi,k,t'))]H(\omega)\} = 0 \tag{5.40}$$

As $n \to +\infty$, we have

$$\mathrm{E}\{[\varphi(Z(k,t)) - \varphi(Z(k,t')) - \int_{t'}^t \varphi'(Z(k,s))q(k,Z(s),Z(2,s),...,Z(k,s),s) + \\ \frac{1}{2}\varphi''(Z(k,s))r(k,s)ds]H(\omega)\} = 0 \tag{5.41}$$

Thus we conclude that

$$M(\varphi, Z(k,t), t) = \varphi(Z(k,t)) - \varphi(Z(k,0)) \\ - \int_0^t \varphi'(Z(k,s))q(k, Z(s), Z(2,s), ..., Z(k,s), s) + \frac{1}{2}\varphi''(Z(k,s))r(k,s)ds \tag{5.42}$$

is a $(\mathcal{F}_t)_{t\geq 0}$ - martingale. This ends the proof of Theorem 5.1.

Chapter 6

The joint process $\mathbf{Z}(k, t)$, $k \in \mathbb{N}$

In Chapter 4 ($k = 1$) and 5 ($k \geq 2$) we have shown that for each $k \in \mathbb{N}$, $(\mathbf{Z}_n(k, t))_{t \geq 0}$ defined in (5.1) is tight. In the case $k = 1$ we know explicitly it is a linear diffusion (Theorem 4.6). In addition, if we know that if $(\mathbf{Z}_n(k-1, t))_{t \geq 0}$ converges in distribution to $\mathbf{Z}(k-1, t)$, then any limit point $Z(k, t)$ of $Z_n(k, t)$ is a diffusion with coefficients depending on $\mathbf{Z}(k-1, t)$ and itself, linear in the variable representing itself (Theorem 5.1).

Theorem 4.6 is the verification step and Theorem 5.1 is the induction step of the identification of the joint law of the process $(\mathbf{Z}(k, t))$ as a linear diffusion. This is accomplished in Theorem 6.3. The result is possible because the coefficients of $Z(k, t)$ are depending, for any $k > 1$, on $Z(j, t)$ with $1 \leq j \leq k-1$ only. In that sense we characterize the process hierarchically over $k \geq 1$.

In our Theorem 6.3, we will give an explicit formula for the coefficients of $(\mathbf{Z}(k, t))$. Moreover, for Gaussian initial value, the process is Gaussian. This is because, under proper assumptions for initial conditions, $Z(t) = Z(1, t)$ is a Gaussian process, with an explicit formula for the covariance. Since the vector given as $\mathbf{Z}(k, t) = (Z(1, t), Z(2, t), ..., Z(k, t))$, for each $k > 1$ (recall that $Z(1, t) = Z(t)$) is a multidimensional linear process, it is a

Gaussian process.

6.1 The martingale part

In the presentation of Proposition 6.1, we are looking at the joint distribution of the multi-dimensional martingale $(M(Z(1,t),t), M(Z(2,t),t), ..., M(Z(k,t),t))$. Each $M(Z(j,t),t)$, $1 \leq j \leq k$ a martingale that had already been defined in eq. (5.42) for $k > 1$ and eq. (4.52) for $k = 1$.

Proposition 6.1. *For every $k \in \mathbb{N}$, the $(\mathcal{F}_t)_{t\geq 0}$ - martingales*

$$(M(Z(1,t),t), M(Z(2,t),t), ..., M(Z(k,t),t)) \qquad t \geq 0$$

obtained in (5.42) as limits of the martingales in Definition 5.4, are continuous, square integrable with absolutely continuous cross variation

$$\langle M(Z(i_1,t),t), M(Z(i_2,t),t)\rangle = \frac{1}{4}[\langle M(Z(x^{i_1}+x^{i_2},t),t)\rangle - \langle M(Z(x^{i_1}-x^{i_2},t),t)\rangle] \tag{6.1}$$

Based on Proposition 6.1, we can define a square integrable matrix in $t \geq 0$

$$A_2(t) = \begin{pmatrix} h_1(t) & h_2(t) & h_3(t) & \dots & h_k(t) \\ h_1(2,t) & h_2(2,t) & h_3(2,t) & \dots & h_k(2,t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1(k,t) & h_2(k,t) & h_3(k,t) & \dots & h_k(k,t) \end{pmatrix}$$

with the property that $A_2(t)A_2^T(t)$ equals, componentwise for $1 \le i_1, i_2 \le k$,

$$\langle M(Z(i_1,t),t), M(Z(i_2,t),t)\rangle = \sum_{j=1}^{k}\int_0^t h_j(i_1,s)h_j(i_2,s)ds\,. \tag{6.2}$$

In fact, based on (6.6) and (6.7), we shall give the exact form of the deterministic continuous functions $h_j(i,t)$, $1 \le i \le k$, $1 \le j \le k$, $t \ge 0$

$$\begin{aligned}\sum_{j=1}^{k} h_j^2(k,s) = (1-p(\bar{x}(s)))[\binom{k}{1}^2 \bar{x}(2k-2,s) + 2\binom{k}{1}\binom{k}{2}\bar{x}(2k-3,s) + \cdots + 1]\\ + (1-\gamma^k)^2 p(\bar{x}(s))\bar{x}(2k,s)\end{aligned} \tag{6.3}$$

Corollary 6.2. *There exists a k-dimensional Brownian motion*

$$W_k(t) = (W_k^1(t), W_k^2(t), ..., W_k^k(t))^T\,, \qquad t \ge 0\,,$$

possibly adapted an extenson $(\tilde{\mathcal{F}}_t)_{t\ge0}$ of the filtration $(\mathcal{F}_t)_{t\ge0}$ such that

$$M(Z(i,t),t) = \sum_{j=1}^{k}\int_0^t h_j(i,s)dW_k^j(s)\,. \tag{6.4}$$

We shall prove the proposition and corollary together.

Proof. (Proposition 6.1 and Corollary 6.2)

Consider $\varphi(z) = z$. Repeating the estimates from the proof of Theorem 5.1 we have

$$\begin{aligned}\langle M_n(\varphi, Z(k,t),t)\rangle = \int_0^t \varphi'(Z_n(k,s))\{(1-p(\bar{x}_n(s)))[\binom{k}{1}^2\bar{x}_n(2k-2,s)+\\ 2\binom{k}{1}\binom{k}{2}\bar{x}_n(2k-3,s)+\cdots+1] + (1-\gamma^k)^2 p(\bar{x}_n(s))\bar{x}_n(2k,s)\}ds + R_n(3,\varphi,k,t)\end{aligned} \tag{6.5}$$

With $R_n(3, \varphi, k, t)$ as the remainder term. Using a similar argument as we did for $R_n(3, \varphi, t)$ in Theorem 5.1, as $n \to +\infty$, $\mathrm{E}[\left|R_n(3, \varphi, k, t)\right|^2]$ vanishes uniformly in t, so we have that (we recall that now $\varphi(z) = z$ and Definition 5.4)

$$\begin{aligned}\langle M(Z(k,t),t)\rangle = \int_0^t (1 - p(\bar{x}(s)))[\binom{k}{1}^2 \bar{x}(2k-2,s) + 2\binom{k}{1}\binom{k}{2}\bar{x}(2k-3,s) \\ + \cdots + 1] + (1-\gamma^k)^2 p(\bar{x}(s))\bar{x}(2k,s)ds\,.\end{aligned} \tag{6.6}$$

For each k, we have a corresponding martingale $M(Z(k,t),t)$. As had already been given in Section 2.4, when $1 \leq i_1, i_2 \leq k$,

$$Z_n(x^{i_1} + x^{i_2}, t) = Z_n(i_1, t) + Z_n(i_2, t)\,. \tag{6.7}$$

We have already shown that both $Z_n(i_1, t)$ and $Z_n(i_2, t)$ and their corresponding martingales $M(Z(i_1,t),t)$ and $M(Z(i_2,t),t)$ are tight. Considering a limit point for the pair, we may consider, without loss of generality, that all converge in distribution as $n \to +\infty$. In the same manner $Z_n(x^{i_1} + x^{i_2}, t)$ and $Z_n(x^{i_1} - x^{i_2}, t)$, as applied to test functions $\phi(x) = x^{i_1} + x^{i_2}$, respectively $\phi(x) = x^{i_1} - x^{i_2}$ converge in distribution, together with their martingales.

Using a similar argument as we did for $Z(k,t)$, we have that the quadratic variations for martingale parts of $Z(x^{i_1} + x^{i_2}, t)$ and $Z(x^{i_1} - x^{i_2}, t)$ are both integrations with respect to continuous deterministic functions.

The cross variation between $M(Z(i_1,t),t)$ and $M(Z(i_2,t),t)$ ([22], Sec 1.5), is given by (6.1). We deduce that the cross-variation between $M(Z(i_1,t),t)$ and $M(Z(i_2,t),t)$ is absolutely continuous with respect to t.

By Theorem 7.9, there exists a k-dimensional Brownian motion, denoted as $W_k(t) =$

$(W_k^1(t), W_k^2(t), ..., W_k^k(t))^T$, such that

$$M(Z(i,t),t) = \sum_{j=1}^{k} \int_0^t h_j(i,s)dW_k^j(s)\,,$$

and the cross-variation, by Ito formula is exactly

$$\langle M(Z(i_1,t),t), M(Z(i_2,t),t)\rangle = \sum_{j=1}^{k} \int_0^t h_j(i_1,s)h_j(i_2,s)ds\,.$$

$\square$

6.2 Explicit form of the k - dimensional diffusion

The goal is to show Theorem 6.3, which claims that the limit point $(\mathbf{Z}(k,t))_{t\geq 0}$ can be identified as a k - dimensional linear diffusion. We describe its drift coefficients with the following notation, for $s \geq 0$

$$\begin{aligned} g_1(s) = g_1(1,s) &= -p'(\bar{x}(s)) - (1-\gamma)\Big[p(\bar{x}(s)) + \bar{x}(s)p'(\bar{x}(s))\Big] \\ g_k(k,s) &= -(1-\gamma^k)p(\bar{x}(s)), \qquad k \geq 2\,. \end{aligned} \tag{6.8}$$

In here, $g_1(k,s)$,$g_2(k,s)$,...,$h_k(k,s)$ are all continuous deterministic functions.

Let $\mathbf{U}(k,t)$ be the solution to the equation with coefficients (6.8), respectively (6.3)

$$U(k,t) = U(k,0) + \sum_{j=1}^{k} \int_0^t g_j(k,s)U(j,s)ds + \sum_{j=1}^{k} \int_0^t h_j(k,s)dW_k^j(s) \tag{6.9}$$

with

$$\mathbf{W}_k(t) = (W_k^1(t), W_k^2(t), ..., W_k^k(t))^T$$

a k-dimensional Brownian motion.

By (6.9), the vector process

$$\mathbf{U}(k,t) = (U(1,t), U(2,t), U(3,t), ..., U(k-1,t), U(k,t))^T$$

is a solution to the stochastic differential equation

$$\begin{aligned} \mathbf{Y}(t) &= \mathbf{Y}(0) + \int_0^t A_1(s)\mathbf{Y}(s)ds + \int_0^t A_2(s)d\mathbf{W}_k(s)\,, \\ \mathbf{Y}(0) &= (U(1,0), U(2,0), ..., U(k,0))^T\,. \end{aligned} \tag{6.10}$$

In matrix form, the coefficients are

$$A_1(t) = \begin{pmatrix} g_1(t) & 0 & 0 & \dots & 0 \\ g_1(2,t) & g_2(2,t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_1(k,t) & g_2(k,t) & g_3(k,t) & \dots & g_k(k,t) \end{pmatrix} \tag{6.11}$$

and $A_2(t)$ from (6.1).

The matrices $A_1(t) \in \mathbb{R}^{k\times k}$, $A_2(t) \in \mathbb{R}^{k\times k}$. $A_1(t)$, $A_2(t)$ are continuous, deterministic, measurable and locally bounded. Then

$$\mathbf{U}(k,t) = (U(1,t), U(2,t), \dots, U(k-1,t), U(k,t))$$

is a k - dimensional linear diffusion process according to Definition 7.7. Then, if the initial value is Gaussian, so is $(\mathbf{U}(k,t))_{t\geq 0}$. This is based on a multidimensional argument similar to our proof of Proposition 4.7 but results directly from the literature, e.g. [22], Sec 5.6.

It remains to identify $\mathbf{Z}(k,t)$ with the solution $\mathbf{U}(k,t)$.

Theorem 6.3. *For every $k \in \mathbb{N}$, $\mathbf{Z}(k,t) = (Z(t), Z(2,t), ..., Z(k,t))^T$ defined in (5.3) has the same distribution as the unique solution of the multidimensional stochastic differential equation (6.10), implying that $(Z(t), Z(2,t), ..., Z(k,t))$ is a linear diffusion process. If $\mathbf{Z}(k,0)$ is Gaussian, then $(\mathbf{Z}(k,t))$ is a Gaussian process.*

Proof. By our Theorem 5.1, we conclude that

$$\begin{aligned} \varphi(Z(k,t)) = \varphi(Z(k,0)) + \int_0^t \varphi'(Z(k,s))\{\sum_{j=1}^{k}[(1-p(\bar{x}(s)))\binom{k}{j}Z(k-j,s)- \\ p'(\bar{x}(s))Z(s)\binom{k}{j}\bar{x}(k-j,s)] - (1-\gamma^k)[p(\bar{x}(s))Z(k,s) + p'(\bar{x}(s))Z(s)\bar{x}(k,s)]\}ds \\ +\frac{1}{2}\int_0^t \varphi''(Z(k,s))\{(1-p(\bar{x}(s)))[\binom{k}{1}^2\bar{x}(2k-2,s) + 2\binom{k}{1}\binom{k}{2}\bar{x}(2k-3,s) + \cdots + 1] \\ +(1-\gamma^k)^2 p(\bar{x}(s))\bar{x}(2k,s)\}ds + M(\varphi, Z(k,t),t) \end{aligned} \tag{6.12}$$

with

$$\begin{aligned} \langle M(\varphi, Z(k,t),t)\rangle = \int_0^t \varphi'(Z(k,s))\{(1-p(\bar{x}(s)))[\binom{k}{1}^2\bar{x}(2k-2,s) \\ +2\binom{k}{1}\binom{k}{2}\bar{x}(2k-3,s) + \cdots + 1] + (1-\gamma^k)^2 p(\bar{x}(s))\bar{x}(2k,s)\}ds \end{aligned} \tag{6.13}$$

choose $\varphi(x) = x$, we have

$$\begin{aligned} Z(k,t) = Z(k,0) + \int_0^t \sum_{j=1}^{k}[(1-p(\bar{x}(s)))\binom{k}{j}Z(k-j,s) - p'(\bar{x}(s))Z(s)\binom{k}{j}\bar{x}(k-j,s)] \\ -(1-\gamma^k)[p(\bar{x}(s))Z(k,s) + p'(\bar{x}(s))Z(s)\bar{x}(k,s)]ds + M(Z(k,t),t) \end{aligned} \tag{6.14}$$

Recall that

$$(A_2(t)A_2^T(t))_{i_1 i_2} = \langle M(Z(i_1,t),t), M(Z(i_2,t),t)\rangle .$$

By our (6.14), we conclude that for every $k \in \mathbb{N}$, $Z(k,t)$ have the form as (6.9). $\square$

6.3 The fluctuation field $\xi(t)$, indexed by polynomials, is Gaussian

To finish the proof of our Theorem 3.4, we still need to show that for a general polynomial test function ϕ, $Z_n(\phi, t)$ converges in distribution to $Z(\phi, t)$, which is our Corollary 6.4.

Corollary 6.4. *For every polynomial function $\phi(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$, $Z_n(\phi, t)$ converges in distribution to a $Z(\phi, t)$. $Z(\phi, t)$ satisfies the equation given by (3.17).*

Proof. In here, denote $v_k = (a_1, a_2, ..., a_k)$, and $Y(t) = (Z(t), Z(2, t), ..., Z(k, t))^T$. Since

$$Z_n(\phi, t) = \sum_{j=1}^{k} a_j Z_n(j, t)$$

for each j, $Z_n(j, t)$ converges in distribution to $Z(j, t)$, $\sum_{j=1}^{k} a_j Z_n(j, t)$ converges in distribution to $\sum_{j=1}^{k} a_j Z(j, t)$. So $Z_n(\phi, t)$ converges in distribution, which we denote as $Z(\phi, t)$, with

$$Z(\phi, t) = \sum_{j=1}^{k} a_j Z(j, t) = v_k Y(t)$$

By (6.10), we have that

$$d(v_k Y(t)) = v_k A_1(t) Y(t) dt + v_k A_2(t) dW_k(t) \tag{6.15}$$

By our Theorem 6.3,

$$v_k A_2(t) dW_k(t) = \sum_{i=1}^{k} \sum_{j=1}^{k} a_j h_i(j, t) dW_k^i(t) \tag{6.16}$$

For $Z(\phi, t)$, by our (6.16),

$$\langle Z(\phi, t)\rangle = \int_0^t \sum_{i=1}^k [\sum_{j=1}^k a_j h_i(j, s)]^2 ds \tag{6.17}$$

which is

$$\langle Z(\phi, t)\rangle = \int_0^t \sum_{i=1}^k \sum_{1\le j_1, j_2\le k} a_{j_1} a_{j_2} h_i(j_1, s) h_i(j_2, s) ds \tag{6.18}$$

By our Proposition 6.1,

$$\begin{aligned}\langle Z(\phi, t)\rangle &= \sum_{1\le j_1, j_2\le k} \int_0^t \sum_{i=1}^k a_{j_1} a_{j_2} h_i(j_1, s) h_i(j_2, s) ds \\ &= \sum_{1\le j_1, j_2\le k} a_{j_1} a_{j_2} \langle M(Z(j_1, t), t), M(Z(j_2, t), t)\rangle\end{aligned} \tag{6.19}$$

In Chapter 3, (3.15) gives a Gaussian random field $W(\phi, t)$, in here, denote $q_j(x) = a_j x^j$. As we have already given a sequence of martingales $\{\sqrt{n} M_n(\phi, t)\}$ in Chapter 3, and

$$\begin{aligned}\langle \sqrt{n} M_n(\phi, t)\rangle = \frac{1}{n} \sum_{i=1}^n \int_0^t \{(1 - p(\bar{x}_n(s))[\sum_{j=1}^k (q_j(x_i(s) + 1) - q_j(x_i(s)))]^2 \\ + p(\bar{x}_n(s))[\sum_{j=1}^k (q_j(\gamma x_i(s)) - q_j(x_i(s)))]^2\} ds \\ = \frac{1}{n} \sum_{i=1}^n \int_0^t \{p(\bar{x}_n(s)) \sum_{1\le j_1, j_2\le k} (q_{j_1}(\gamma x_i(s)) - q_{j_1}(x_i(s)))(q_{j_2}(\gamma x_i(s)) - q_{j_2}(x_i(s))) + \\ (1 - p(\bar{x}_n(s)) \sum_{1\le j_1, j_2\le k} (q_{j_1}(x_i(s) + 1) - q_{j_1}(x_i(s)))(q_{j_2}(x_i(s) + 1) - q_{j_2}(x_i(s)))\} ds \\ = \frac{1}{n} \sum_{i=1}^n \sum_{1\le j_1, j_2\le k} \int_0^t \{p(\bar{x}_n(s))(q_{j_1}(\gamma x_i(s)) - q_{j_1}(x_i(s)))(q_{j_2}(\gamma x_i(s)) - q_{j_2}(x_i(s))) + \\ (1 - p(\bar{x}_n(s))(q_{j_1}(x_i(s) + 1) - q_{j_1}(x_i(s)))(q_{j_2}(x_i(s) + 1) - q_{j_2}(x_i(s)))\} ds\end{aligned} \tag{6.20}$$

In here, we notice that

$$\langle \sqrt{n}M_n(\phi, t)\rangle = \int_0^t \langle \mu_n(s, dx), \mathcal{D}_{s,n}\phi\rangle ds$$

For every $1 \leq j_1, j_2 \leq k$, by what we have proved in Chapter 5, below two formulas converge in distribution as $n \to \infty$.

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\int_0^t \{p(\bar{x}_n(s))(q_{j_1}(\gamma x_i(s)) - q_{j_1}(x_i(s)))^2 + \\ &(1 - p(\bar{x}_n(s))(q_{j_1}(x_i(s) + 1) - q_{j_1}(x_i(s)))^2\}ds\end{aligned} \tag{6.21}$$

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\int_0^t \{p(\bar{x}_n(s))(q_{j_2}(\gamma x_i(s)) - q_{j_2}(x_i(s)))^2 + \\ &(1 - p(\bar{x}_n(s))(q_{j_2}(x_i(s) + 1) - q_{j_2}(x_i(s)))^2\}ds\end{aligned} \tag{6.22}$$

and their weak limits represent for $\langle M(Z(j_1, t), t)\rangle$ and $\langle M(Z(j_2, t), t)\rangle$.

By the cross variation formula, for (6.23),

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\int_0^t \{p(\bar{x}_n(s))(q_{j_1}(\gamma x_i(s)) - q_{j_1}(x_i(s)))(q_{j_2}(\gamma x_i(s)) - q_{j_2}(x_i(s))) + \\ &(1 - p(\bar{x}_n(s))(q_{j_1}(x_i(s) + 1) - q_{j_1}(x_i(s)))(q_{j_2}(x_i(s) + 1) - q_{j_2}(x_i(s)))\}ds\end{aligned} \tag{6.23}$$

(6.23) converges in distribution.

By our formula (6.20), $\langle \sqrt{n}M_n(\phi, t)\rangle$ converges in distribution, which is equal to $\langle W(\phi, t)\rangle$, by (6.20), we have

$$\langle W(\phi, t)\rangle = \sum_{1\leq j_1, j_2\leq k} a_{j_1}a_{j_2}\langle M(Z(j_1, t), t), M(Z(j_2, t), t)\rangle \tag{6.24}$$

By (6.19) and (6.24), $\langle Z(\phi, t)\rangle = \langle W(\phi, t)\rangle$.

Also, using similar argument, in here,

$$v_k A_1(t)Y(t) = \sum_{j=1}^{k} a_j(\sum_{i=1}^{k} g_i(j,t)Z(i,t)) \tag{6.25}$$

and by the linearity of the inner product between $\xi(t,dx)$ and the $\mathcal{L}_t^\circ \phi$,

$$\langle \xi(t,dx), \mathcal{L}_t^\circ \phi \rangle = \sum_{j=1}^{k} \langle \xi(t,dx), \mathcal{L}_t^\circ q_j \rangle \tag{6.26}$$

For every $1 \le j \le k$,

$$\sum_{i=1}^{k} g_i(j,t)Z(i,t) = g_1(j,t)Z(t) + g_2(j,t)Z(2,t) + \cdots + g_j(j,t)Z(j,t) \tag{6.27}$$

and

$$\begin{aligned} \langle \xi(t,dx), \mathcal{L}_t^\circ(x^j) \rangle = \langle \xi(t,dx), \mathcal{L}_t(x^j) \rangle + \langle \xi(t,dx), \mathcal{G}_t(x^j) \rangle = \\ (1 - p(\bar{x}(s)))(\binom{j}{1} Z(j-1,t) + \binom{j}{2} Z(j-2,t) + \cdots + \binom{j}{j-1} Z(t)) - (1-\gamma^j) p(\bar{x}(t)) Z(j,t) \\ -p'(\bar{x}(t)) Z(t)((1-\gamma^j)\bar{x}(j,t) + \binom{j}{1}\bar{x}(j-1,t) + \cdots + \binom{j}{j-1}\bar{x}(t) + 1) \end{aligned} \tag{6.28}$$

By our (6.14), $g_i(j,t)$ in (6.27) equals the coefficient for each $Z(i,t)$ in (6.28).

So we have for every $1 \le j \le k$,

$$a_j \sum_{i=1}^{k} g_i(j,t)Z(i,t) = \langle \xi(t,dx), \mathcal{L}_t^\circ q_j \rangle \, .$$

That is, $v_k A_1(t)Y(t) = \langle \xi(t,dx), \mathcal{L}_t^\circ \phi \rangle$. Also, $W(\phi,t) = \int_0^t v_k A_2(s) dW_k(s)$. So $Z(\phi,t)$ satisfies (3.17). $\square$

Chapter 7

Appendix

We give a brief overview of some general concepts and definitions used throughout this paper, adapted to our specific context.

7.1 Gaussian processes and random fields

In here we will present Gaussian random fields based on [22] and [1]. We will present the definition of Gaussian random fields from both sources.

Definition 7.1. *Let a probability space* $(\Omega, \mathcal{F}, P)$*, a parameter set,* $\mathcal{T} \neq \emptyset$*, and a metric space* M *be given. A stochastic process on* M*, indexed by* $\mathcal{T}$*, is a family of* M *- valued random variables* $X(\tau, \omega)$*, i.e. for every fixed* $\tau \in \mathcal{T}$*,* $X : \Omega \to M$ *is a measurable function from* $\mathcal{F}$ *to the Borel sets of* M*, denoted* $\mathcal{B}_M$*.*

The most common choices of M are the Euclidean space but also other spaces, as for example the path space (Skorokhod for rcll valued processes, or simply the space of continuous functions for diffusions). An important case is when M is a Banach space, and specifically a Hilbert space.

The Kolmogorov Extension Theorem shows that the finite dimensional distributions are sufficient to define a stochastic process if a consistency condition is satisfied ([5]).

Among stochastic processes, a Gaussian process can be given by only the mean and covariance, since the two determine completely the finite dimensional distributions. Gaussian processes play a central role in stochastic processes. First we introduce the Definition 7.3 and Definition 7.4 based on [22].

Below we will introduce the concept of multivariate Gaussian distribution first, as we will use this definition to define a Gaussian random field.

Definition 7.2. *An n - dimensional vector-valued random variable* $X = (X_1, X_2, ..., X_n)$ *on* $M = \mathbb{R}^n$ *is said to have a multivariate normal (or Gaussian) distribution with mean* $\mu \in \mathbb{R}^n$ *and covariance matrix* $\Sigma \in S_n^{++}$ *if its probability density function is given by*

$$p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} exp\Big(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\Big) \tag{7.1}$$

In here, S_n^{++} is the space of symmetric positive definite $n \times n$ matrices, defined as $S_n^{++} = \{A \in \mathbb{R}^{n\times n} : A = A^T \text{ and } x^T A x > 0 \text{ for all } x \in \mathbb{R}^n \text{ such that } x \neq 0\}$. In here,

$$\mu = E[X]\,, \qquad (\Sigma)_{i,j} = Cov(X_i, X_j)$$

Definition 7.3 (Gaussian process)**.** *Let* $M = \mathbb{R}^d$ *with the Euclidean norm. A stochastic process* $X(\tau, \omega)$ *indexed by* $\mathcal{T}$ *(which needs not be a time interval) on* M *is Gaussian if for any finite collection* $\tau_1, \ldots, \tau_k \in \mathcal{T}$,

$$(X(\tau_1, \omega), \ldots, X(\tau_k, \omega)) \quad \textit{is jointly Gaussian (normal)}\,.$$

Equivalently, we can define as follows.

Definition 7.4 (Gaussian process - Alternative definition)**.** *A stochastic process is Gaussian if for any finite collection $t_1, \ldots, t_k \in T$ and $\alpha \in \mathbb{R}^k$ the random variable*

$$X(\omega) = \sum_{i=1}^{k} \alpha_i X(t_i, \omega) \quad \textit{is Gaussian (normal)}$$

Remark. The index set $\mathcal{T}$ is not necessarily a subset of $\mathbb{R}$ but often it is, in fact, representing time.

A special case is the following, in which $I = [0, T]$ a time interval and $M = \mathbb{R}^d$. An $\mathbb{R}^d$-valued random field $X(t, \omega)$ with index set $[0, T] \ni t$ is called Gaussian process if, for any integer $k > 1$ and real numbers $0 < t_1 < t_2 < ... < t_k < \infty$, the random vector $(X(t_1), X(t_2), ..., X(t_k))$ has a joint normal distribution.

Some of the most important processes are Gaussian. For example, Brownian motion and Ornstein-Uhlenbeck process. As examples, some of their properties will be stated below.

The finite-dimensional distributions of a Gaussian random field $X(t, \omega)$ are determined by its expectation vector $m(t) \triangleq \mathrm{E}X(t)$, $t > 0$, and its covariance matrix $cov(X(s), X(t)) = E[(X(s) - m(s))(X(t) - m(t))^T]$, $s, t > 0$, where the superscript T indicates transposition. If $m(t) = 0$, $t > 0$, we say that $X(t, \omega)$ is a zero-mean Gaussian process.

One-dimensional Brownian motion is a zero-mean Gaussian random field with covariance function

$$cov(X(s), X(t)) = s \wedge t, s, t > 0 \tag{7.2}$$

Conversely, any zero-mean Gaussian random field $X = \{X(t, \omega), \mathcal{F}_t^X : 0 \leq t < \infty\}$ with a.s. continuous paths and covariance function given by (7.2) is a one dimensional Brownian motion.

We now give the definition of a Gaussian random field adapted to our context. Let M be a Hilbert space whose elements will be denoted by ϕ, ψ (like test functions) with inner product denoted by the bracket $\langle\cdot,\cdot\rangle$, norm $\|\cdot\|$ and I be a subset of M.

Definition 7.5 (Gaussian random field)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space. We say that $\xi = \xi(\omega) : \Omega \to M$, a random variable on M, is a Gaussian random field on M indexed by $I \subseteq M$, denoted by $(\xi(\phi,\omega))_{\phi\in I}$, if the family of real valued random variables $\xi(\phi,\omega) := \langle\xi(\omega),\phi\rangle$, $\phi \in I$ is a Gaussian process indexed by I with covariance $Cov(\xi(\phi,\omega),\xi(\psi,\omega)) = \langle\phi,\psi\rangle$.*

The random field at the center of this work presented in Chapter 3 is described in Definition 3.1 and Proposition 3.2 using a Gaussian random field on a special Hilbert space.

Let $\phi(t,x), \psi(t,x)$, $t \in [0,\infty)$ and $x \in (0,\infty)$ be two variable test functions bounded in t satisfying the exponential growth bound in x as stated in Definition 1.4. Recall $\mathcal{D}_s$ is the bilinear form defined in (3.14) and $\mu(\cdot, dx)$ is the solution, belonging to $C([0,\infty); M_1((0,\infty))$ of the fluid limit equation from Theorem 2.3. Fix $T > 0$ and consider the inner product

$$\langle\phi,\psi\rangle = \int_0^T \int_{(0,\infty)} \mathcal{D}_s(\phi(s,x),\psi(s,x))\,\mu(s,dx)ds\,, \tag{7.3}$$

where $\phi(t,x)$, $\psi(t,x)$ are test functions with exponential growth bound, hence $\langle\phi,\phi\rangle < \infty$. The Hilbert space is the closure under the norm induced by (7.3) of the vector space of test functions.

Of course this space is abstract and we hope to pursue its study in future work. Fortunately, to obtain our main results we only need functions $\phi(t,x) = \mathbf{1}_{[0,t]}\phi_1(x)$ where $\phi_1 \in \mathbb{R}[x]$ (polynomial) which belong to the space of test functions with exponential growth bound. This is a consequence of the basic bounds from Definition 1.4 and further down, the bounds from Section 2.2, more specifically Lemma 2.6. The analysis can be done in finite dimensional setting (cf. Theorem 3.6) because polynomials are invariant to the operators $\mathcal{L}_s$, $\mathcal{D}_s$ and the degree can only be lowered - see Lemma 3.5.

7.2 Linear diffusions

The central topic in our model is a formal discussion of a system of linear stochastic differential equations, with solutions Gaussian processes of a special type, namely *Linear Diffusions*. In Chapter 5 and Chapter 6, we focused on such form of time dependent linear case process, in our case, we consider the k-dimensional stochastic differential equation, in the form of

$$dY(t) = A_1(t)Y(t)dt + A_2(t)dW_d(t) \tag{7.4}$$

with the initial distribution $Y(0)$ well defined. $W_d(t)$ is a d-dimensional Brownian motion, independent of the k-dimensional initial vector $Y(0)$, and the $(k \times k)$, $(k \times d)$ matrices $A_1(t)$, $A_2(t)$ are deterministic, measurable and locally bounded. When the initial distribution $Y(0)$ is normally distributed, then $Y(t)$ is a Gaussian random field and the finite dimensional distribution of this Gaussian random field given in (7.4) are completely determined by the mean and covariance functions.

Ornstein-Uhlenbeck process is the special case of the equation (7.4), when $d = 1$, and the matrix $A_1(t)$ is replaced by a negative constant, $A_2(t)$ by a positive constant. The character of Ornstein-Uhlenbeck process will be given in detail in the next section.

An important character for the Ornstein-Uhlenbeck process is $A_1(t)$ is replaced by a negative constant. In this paper, we did not discuss the stationary Gaussian process, which is an important feature of the Ornstein-Uhlenbeck process. In our model, $A_1(t)$ is a matrix with all eigenvalues negative for every $t \geq 0$, and this is one of the most important characters for our AIMD model. All eigenvalues of the matrix $A_1(t)$ are negative, and this is the reason we call the fluctuation limit of our AIMD model has a similar form to the Ornstein-Uhlenbeck process.

In our work, Gaussian random fields are indexed also by test functions. For test functions, denoted as ϕ_1, ϕ_2, and parameters t_1 and t_2, covariance between Gaussian random fields $X(\phi_1, t_1)$ and $X(\phi_2, t_2)$ had been analyzed in our Chapter 4-Chapter 6. When $\phi_1 = \phi_2$, denoted commonly as ϕ, which is the special case, covariance between $X(\phi, t_1)$ and $X(\phi, t_2)$ is much more simpler in our model, with explicit formula been given.

In the next section we will introduce the Ornstein-Uhlenbeck process, which is the most important kind of Gaussian random fields. Materials from [22] largely be cited in order to give a revelation of their relationships.

7.3 Multidimensional linear diffusion processes

Now let us talk briefly about linear processes and Ornstein-Uhlenbeck process. We start with the one dimensional classical Ornstein-Uhlenbeck process, based on [22].

Definition 7.6. *A stochastic process $\{Y(t) : t \geq 0\}$ is an Ornstein-Uhlenbeck process if*

$$dY(t) = -\rho(Y(t) - \mu)dt + \sigma dW(t), \quad Y(0) = Y_0 \tag{7.5}$$

where $\{W(t) : t \geq 0\}$ is a one dimensional Brownian motion with unit parameter and μ, ρ, σ are constants, with $\rho > 0$ and $\sigma > 0$.

When $\mu = 0$, this corresponds to the Langevin (1908) equation for the Brownian motion of a particle with friction. The solution of this equation is

$$Y(t) = Y(0)e^{-\rho t} + \sigma \int_0^t e^{-\rho(t-s)} dW(s), 0 \leq t < \infty. \tag{7.6}$$

If $\mathrm{E}Y^2(0) < \infty$, the expectation, variance, and covariance functions are follows, with notations consistent to the above

$$m(t) \triangleq \mathrm{E}Y(t) = m(0)e^{-\rho t} \tag{7.7}$$

$$V(t) \triangleq Var(Y(t)) = \frac{\sigma^2}{2\rho} + (V(0) - \frac{\sigma^2}{2\rho})e^{-2\rho t} \tag{7.8}$$

$$cov(Y(t), Y(r)) = [V(0) + \frac{\sigma^2}{2\rho}(e^{2\rho(t\wedge r)} - 1)]e^{-\rho(t+r)} \tag{7.9}$$

If the initial random variable $Y(0)$ has a normal distribution with mean zero and variance $\frac{\sigma^2}{2\rho}$, then $Y(t)$ is a stationary, zero-mean Gaussian random field with covariance function $cov(Y(t), Y(r)) = \frac{\sigma^2}{2\rho}e^{-\rho|t-r|}$.

A detailed description of Ornstein-Uhlenbeck process and its generalization is in [19], [18] and [23]. In Chapter 4, we discussed a stochastic process with a similar form to Ornstein-Uhlenbeck process, though in our case, coefficients are deterministic functions with the time parameter as the variable, not constants, as given below

$$dY(t) = -\gamma_1(t)Y(t)dt + \gamma_2(t)dW(t) \tag{7.10}$$

where $\gamma_1(t)$ is a deterministic positive function of time, $\gamma_2(t)$ is a deterministic function of time. In here, we also require that both $\gamma_1(t)$ and $\gamma_2(t)$ are bounded by a constant. Actually, in our fluctuation limit case, both $\gamma_1(t)$ and $\gamma_2(t)$ converge as $t \to +\infty$.

Please notice that in our (7.10), $\gamma_1(t)$ is a positive function of time, which is an important character for this kind of stochastic differential equation. For the higher dimensional case, those differential equations are a bit different, as will be carefully described below.

We give the definition of a linear diffusion, according to [22].

Definition 7.7. *If a k-dimensional stochastic process $Y(t)$ satisfies the k-dimensional stochastic differential equation, in the form of*

$$dY(t) = [A_1(t)Y(t) + a(t)]dt + A_2(t)dW_d(t), Y(0) = \xi_0 \tag{7.11}$$

$\{W_d(t)\}_{t\geq 0}$ *is a d-dimensional Brownian motion, independent of the k-dimensional initial vector $Y(0) = \xi_0$, and the $(k \times k)$, $(k \times d)$, $(k \times 1)$ matrices $A_1(t)$, $A_2(t)$ and $a(t)$ are deterministic, measurable and locally bounded.*

Then $Y(t)$ is a linear diffusion process.

7.4 Tightness

First let us state the general conditions for the tightness on the Skorohod space of right continuous left limit functions. We refer to Billingsley [5] for the study of the Skorokhod space and Jacod-Shireaev [21] for C - tightness.

Definition 7.8 (C - tightness)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider a family of random variables on $D([0, +\infty); \mathbb{R})$, i.e. real - valued continuous time right continuous with left limit random processes, denoted as $\{Y_n(t)\}_{t\geq 0}$, indexed by $n \in \mathbb{N}$. If*

$$\lim_{K\to+\infty} \lim_{n\to+\infty} P(|Y_n(t)| > K) = 0 \tag{7.12}$$

and for every $T > 0$, for any $\epsilon > 0$,

$$\lim_{\lambda\to 0}\limsup_{n\to+\infty} P(\sup_{0<t-s<\lambda, 0\le s\le t\le T} |Y_n(t) - Y_n(s)| > \epsilon) = 0\,. \tag{7.13}$$

Then $\{Y_n(t)\}_{t\ge 0}$ is tight in $D([0,+\infty);\mathbb{R})$ and any limit point is in $C([0,+\infty);\mathbb{R})$.

7.5 Martingale Representation Theorem

We discussed a multidimensional martingale, in which we applied the result of [22], Chapter 3, Sec 3.4. This result is presented here, which is a version of Martingale Representation Theorem.

Theorem 7.9 (from [22])**.** *Suppose $M = \{(M(1,t), M(2,t), ..., M(d,t))\}$, $0 \le t < +\infty$ is defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ with each $M(i,t)$ a continuous and local $\mathcal{F}_t$ - martingale, $1 \le i \le d$. Suppose also for each $1 \le i_1, i_2 \le d$, the cross variaion $\langle M(i_1,t), M(i_2,t)\rangle$ is absolutely continuous with respect to t, for P-almost sure ω. Then there is an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ of $(\Omega, \mathcal{F}, P)$ on which is defined a d-dimensional Brownian motion $W = \{W_d(t) = (W_d^1(t), W_d^2(t), ..., W_d^d(t))^T, \widetilde{\mathcal{F}}_t, 0 \le t < +\infty\}$, and a matrix $X = \{(X^{i,j}(t))_{i,j=1}^d, \widetilde{\mathcal{F}}_t, 0 \le t < +\infty\}$ of measurable, adapted processes with*

$$\widetilde{P}[\int_0^t (X^{(i,j)}(s))^2 ds < \infty] = 1, \quad 1 \le i, j \le d, 0 \le t < \infty$$

such that we have $\widetilde{P}$-a.s. the representation

$$M(i,t) = \sum_{j=1}^d \int_0^t X^{(i,j)}(s) dW_d^j(s), \quad 1 \le i \le d, \;\; 0 \le t < \infty \tag{7.14}$$

and

$$\langle M(i_1,t), M(i_2,t)\rangle = \sum_{j=1}^{d} \int_0^t X^{(i_1,j)}(s)X^{(i_2,j)}(s)ds, \quad 1 \le i_1, i_2 \le d, \quad 0 \le t < \infty\,. \qquad (7.15)$$

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