2016-06-18

Dihedral Symmetries of Non-crossing Partition Lattices

Ziqian Ding
University of Miami, joeding1988@gmail.com

UNIVERSITY OF MIAMI

DIHEDRAL SYMMETRIES OF NON-CROSSING PARTITION LATTICES

By

Ziqian Ding

A DISSERTATION

Submitted to the Faculty
of the University of Miami
in partial fulfillment of the requirements for
the degree of Doctor of Philosophy

Coral Gables, Florida

August 2016

UNIVERSITY OF MIAMI

A dissertation submitted in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

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Ziqian Ding

Approved:

Drew Armstrong, Ph.D.
Associate Professor of Mathematics

Ken Baker, Ph.D.
Associate Professor of
Mathematics

Alexander Dvorsky, Ph.D.
Associate Professor of Mathematics

Guillermo J. Prado, Ph.D.
Dean of the Graduate School

Victor Milenkovic, Ph.D.
Professor of Computer Science

DING, ZIQIAN (Ph.D., Mathematics)
Dihedral Symmetries of Non-crossing Partition Lattices (August 2016)

Abstract of a dissertation at the University of Miami.

Dissertation supervised by Professor Drew Armstrong.
No. of pages of text. (68)

The idea of the lattice of non-crossing partitions, $NC(n)$, is inspired by early work of Kreweras [7]. In this thesis we study the action of dihedral group D_{2n} on $NC(n)$, especially the sublattice $NC(n)^F$ in which all the elements are fixed by a reflection F, and then we extend our work to the characters of the dihedral group acting on $NC(n)$. We start from enumerative properties of the lattice $NC(n)^F$. Next we investigate the recursive structure on the lattice $NC(n)^F$ related to central binomial coefficients and the Catalan numbers. We proceed to look into combinatorial structure of a graded sublattice, $NC(n)^F_{pr}$, which is named "the pruned sublattice". Two characters α_S, β_S introduced by Stanley [18] of dihedral groups acting on $NC(n)$ are computed with respect to certain rank-selected subposet $NC(n)_S \subset NC(n)$. We first recall Montenegro's computation of $\beta_{[n-2]}$ from his unpublished manuscript [8]. Based on the cyclic sieving phenomenon of Reiner, Stanton and White [10], we obtain a general result for all α_S's, where S is a subset of $[n]$ of size 1 or $n-2$.

Acknowledgments

First and foremost, I would like to express all my appreciation and thanks to my advisor Dr. Drew Armstrong, without whose tremendous assistance I would never have finished this thesis. I am indebted to him for his selfless devotion, constant support, and endless patience.

I would also like to thank my committee members, Dr. Ken Baker, Dr. Alexander Dvorsky and Dr. Victor Milenkovic for serving as my committee members and for their encouragement, insightful comments all the way.

I am also incredibly grateful to Dr. Shuliman Kaliman and Dr. Marvin Mielke for their gracious guidance throughout my graduate study. Meanwhile, I would like to thank many other faculty and staff in Math department at UM who have helped a lot during the past five years.

Finally, I thank all my family and I am sorry for not being able to be with you these years. And I have numerous thanks to all my friends no matter how far we are apart.

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Chapter 1

Introduction

The idea of non-crossing partitions of the set $[n] := \{1, 2, \ldots, n\}$ comes from Germain Kreweras. In his 1972 paper *Sur les partitions non croisesé d'un cycle* [7], he investigated non-crossing partitions under the refinement order relation. His paper set a good background for further enumerative results, and for new connections between non-crossing partitions, the combinatorics of partially ordered sets and algebraic combinatorics, all of which are the key topics in this thesis.

In this first chapter, we will go over some preliminary knowledge with respect to the classical non-crossing partitions. First we will introduce the idea of non-crossing partitions and the basic combinatorial properties behind them. Then we will introduce the dihedral groups, the cyclic sieving phenomenon of Reiner, Stanton and White [10], and Richard Stanley's α and β characters of groups acting on finite posets [18].

For general information on non-crossing partitions, readers may refer to Chapter 3 and Chapter 4 of [2]. For general information about finite posets, see Chapter 3 of [16].

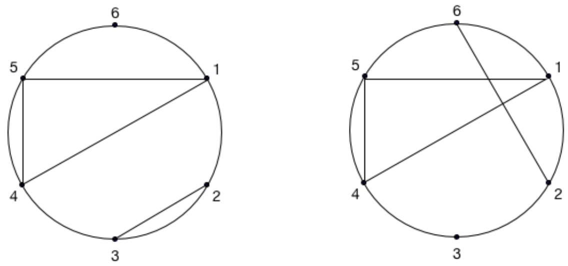


Figure 1.1: A non-crossing and a crossing partition of the set [6]

1.1 The Classical Non-crossing Partitions

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and $\pi = \{B_1, B_2, \ldots, B_k\}$ be a partition of the set $[n]$ with k blocks. Each $B_i \subset [n]$ is called a block of π, and we have a disjoint union $[n] = \sqcup_{i=1}^{k} B_i$.

Definition 1.1.1. Given a partition π of the set $[n]$, for two different blocks $B_i \neq B_j$ in π, with $1 \leq i, j \leq n$. We say B_i and B_j are *crossing* if there exist $a, b, c, d \in [n]$, such that $1 \leq a < b < c < d \leq n$ with $\{a, c\} \in B_i$ and $\{b, d\} \in B_j$. We say $\pi = \{B_1, B_2, \ldots, B_k\}$ is a *non-crossing partition* of $[n]$ if any two blocks B_i and B_j do not cross for all $1 \leq i < j \leq k$. Let $NC(n)$ denote the set of all non-crossing partition of the set $[n]$.

This definition becomes clearer is if we think of $[n]$ as a regular n-ploygon with n vertices labelled clockwise and identify each block of π with the convex hull of its corresponding vertices. Then π is non-crossing if and only if its blocks are pairwise disjoint.

Example 1.1.2.

In Figure 1.1, the left picture represents the non-crossing partition

$\{\{1,4,5\},\{2.3\},\{6\}\}$, while the right one represents $\{\{1,4,5\},\{2,6\},\{3\}\}$ which is crossing.

When describing a specific partition of $[n]$, we will usually list the blocks in increasing order of their minimum elements. For example, in Figure 1.1, we express the partition on the left as $\pi = \{\{1,4,5\},\{2,3\},\{6\}\}$ instead of $\pi = \{\{5,1,4\},\{6\},\{2,3\}\}$.

Given two non-crossing partition $\pi = \{B_1, B_2, \ldots, B_k\}$ and $\tau = \{\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_l\}$ in $NC(n)$, we say τ is a refinement of π if $\forall\ \tilde{B}_i \in \tau, \exists$ some $B_s \in \pi$ so that $\tilde{B}_i \subset B_s$, and we write $\tau \leq \pi$.

Definition 1.1.3. A partially ordered set P, which is usually called *poset* P for short, is a set together with a binary relation denoted $\leq$ (or $\leq_P$) satisfying the following three axioms:

1. Reflexivity: For all $a \in P, a \leq a$;
2. Antisymmetry: If $a \leq b$ and $b \leq a$, then $a = b$;
3. Transitivity: If $a \leq b$ and $b \leq c$, then $a \leq c$.

We say that two elements a and b of P are comparable if either $a \leq b$ or $b \leq a$, otherwise a and b are incomparable. It is not necessary that any two elements of P are comparable. Hence the relation $\leq$ is called a partial order. If all elements of a poset are comparable, we call it a total order or a chain. (see Definition 1.1.8)

Example 1.1.4. $NC(n)$ forms a poset under the refinement of partitions, with maximum element $\hat{1}_n = \{\{1,2,\ldots,n\}\}$ and minimum element $\hat{0}_n = \{\{1\},\{2\},\ldots,\{n\}\}$.

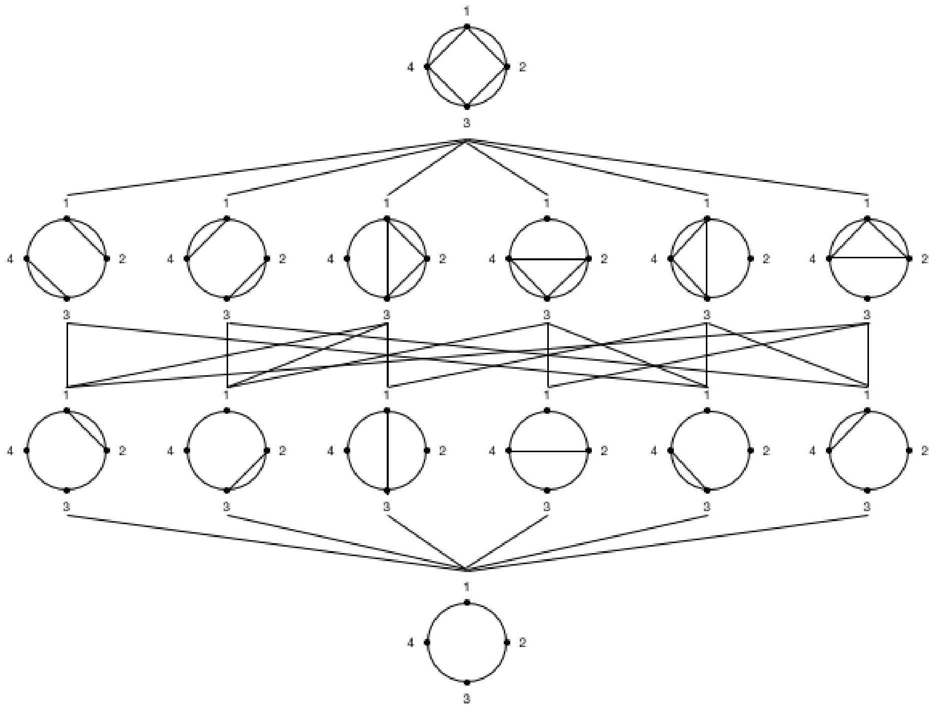


Figure 1.2: Lattice of NC(4)

Definition 1.1.5. Given two elements a, b in a poset P, we say b *covers* a if $a < b$ and there does not exist an element $c \in P$ such that $a < c < b$, in which case we write $a \lessdot b$.

We can usually visualize a finite poset P by its *Hasse diagram* whose vertices are the elements of P and edges are the covering relations such that if $a < b$ then b is drawn above a.

Example 1.1.6. Figure 1.2 exhibits the Hasse diagram of the poset $NC(4)$.

Definition 1.1.7. Let a and b be two elements in a poset P, an *upper bound* of a and b is an element $s \in P$ such that $s \geq a$ and $s \geq b$. A *least upper bound* (or *join* or *supremum*) of a and b is an upper bound s of a and b such that every upper bound s' of a and b satisfies $s' \geq s$. If a least upper bound of a and b exists, then it is unique (by the antisymmetry of a poset) and we denote it as $a \vee b$.

Similarly, we can define the *greatest lower bound* (or *meet* or *infimum*) of a and b and denote it as $a \wedge b$ if it exists.

A *lattice* is a poset L for which every pair of elements has a least upper bound and greatest lower bound.

It is easy to check that the poset $NC(n)$ is a lattice. Indeed, the meet of any two non-crossing partitions π and τ is just their coarsest common refinement whose blocks are obtained by intersecting the blocks of π with those of τ. It is a standard fact that a finite poset with meets and a top element $\hat{1}$ also has joins [16, Proposition 3.3.1].

Hence Figure 1.2 is also the lattice of $NC(4)$.

Definition 1.1.8. A *chain* in a poset P is a subset of P in which any two elements are comparable. A chain C of P is called *maximal* if it is not contained in a longer chain C' of P such that $C \subset C'$.

For any finite chain C of P, we may define the length $\ell(C)$ of this chain by $\ell(C) := \#C - 1$, i.e. the length of a chain C is equal to the number of covering relations in C.

Definition 1.1.9. A poset P is *graded of rank n* if all the maximal chains of P have the same length n. In this case, there exists a unique *rank function* $r : P \to [n]$ such that $ra = 0$ if a is a minimal element of P and $r(b) = r(a) + 1$ if $a \lessdot b$.

The lattice $NC(n)$ is graded of rank n with rank function as

$$r(\pi) := n - |\pi|,$$

where $|\pi|$ is the number of blocks of π.

Given $\pi \in NC(n)$, Let $next(\pi)= min\{i : 1, i \text{ in the same block}\}$, and $NC_i(n) = \{\pi \in NC(n) : next(\pi) = i\}$. Then we have a disjoint union

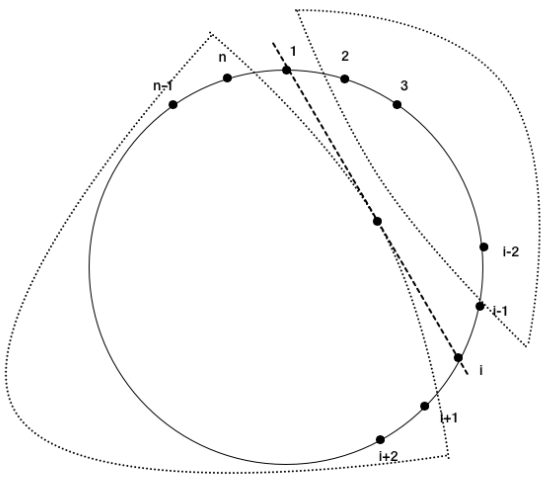
$$NC(n) = \bigsqcup_i NC_i(n)$$

Since all the $NC_i(n)$ are disjoint, it is easy to see that $\#NC(n) = \sum_i \#NC_i(n)$.

By defining $\#NC(-1) = \#NC(0) = 1$ by convention, we have the following proposition:

Proposition 1.1.10. $\#NC_i(n) = \#NC(i-2)\#NC(n-i+1)$.

Proof. Given a fixed $i \in [n]$, for $\forall\ \pi \in NC_i(n)$, blocks of π form two non-crossing partitions on sets $\{2, 3, \ldots, i-2\}$ and $\{1, i, i+1, \ldots, n\}$ respectively. Note that in the set $\{1, i, i+1, \ldots, n\}$, there are actually $n-i+1$ vertices, since vertex 1 and i must be in the same block and hence these two vertices can be regarded as one "big" vertex. Hence $\#NC_i(n) \leq \#NC(i-2)\#NC(n-i+1)$.



On the other hand, if we give any two non-crossing partitions on sets $\{2, 3, \ldots, i-2\}$ and $\{1, i, i+1, \ldots, n\}$ by connecting vertices 1 and i, we obtain a non-crossing partition in $NC_i(n)$. Then, Hence $\#NC_i(n) \geq \#NC(i-2)\#NC(n-i+1)$. The equality follows.

□

With the proposition above, it is clear that

$$\#NC(n) = \sum_{i=1}^{n} \#NC(i-2)\#NC(n-i+1).$$

Define $nc(n) := \#NC(n)$, then we have

$$nc(n) = \sum_{k+l=n-1} nc(k)nc(l)$$

with the initial condition $nc(0) = 1$.

Based on the proposition above, we have a very important enumerative result as follows:

Theorem 1.1.11. *[7, Theorem 7] NC(n) is counted by the classic Catalan Number, that is,*

$$nc(n) = \mathrm{Cat}(n) := \frac{1}{n+1}\binom{2n}{n}$$

Proof. Define $nc(0) = 1$ and notice that

$$nc(n+1) = \sum_{i=0}^{n} nc(i)nc(n-i),$$

the equation above together with the initial condition that $\#NC(1) = 1$ is exactly the recurrence relation defining the Catalan numbers. [15] □

We have seen that the lattice of $NC(n)$ is graded, i.e. the elements of the same rank in $NC(n)$ have the same number of blocks as well. The next question is what is the number of the elements of $NC(n)$ with a certain rank?

Theorem 1.1.12. *[5] #NC(n) with k blocks is counted by the classic Narayana Number, that is,*

$$\#\{\pi \in NC(n) : \pi \text{ has } k \text{ blocks}\} = \mathrm{Nar}(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Remark. This theorem is also implied by a more general formula of Kreweras [7, Theorem 4].

1.2 Characters of Dihedral Groups

The dihedral group D_{2n} is the symmetry group of an n-sided regular polygon for $n \geq 3$. A regular n-sided polygon has $2n$ different symmetries: n rotational symmetries and n reflectional symmetries. The associated rotations and reflections make up the dihedral group D_{2n}.

If we fix a vertex of an n-sided polygon at 12-o'clock and label from this vertex clockwise as 1... n, R is the action of the clockwise rotation by $2\pi/n$ and F is the reflection with respect to the vertical symmetric axis crossing vertex 1.

By the description above, we can define D_{2n} abstractly in terms of generators and relations as follows:

Definition 1.2.1. The dihedral group D_{2n}= $\langle R, F | R^n = F^2 = RFRF = 1\rangle$, where R denotes the rotational generator with order n and F denotes the reflectional generator with order 2,.

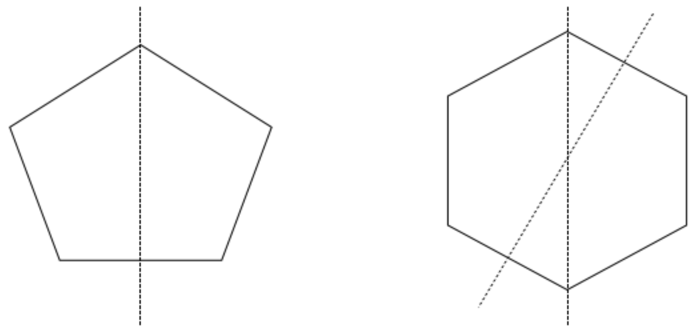


Figure 1.3: Reflections of NC(5) and NC(6)

We note that D_{2n} is the semi-direct product of the cyclic groups $\langle R\rangle \approx \mathbb{Z}/n$ and $\langle F\rangle \approx \mathbb{Z}/2$, that is, $D_{2n} = \langle R\rangle \ltimes \langle F\rangle$. Clearly, the size of D_{2n} is $2n$.

In Figure 1.3, we may see that while the number of reflections in D_{2n} $(n \geq 3)$ is n for all cases, the geometric description of reflections depends on the parity of n: for odd n all symmetric axes look the same, but for even n these symmetric axes fall into two types. There are n reflections which form one or two conjugacy classes, depending on the parity of n.

The different geometric descriptions of reflections in D_{2n} for odd and even n distinguish them algebraically when we describe the conjugacy classes of D_{2n}.

Theorem 1.2.2. *The conjugacy classes in D_{2n} are as follows:*

Case 1*: When n is odd, there are $\frac{n+3}{2}$ conjugacy classes:*

1. $\{id\}$
2. *$\frac{n-1}{2}$ conjugacy classes of size 2:* $\{R, R^{n-1}\}, \{R^2, R^{n-2}\}, \ldots, \{R^{\frac{n-1}{2}}, R^{\frac{n+1}{2}}\}$
3. *one conjugacy class of reflections:* $\{F, RF, \ldots, R^{n-1}F\}$

Case 2: *When n is even, there are $\frac{n+6}{2}$ conjugacy classes:*

1. $\{id\}$
2. $\{R^{\frac{n}{2}}\}$
3. $\frac{n-2}{2}$ *conjugacy classes of size 2:* $\{R, R^{n-1}\}, \{R^2, R^{n-2}\}, \ldots, \{R^{\frac{n}{2}-1}, R^{\frac{n}{2}+1}\}$
4. *the reflections fall into two conjugacy classes:*
   - *one with F:* $\{\ F, R^2F, R^4F, \ldots\}$
   - *one with RF:* $\{\ RF, R^3F, R^5F, \ldots\}$

Proof. Note that each element in D_{2n} is of form R^k or R^kF for some integer k. Hence, in order to find the conjugacy class of an element g we may compute R^kgR^{-k} and $(R^kF)g(R^kF)^{-1}$.

$$R^kR^lR^{-k} = R^l, \quad (R^kF)R^l(R^kF)^{-1} = R^{-l}.$$

As k varies, this shows that the only conjugates of R^l in D_{2n} are R^l and R^{-l}.

To find the conjugacy class of F, we compute

$$R^kFR^{-k} = R^{2k}F, \quad (R^kF)F(R^kF)^{-1} = R^{2k}F.$$

As k varies, $R^{2k}F$ goes through all the reflections in which R occurs with an exponent divisible by 2. If n is odd then every integer modulo n is a multiple of 2, since 2 is invertible mod n so we can solve $a \equiv 2k \bmod n$ for k given any a.

Hence when n is odd,

$$\{R^{2k}F : k \in \mathbb{Z}\} = \{R^kF : k \in \mathbb{Z}\},$$

and every reflection in D_{2n} is conjugate to F.

When n is even, however, we only get half the reflections as conjugates of F. The other half are conjugate to RF:

$$R^k(RF)R^{-k} = R^{2k+1}F, \quad (R^kF)(RF)(R^kF)^{-1} = R^{2k-1}F.$$

As k varies, this gives us $RF, R^3F, \ldots, R^{n-1}F$.

$\square$

Based on the geometric interpretation of D_{2n}, we may define a map

$$\varphi : D_{2n} \longrightarrow O(2)$$

where $O(2)$ is the orthogonal group in dimension 2, by

$$R \mapsto \begin{pmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}, \text{ and } F \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

It is easy to check the map φ is an injective group homomorphism.

By abuse of notation, we also use D_{2n} to refer to its image of the homomorphism φ and get

$$D_{2n} \subseteq O(2)$$

Remark. Actually D_{2n} can be regarded as a Coxeter group [2, section 2.1] $D_{2n} = \langle s, t | s^2 = t^2 = (st)^n = 1\rangle$, where $s = F$ and $t = RF$. That is, D_{2n} is also generated by F, RF, both with determinant -1.

It is easy to see that $\langle F, RF\rangle \subseteq \langle R, F\rangle$. On the other hand, $R = (RF)F \in \langle F, RF\rangle$ which implies that $\langle R, F\rangle \subseteq \langle F, RF\rangle$.

Note that $\forall\ g \in D_{2n}$, if $\det(\varphi(g)) = 1$, g is a product of even number of generators $s\&t$, which acts as a rotation, if $\det(\varphi(g)) = -1$, g is a product of odd number of $s\&t$ which acts as a reflection.

Definition 1.2.3. A *matrix representation* of a group G is a group homomorphism

$$\phi : G \to GL_d,$$

where $GL_d := GL_d(\mathbb{C}) = \{A \in \text{Mat}(\mathbb{C})_d | A \text{ is invertible}\}$, and d is called the dimension or degree of the representation.

Clearly, under the map φ above, we obtained a 2-dimensional representation of D_{2n}.

Definition 1.2.4. Let $\phi : G \to GL_d(\mathbb{C})$, be a matrix representation. The *character* of ϕ is

$$\chi(g) = \text{tr } \phi(g),$$

where tr denotes the trace of a matrix.

Definition 1.2.5. A matrix representation ϕ is called reducible if there exists a basis in which

$$\phi(g) = \left(\begin{array}{c|c} A(g) & C(g) \\ \hline 0 & B(g) \end{array}\right)$$

for all $g \in G$. Otherwise, it is called an irreducible representation.

Similarly, we can define a matrix representation ϕ to be decomposible if there exists a basis in which

$$\phi(g) = \left(\begin{array}{c|c} A(g) & 0 \\ \hline 0 & B(g) \end{array}\right)$$

and it is called indecomposable otherwise.

Remark. In general, the notions of indecomposable and irreducible representations differ, but when $|G| < \infty$ (i.e. G is finite) over the field $\mathbb{C}$, the two notions coincide [12].

Theorem 1.2.6. *[12] The number of irreducible representations of G (up to isomorphism) is equal to the number of conjugacy classes of G.*

Based on the theorem above and Theorem 1.2.2, we can solve for all the irreducible representations of the dihedral group and compute the characters explicitly.

When n is odd:

There are two one-dim representations:

1. Trivial representration: all elements $\mapsto 1$
2. Determinant representation: all elements in $< R > \mapsto 1$, otherwise -1

There are $\frac{n-1}{2}$ irreducible 2-dim representations, where the k-th representation ϕ_k is defined as

$$R \mapsto \begin{pmatrix} \cos\frac{2k\pi}{n} & \sin\frac{2k\pi}{n} \\ -\sin\frac{2k\pi}{n} & \cos\frac{2k\pi}{n} \end{pmatrix} \quad F \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Geometrically, R is the clockwise rotation by $\frac{2k\pi}{n}$ and F is the reflection across the vertical symmetric axis.

We may construct a character table of D_{2n} in which we list elements of D_{2n} on the first row and irreducible representations on the first column. Each entry of the

D_{2n}	1	F	R	R^2	$\cdots$	$R^{\frac{n-1}{2}}$
Triv	1	1	1	1	1	1
Det	1	-1	1	1	1	1
ϕ_1	2	0	$2\cos\frac{2\pi}{n}$	$2\cos\frac{4\pi}{n}$	$\cdots$	$\cos\frac{(n-1)\pi}{n}$
ϕ_2	2	0	$2\cos\frac{4\pi}{n}$	$2\cos\frac{8\pi}{n}$	$\cdots$	$\cos\frac{2(n-1)\pi}{n}$
$\vdots$						
$\phi_{\frac{n-1}{2}}$	2	0	$2\cos\frac{(n-1)\pi}{n}$	$2\cos\frac{2(n-1)\pi}{n}$	$\cdots$	$\cos\frac{(n-1)^2\pi}{2n}$

Figure 1.4: Character Table of D_{2n}, for n odd

character table show the character of certain irreducible representation evaluated at some element of D_{2n}.

Example 1.2.7. Character Table of $D_{2\cdot 3}$ is:

$D_{2\cdot 3}$	1	F	R
Triv	1	1	1
Det	1	-1	1
ϕ	2	0	-1

When n is even:

There are four one-dim representations:

1. Trivial representation: all elements $\mapsto 1$
2. Determinant representation: all elements in $< R > \mapsto 1$, otherwise -1
3. Lin1: $R \mapsto -1, r \mapsto 1, < R^2, F > \mapsto 1$
4. Lin2: $R \mapsto -1, r \mapsto -1, < R^2, F > \mapsto 1$

Two-dim representations:

D_{2n}	1	F	RF	R	R^2	$\cdots$	$R^{\frac{n}{2}}$
Triv	1	1	1	1	1	1	1
Det	1	-1	-1	1	1	1	1
Lin1	1	1	-1	-1	1	$\cdots$	$(-1)^{n/2}$
Lin2	1	1	1	-1	1	$\cdots$	$(-1)^{n/2}$
ϕ_1	2	0	0	$2\cos\frac{2\pi}{n}$	$2\cos\frac{4\pi}{n}$	$\cdots$	$\cos\frac{n\pi}{n}$
ϕ_2	2	0	0	$2\cos\frac{4\pi}{n}$	$2\cos\frac{8\pi}{n}$	$\cdots$	$\cos\frac{2n\pi}{n}$
$\vdots$							
$\phi_{\frac{n-2}{2}}$	2	0	0	$2\cos\frac{(n-1)\pi}{n}$	$2\cos\frac{2(n-1)\pi}{n}$	$\cdots$	$\cos\frac{n(n-1)\pi}{2n}$

Figure 1.5: Character Table of D_{2n}, for n even

There are $\frac{n-2}{2}$ irreducible 2-dim representations, where the $k-th$ representation ϕ_k is defined as

$$R \mapsto \begin{pmatrix} \cos\frac{2k\pi}{n} & \sin\frac{2k\pi}{n} \\ -\sin\frac{2k\pi}{n} & \cos\frac{2k\pi}{n} \end{pmatrix}, F \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example 1.2.8. The Character Table of $D_{2\cdot 4}$ is:

$D_{2\cdot 4}$	1	F	RF	R	R^2
Triv	1	1	1	1	1
Det	1	-1	-1	1	1
Lin1	1	1	-1	-1	1
Lin2	1	-1	1	-1	1
ϕ	2	0	0	0	-2

1.3 Cyclic Sieving Phenomenon

Suppose that we have a cyclic group G acting on a set X. In combinatorics, it is natural to study the number of fixed points $|X^g| = |\{x \in X : gx = x\}|$. In their

2004 paper [10], Reiner, Stanton and White described a phenomenon where one polynomial encodes the numbers of fixed elements for a given cyclic action. They called this the cyclic sieving phenomenon (CSP).

Definition 1.3.1. Let G be a cyclic group generated by an element of g of order n. Suppose G acts on a set X. Let X(q) be a polynomial with integer coefficients. Then the triple $(X, X(q), G)$ is said to exhibit the cyclic sieving phenomenon (CSP) if for all integers d, the evaluation $X(e^{2\pi i \frac{d}{n}})$ equals the number of elements of X fixed by g^d.

In particular, $X(1)$ is the cardinality of X, so that $X(q)$ can be regarded as a $q-$analogue of $|X|$.

Example 1.3.2. Let G be the cyclic group of order n which acts by adding 1 to each element of the set, modulo n. Let X be the collection of all the k-element subsets of $\{1, 2, \ldots, n\}$, and let $X(q)$ be the q-binomial coefficient defined by

$$X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$$

where $[n]_q! := [n]_q[n-1]_q \cdots [2]_q[1]_q$ and $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$.

It is easy to see that $X(1) = \binom{n}{k}$, hence $X(q)$ is a $q-$analogue for the number of subsets of $\{1, 2, \ldots, n\}$ of size k.

Then the triple $(X, X(q), G)$ exhibits the CSP [10, Theorem 1.1(b)].

Consider dissections of a convex n-gon using k non-crossing diagonals. The number of the dissections is given by the formula [17]:

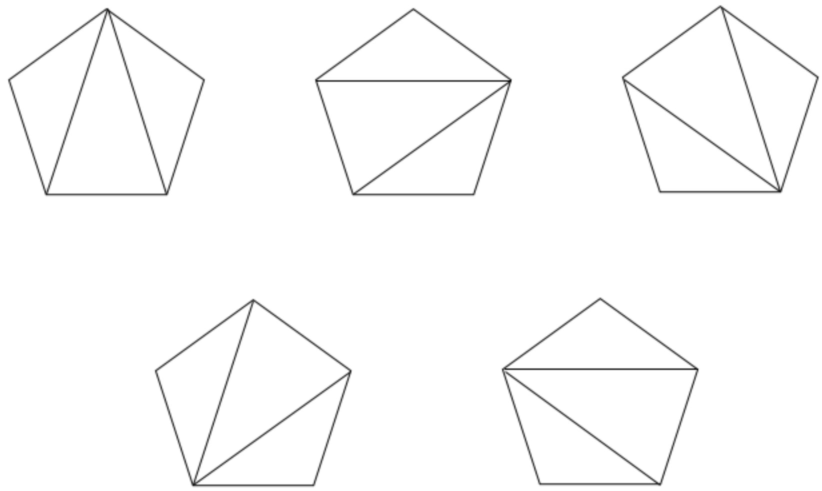


Figure 1.6: Dissections of A Pentagon

$$f(n,k) = \frac{1}{n+k}\binom{n+k}{k+1}\binom{n-3}{k}.$$

For example, if we use two non-crossing diagonals to dissect a pentagon, what we get is shown in Figure 1.6. It is easy to see that $f(5,2) = \frac{1}{7}\binom{7}{3}\binom{2}{2} = 5$.

A q-analogue of $f(n,k)$ is given by $f(n,k;q)$ with

$$f(n,k;q) := \frac{1}{[n+k]_q}\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q \begin{bmatrix} n-3 \\ k \end{bmatrix}_q.$$

Theorem 1.3.3. *[10] Let X be the set of dissections of a convex n-gon using k non-crossing diagonals. Let G be the cyclic group of order n acting on X by rotation. Let $X(q) := f(n,k;q)$. Then the triple $(X, X(q), G)$ exhibits the cyclic sieving phenomenon.*

Recall that the non-crossing partitions are enumerated by the Catalan numbers

$$|NC(n)| = \mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Let $NC(n,k)$ be the set of non-crossing partitions of the set $[n]$ with k blocks. $|NC(n,k)|$ is counted by the Narayana numbers

$$|NC(n,k)| = \mathrm{Nar}(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

A q-analogue of $\mathrm{Cat}(n)$ is given by

$$\mathrm{Cat}(n) = \frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q,$$

and a q-analogue of $\mathrm{Nar}(n)$ is given by

$$\mathrm{Nar}(n,k) = \frac{1}{[n]_q}\begin{bmatrix}n\\ k\end{bmatrix}_q\begin{bmatrix}n\\ k-1\end{bmatrix}_q.$$

Then we have a very important result as follows:

Theorem 1.3.4. *[10] Let G be the cyclic group of order n acting on $NC(n)$ by rotation. Then the triples $(NC(n), \mathrm{Cat}(n), G)$ and $(NC(n), \mathrm{Nar}(n,k), G)$ exhibit the cyclic sieving phenomenon.*

1.4 Representations of Groups Acting on Finite Posets

Definition 1.4.1. Let G be a group and let P be a poset (partially ordered set), a group action ϕ of G on P is a function: $\phi : G \to \mathrm{Aut}(P)$ which satisfies the following three axioms: (with abuse of notations we write $\phi(g)$ as g)

1. Identity: $1_G \cdot a = a, \forall\ a \in P$;

2. Compatability: $gh \cdot a = g(h \cdot a), \forall g, h \in G$ and $a \in P$;

3. Preserving order: If $a \leq_P b$, then $g(a) \leq_P g(b), \forall\ a, b \in P, \forall\ g \in G$.

Let P be a bounded finite poset, i.e. there are two elements $\hat{0}$ and $\hat{1}$ in P, such that $\hat{0} \leq a$, and $a \leq \hat{1}, \forall\ a \in P$, and let G be a finite group of automorphism of P. G is actually a subgroup of $\mathrm{Aut}(P)$ where $\mathrm{Aut}(P)$ denotes the full group of automorphism of P. Each element of G permutes the elements of P, and hence the action of G on P defines a certain permutation representation of G. Note that if P is graded, then the action necessarily preserves the grading.

Remark. The character of a permutation representation at some $g \in G$ is equal to the number of elements in P fixed by the action of g, which is equal to the number of 1's on the diagonal of the matrix under a certain basis. All the other entries on the diagonal are all 0, since there is at most one 1 on each row and column of the matrix.

Let P be a bounded graded poset. S is a subset of $[n-1]$, the rank-selected subposet of P is defined by

$$P_S = \{x \in P : x = \hat{0} \text{ or } \hat{1}, \text{or the rank of } x : r(x) \in S\}.$$

Definition 1.4.2. [18] Let G act on the maximal chains of P_S. Let α_S^P (for simplicity, just write as α_S) be the character of this action. The character of this representation evaluated at $g \in G$ is denoted as $\alpha_S(g)$.

In other words, $\alpha_S(g)$ counts the number of maximal chains of P_S fixed by the element $g \in G$.

In particular, $\alpha_S(1)$ is just the number of maximal chains in P_S.

Definition 1.4.3. Define β_S based on α_S satisfying the Principle of Inclusion-Exclusion, i.e.

$$\alpha_S = \sum_{T \subset S} \beta_T,$$

$$\beta_S = \sum_{T \subset S} (-1)^{|S-T|} \alpha_T.$$

In general, β_S can be regarded as a *virtual* representation of G. Let P be any poset with $\hat{0}$ and $\hat{1}$. Define the *order complex* $\Delta(P)$ to be the simplicial complex whose vertices are the elements of $P - \{\hat{0}, \hat{1}\}$ and whose faces are the chains in $P - \{\hat{0}, \hat{1}\}$.

Sometimes β_S is not only virtual but acutal (i.e. all the coefficients of irreducible representations are non-negative integers). For example, when P has EL-labeling, β_S is an actual representation [16]. $NC(n)$ also has EL-labeling [13]. Hence, in our discussion, β_S is an actual representation.

Denote the simplicial homology groups by $\tilde{H}_i(\Delta(P), \mathbb{C})($ or just $\tilde{H}_i(\Delta(P)))$. Since every element g of G is order-preserving, G also acts on each homology group $\tilde{H}_i(\Delta(P_S))$, $-1 \leq i \leq |S| - 1$.

Let $\gamma_{S,i} : G \to \mathrm{Hom}(\tilde{H}_i(\Delta(P_S), \tilde{H}_i(\Delta(P_S))$ denote above representation of G. Then we have

$$\beta_S = \sum_i (-1)^{|S|-1-i} \gamma_{S,i}.$$

In particular, when $S = \emptyset$, $\beta_\emptyset$ is just the trivial representation, hence $\beta_\emptyset(g) = 1, \forall\ g \in G$.

Example 1.4.4. Let $D_{2\cdot 4}$ be the dihedral group of order 8 which acts on $NC(4)$. Let r and f be the rotational and reflectional generators of $D_{2\cdot 4}$ respectively. We can compute all $\alpha_S{}'s$ and $\beta_S{}'s$ explicitly for $S \subset [2]$ as follows:

S	$\alpha_S(e)$	$\alpha_S(F)$	$\alpha_S(RF)$	$\alpha_S(R)$	$\alpha_S(R^2)$
$\emptyset$	1	1	1	1	1
$\{1\}$	6	2	2	0	0
$\{2\}$	6	2	2	0	0
$\{1,2\}$	16	2	2	0	0

S	$\beta_S(e)$	$\beta_S(F)$	$\beta_S(RF)$	$\beta_S(R)$	$\beta_S(R^2)$
$\emptyset$	1	1	1	1	1
$\{1\}$	5	1	1	-1	-1
$\{2\}$	5	1	1	-1	-1
$\{1,2\}$	5	-1	-1	1	1

Figure 1.2 gives us a good interpretation of all the α's. For instance, $\alpha_{\{1,2\}}(F)$ is the number of maximal chains which are fixed by the action of $F \in D_{2n}$.

Montenegro once came up with an idea in his unpublished manuscript to compute $\beta_{[n-2]}$ for $NC(n)$. We will take a look into this and use his idea to compute $\beta_{[n-2]}$ explicitly in section 3.1.

1.5 Outline of the Thesis

The main results of this thesis consist of two parts: structural studies of non-crossing partitions which are fixed by reflections, and the characters of the dihedral group D_{2n} acting on the lattice of $NC(n)$.

In Chapter 2, we will temporarily restrict our investigation to the sub-lattice of $NC(n)$ in which all the elements are fixed points under the action of $F \in D_{2n}$

$(NC(n)^F)$.[1]

We have already seen in section 1.2 that when n is odd or even, we have different classifications of conjugacy classes. However, we will later introduce Kreweras Complement which will show that we have the same result for the sublattices of $NC(n)$ fixed by F and RF despite the parity of n (which is actually an anti-isomorphism). Hence, we only need to investigate $NC(n)^F$.

We will first investigate the enumerative properties of $NC(n)^F$ by establishing a nice bijection between $NC(n)^F$ and $NC(n)^{R^{\lfloor n/2 \rfloor}}$ which tells us that the number of $NC(n)^F$ is just equal to the central binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$.

Next, we will prove a recurrence relation $NC(n)^F \overset{\text{as set}}{\cong} \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor}(NC(i) \times NC(n-1-2i)^F)$ using a combinatorial bijection. With the help of this relation and the result from section 2.1 we obtain a nice formula $\binom{n}{\lfloor \frac{n}{2} \rfloor} = \sum_{i=0}^{\lfloor n/2 \rfloor} \mathrm{Cat}(i) \cdot \binom{n-1-2i}{\lfloor \frac{n-1-2i}{2} \rfloor}$.

The last thing we are going to do in the second chapter is to restrict our investigation to a "pruned" sublattice of $NC(n)^F$ in which all the elements live in the maximal chains of length n. We will show that the number of elements in the pruned lattice is counted by the Fibonacci numbers and prove an interesting theorem that the pruned lattice is isomorphic to the lattice of order ideals of Zigzag poset, hence it is a distributive lattice.

In Chapter 3, we will examine the α and β characters defined by Stanley [18] of D_{2n} acting on $NC(n)$.

We will first describe some unpublished results of Montenegro and then we will build on these results and compute all $\beta_{[n-2]}$.

In the following section of this chapter we will compute all the $1-$rank-selected α characters evaluated at all the elements in the dihedral group D_{2n}. In particular, we will compute $\alpha_{[k]}(F)$ for some $k \in [n]$, which is the number of non-crossing

[1] Some notations which appear in this section will be introduced in latter chapters.

partitions fixed by reflection with exactly k blocks and show that this is equal to the $q-$Narayana number $\mathrm{Nar}_q(n,k)$ with q evaluated at -1.

For the last section of Chapter 3, we will compute $\alpha_{[n-2]}$ evaluated at all $g \in D_{2n}$, where an interesting result is that $\alpha_{[n-2]}$ evaluated at F is counted by the Euler number.

In the last section, we will discuss some open problems and suggestions for future research.

Chapter 2

Poset Structure on Non-crossing Partitions Fixed by a Reflection

In the lattice of $NC(n)$, the set of non-crossing partitions fixed by rotations are basically understood enumeratively and algebraically [9, 10]. However, the set of non-crossing partitions fixed by a reflection has not been investigated that much.

Definition 2.0.1. Let $NC(n)^g$ denote the set of non-crossing partitions of the set $[n]$ fixed by action of g, where $g \in D_{2n}$.

Theorem 2.0.2. *For all $g \in D_{2n}$, $NC(n)^g$ is a lattice.*

Proof. Given $g \in D_{2n}$, we define $NC(n)^g = \{\pi \in NC(n) : g(\pi) = \pi\}$.

For all $\tau, \sigma \in NC(n)^g$, by definition of join and meet of two elements, we have $\tau \leq \tau \vee \sigma$ and $\sigma \leq \tau \vee \sigma$, which implies that $\tau = g(\tau) \leq g(\tau \vee \sigma)$ because $g \in \mathrm{Aut}(NC(n))$ which preserves ordering. Similarly, $\sigma = g(\sigma) \leq g(\tau \vee \sigma)$, and hence $\tau \vee \sigma \leq g(\tau \vee \sigma)$.

On the other hand, $g^{-1} \in D_{2n}$ and $g^{-1}(\tau) = \tau, g^{-1}(\sigma) = \sigma$, which shows that g^{-1} is also a stabilizer for both τ and σ. We may conclude that $\tau = g^{-1}(\tau) \leq g^{-1}(\tau \vee \sigma)$ and $\sigma = g^{-1}(\sigma) \leq g^{-1}(\tau \vee \sigma)$, which shows that $\tau \vee \sigma \leq g^{-1}(\tau \vee \sigma)$. By applying g on both sides, we obtain $g(\tau \vee \sigma) \leq \tau \vee \sigma$.

To sum up, we have $\tau \vee \sigma = g(\tau \vee \sigma)$, hence $\tau \vee \sigma \in NC(n)^g$.

Similarly, we may prove that $\tau \wedge \sigma \in NC(n)^g$ as well.

Hence, for all $\tau, \sigma \in NC(n)^g$, there is a join and meet in $NC(n)^g$ (actually it is their original join and meet in $NC(n)$). Therefore $NC(n)^g$ is a lattice for all $g \in D_{2n}$. □

Recall from Theorem1.2.2 that when n is even the reflections fall into two conjugacy classes:

- one with F: $\{F, R^2F, R^4F, \ldots\}$
- one with RF: $\{RF, R^3F, R^5F, \ldots\}$

We will show that the lattices $NC(n)^F$ and $NC(n)^{RF}$ are anti-isomorphic under a map which is is called the *Kreweras Complement* [7] and hence we only need to study the behavior of $NC(n)^F$.

To define the Kreweras complement of a non-crossing partition $\pi \in NC(n)$, we first add n imaginary vertices between the existing n vertices. By connecting those imaginary vertices in the maximal way of not crossing the blocks of π, we obtain a non-crossing partition of $[n]$ on the imaginary vertices. Define this map to be K and we established a bijection

$$K : NC(n) \to NC(n),$$

which is called the *Kreweras Complement*.

Example 2.0.3. Let $\pi = \{\{1,4,5\},\{2,3\},\{6\}\}$. Figure 2.1 shows $K(\pi) = \{\{1,6\},\{2,4\}\}$.

Theorem 2.0.4. *[2, section 4.2] K defines an anti-isomorphism between $NC(n)^F$ and $NC(n)^{RF}$.*

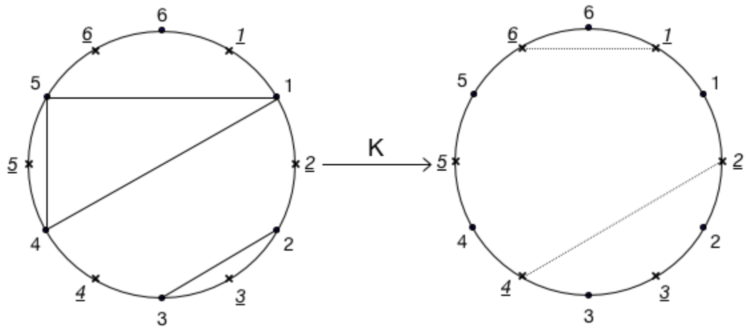


Figure 2.1: Kreweras Complement

Proof. One must first show that K sends $NC(n)^F$ into $NC(n)^{RF}$. Let $\pi \in NC(n)^F$, for a given symmetric axis of π which passes through vertex 1, by Kreweras Complement, we do the same connections to both sides of the symmetric axis and $K(\pi)$ is still a non-crossing partition. K rotates the symmetric axis by a rotation R, hence $K(\pi) \in NC(n)^{RF}$.

Secondly, for any π and $\tau \in NC(n)^F$ satisfying $\pi \leq \tau$. By the definition of Kreweras Complement, if two imaginary vertices are connected in $K(\tau)$, so are those in $K(\pi)$. The other direction might not be true. Hence $K(\pi) \geq K(\tau)$.

The result follows. □

Example 2.0.5. In Figure 2.2, we see that the lattice of $NC(4)^F$ is anti-isomorphic to the lattice of $NC(4)^{RF}$.

In this chapter, we will first discuss the enumerative properties of $NC(n)^F$, and then take a look into the poset structures on $NC(n)^F$ and the "pruned sublattice" $NC(n)^F_{pr}$.

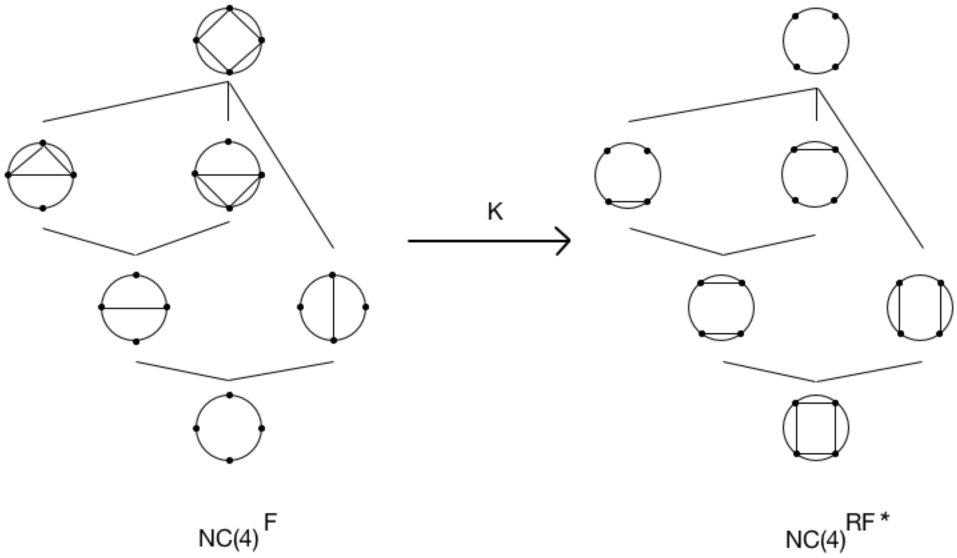


Figure 2.2: Anit-isomorphism under Kreweras Complement

2.1 Enumeration on $NC(n)^F$

Recall from Chapter 1 that the set of non-crosing partitions in $NC(n)$ exhibits the cyclic sieving phenomenon under the cyclic rotations of the $n-$gon, and we have the following important theorem:

Theorem 2.1.1. *[10, Theorem 7.2] The number of non-crossing partitions in $NC(n)$ fixed by R^d is counted by the $q-$Catalan Number evaluated at the d-th root of the unity, i.e.*

$$\#NC(n)^{R^d} = \mathrm{Cat}_q(n)|_{q=e^{2\pi i \frac{d}{n}}},$$

where R is the rotational generator of the dihedral group of D_{2n}.

In particular, when n is even, the number of non-crossing partitions of the set $[n]$ fixed by $R^{\frac{n}{2}}$ is counted by the $q-$Catalan Number evaluated at $q = -1$, which is $\mathrm{Cat}_q(n)|_{q=-1} = \binom{n}{n/2}$.

D.Callan and L.Smiley proved a similar result for $NC(n)^F$.

Theorem 2.1.2. *[3, Theorem 1] The number of self-complementary non-crossing partitions of [n] is* $\binom{n}{\lfloor n/2 \rfloor}$.

A non-crossing partition $\pi \in NC(n)$ is called $self-complementary$ if $F(\pi) = \pi$., which is clearly equivalent to say that $\pi \in NC(n)^F$. Hence we know that $\#NC(n)^F = \binom{n}{\lfloor n/2 \rfloor}$.

However, they did not notice the connection with $q-$Catalan numbers. Callan and Smiley's proof uses a bijection to lattice paths. In this section we will give a new bijective proof by relating $NC(n)^F$ to $NC(n)^{R^{\lfloor n/2 \rfloor}}$.

Recall that the $q-$analogue of n, which is denoted as $[n]_q$, is the polynomial $1 + q + q^2 + \ldots + q^{n-1}$.

Hence

$$[n]_q\big|_{q=-1} = \begin{cases} 1, & n \text{ is odd,} \\ 0, & n \text{ is even.} \end{cases}$$

Also note that the $q-$binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q[k-1]_q \cdots [1]_q}.$$

It is clear that $\frac{[n]_q}{[k]_q}\Big|_{q=-1} = 1$ if both n and k are odd, and that $\frac{[n]_q}{[k]_q}\Big|_{q=-1} = 0$, if n is even and k is odd.

Lemma 2.1.3. For both n and k even, $lim_{q \to -1} \dfrac{[n]_q}{[k]_q} = \dfrac{n}{k}$.

Proof. We may use L'Hospital's Rule to evaluate this limit quickly:

$$\lim_{q \to -1} \frac{[n]_q}{[k]_q} = \lim_{q \to -1} \frac{[n]'_q}{[k]'_q} = \lim_{q \to -1} \frac{1 + 2q + 3q^2 + \ldots + (n-1)q^{(n-2)}}{1 + 2q + 3q^2 + \ldots + (n-1)q^{(k-2)}}$$

$$= \frac{1-2+3-4+\ldots-(n-2)+(n-1)}{1-2+3-4+\ldots-(k-2)+(k-1)}$$
$$= \frac{1+\frac{n-2}{2}}{1+\frac{k-2}{2}} = \frac{\frac{2+n-2}{2}}{\frac{2+k-2}{2}}$$
$$= \frac{n}{k}.$$

□

Lemma 2.1.4. $\mathrm{Cat}_q(n)|_{q=-1} = \binom{n}{\lfloor n/2 \rfloor}$.

Proof. We only prove the case when n is odd, the other case is easy to check by the reader. For any even number $2m$, we have

$$(2m)!! = 2m \cdot (2m-2) \cdot (2m-4) \cdots 2 = 2^m m!.$$

Note that as $q = -1$,

$$\begin{aligned}
\mathrm{Cat}_q(n) &= \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix} = \frac{1}{[n+1]_q} \cdot \frac{[2n]_q[2n-1]_q[2n-2]_q \cdots [n+1]_q}{[n]_q[n-1]_q \cdots [2]_q[1]_q} \\
&= \frac{[2n]_q}{[n+1]_q} \cdot \frac{[2n-1]_q}{[n]_q} \cdot \frac{[2n-2]_q}{[n-1]_q} \cdots \frac{[n+1]_q}{[2]_q} \cdot \frac{1}{[1]_q} \\
&= \frac{2n}{n+1} \cdot 1 \cdot \frac{2n-2}{n-1} \cdot 1 \cdot \cdots \cdot \frac{n+1}{2} \cdot 1 \\
&= \frac{(2n)(2n-2)\cdots(n+1)}{(n+1)(n-1)\cdots(2)} \\
&= \frac{(2n)!!}{(n-1)!!(n+1)!!} \\
&= \frac{n!}{(\frac{n-1}{2})! \cdot (\frac{n+1}{2})!} \\
&= \binom{n}{\lfloor \frac{n}{2} \rfloor}
\end{aligned}$$

□

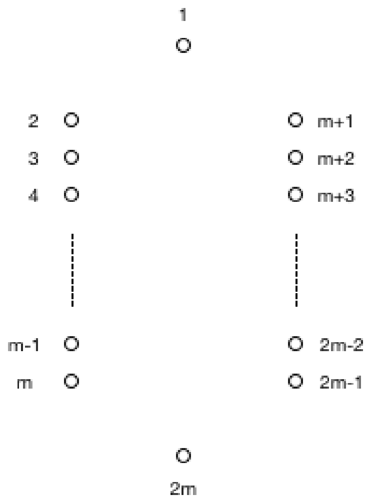


Figure 2.3: Labelling of n-gon

Based on the lemma above and the result from theorem 2.1.1 and establish a different combinatorial bijection to prove the following theorem:

Theorem 2.1.5. *The number of non-crossing partitions in $NC(n)$ fixed by F is counted by the $q-$Catalan Number evaluated at $q = -1$, which is,*

$$\#NC(n)^F = \mathrm{Cat}_q(n)|_{q=-1}.$$

Proof. We will break our proof in two pieces:

Case 1. When n is even ($n = 2m$).

We can build find a nice bijection between $NC(n)^{R^m}$ and $NC(n)^F$.

Since we can regard elements of the set $[n]$ as vertices of an $n-$gon, we can label the vertices of such an $n-$gon as in the Figure 2.3:

Label the top and bottom vertices as 1 and $2m$, left column from 2 to m and right column from $m+1$ to $2m-1$. We call two vertices (a, b) a pair of *anitpodes* if the sum of labels equals $2m+1$.

Clearly, if $\pi \in NC(n)$ is a non-crossing partition fixed by $R^{n/2}$ then $i, j \in \{2, 3, \ldots, m\}$ are in the same block if and only if $2m+1-i$ and $2m+1-j$ are in the same block on the right column.

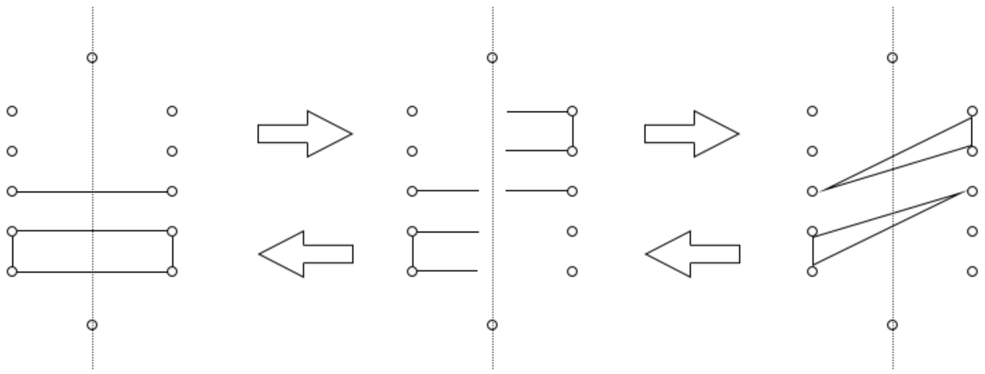
Sub-Case 1. Suppose 1 is a singleton, then $2m$ is also a singleton.

For $\forall\ \pi \in NC(n)^F$, cutting through the symmetric axis connecting 1 and $2m$, we obtain two non-crossing partitions, one is on the set $A = \{2, 3, \ldots, m\}$, the other is on the set $B = \{m+1, m+2, \ldots, 2m-1\}$.

We say that a block in A (B) is an *external block* if this block is previously connected to some vertices in B.

Define a map $\phi : NC(n)^{R^m} \to NC(n)^F$ as follows: Given $\pi \in NC(n)^{R^m}$, reversing the labels of B from $2m-1$ to $m+1$, there will be the same number of external blocks in A and B, since π is invariant under a rotation of $180°$. Hence there will be a unique way to connect all the external blocks of A and B in a "non-crossing" way. It is easy to see $\phi(\pi) \in NC(n)^F$.

This map is also invertible. Consider any non-crossing partition fixed by reflection. Cutting through the symmetric axis and reversing the labelling of B, there is a unique way to connect the external blocks from A and B to make partition non-crossing.



Hence we established a bijection between $NC(n)^{R^{n/2}}$ and $NC(n)^F$, the result follows.

Sub-Case 2. Suppose 1 is a not singleton, neither is $2m$.

Define the map ϕ as above. Then ϕ will send π to some $\pi' \in NC(n)^F$ with 1 or $2m$ singleton depending on whether $2m$ or 1 is connected to some vertex from A.

Case 2. When n is odd ($n = 2m - 1$).

We can do the same labelling as in the first part and add an auxiliary vertex at the bottom and label it as $2m$.

From the first case we know that $\#NC(2m)^F = \#NC(2m)^{R^m} = \mathrm{Cat}_q(n)|_{q=-1} = \binom{2m}{m}$.

Define the path connecting any two vertices which are symmetric via the reflection axis as a *bridge*. Given an element in $NC(2m-1)^F$, by adding the auxiliary vertex $2m$, we may obtain two elements in $NC(2m)^F$: one with vertex $2m$ isolated, the other one with $2m$ in the block connected to the lowest bridge. Hence we obtain two subsets of $NC(2m)^F$ with the same size.

Notice that $\frac{\binom{2m}{m}}{2} = \binom{2m-1}{m}$, which is easy to see via Pascal's triangle. We obtain

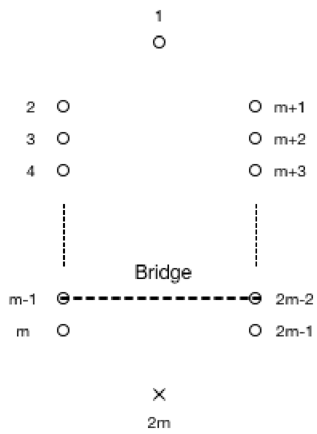


Figure 2.4: Bridge between vertices of a $n-$gon

that $\#NC(2m-1)^F = \binom{2m-1}{m} = \mathrm{Cat}_q(2m-1)\big|_{q=-1} = \binom{n}{\lfloor n/2 \rfloor}$.

□

2.2 Structural Decomposition of $NC(n)^F$

In section 2.1, we discussed the enumeration of $NC(n)^F$ and showed it is counted by the central binomial coefficient. In this section we will go into to $NC(n)^F$ and establish a structural recurrence on it, which leads to a relation between the central binomial coefficients and the Catalan numbers.

Definition 2.2.1. Label vertices of a regular $n-$gon as before, define the *height* of a vertex i as follows:

$$ht(i) = \begin{cases} i & \text{if } i \leq \lceil \frac{n}{2} \rceil, \\ i - \lceil \frac{n}{2} \rceil + 1 & \text{if } i > \lceil \frac{n}{2} \rceil. \end{cases}$$

For example, if $n = 2m$, the heights of vertices are shown in Figure 2.5.

Definition 2.2.2. If two vertices have the same height, they are called a *mirror pair*. The vertex from the left column is called the *left image*. Similarly we can define the *right image*.

Consider the lattice of $NC(n)^F$, we may think of decomposing the lattice into several sublattices and each of the sublattices is isomorphic to the product of a copy of the lattice of non-crossing partitions and a copy of the lattice of non-crossing partitions fixed by reflection, which is proved in the following theorem.

Theorem 2.2.3 (Decomposition of $NC(n)^F$)**.**

$$NC(n)^F \overset{as\ set}{\cong} \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor} (NC(i) \times NC(n-1-2i)^F),$$

where we define $NC(0)$ and $NC(-1)^F$ to be posets with one element.

Proof. We will give a combinatorial proof of the theorem.

Consider the block containing vertex 1. Since for any $\pi \in NC(n)^F$, if vertex 1 is in the same block with a vertex with height k, then this block must contain all vertices with height k. For a fixed n, there are $\lceil \frac{n}{2} \rceil$ different heights. Let j be the maximal height of the vertices in the same block with vertex 1. Then the set of possible heights is actually the set $[\lceil \frac{n}{2} \rceil]$.

Now consider two different set of vertices: one with all vertices of heights higher than j, the other with all vertices of heights lower or equal to j. For a fixed set of vertices with heights lower or equal to j, the remaining $n - 2j + 1$ vertices can be partitioned in a non-crossing way fixed by reflection as in $NC(n - 2j + 1)^F$. Consider all the vertices with heights lower or equal to j. All these vertices are coming from j heights, and the let image and the right image of a mirror pair

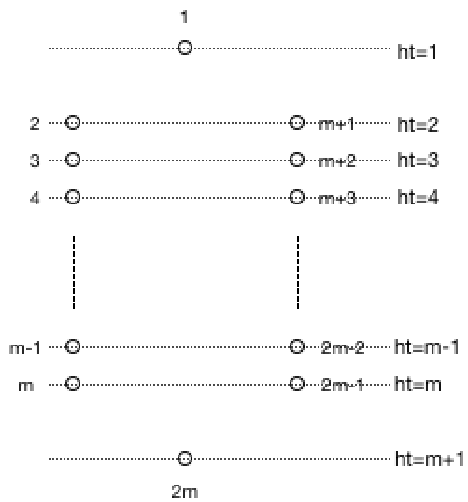


Figure 2.5: Heights of vertices

should have the same behavior, i.e. either they are in the same block or they are both isolated pairs themselves. Hence we may only consider "half" of them, that is, vertices from 1 to j directly (the pattern of the other half could be copied from it). What we need actually is just that those j vertices are partitioned in a non-crossing way. The number of ways we can do to those j vertices is exactly $\#NC(j)$, which is $\mathrm{Cat}(j)$.

The biggest such j we can find is $\lceil \frac{n}{2} \rceil$ and the smallest j is 1. Let $i = j - 1$, the result follows.

□

It might be clearer if we take a look a the picture of the case in $NC(6)$ as shown in Figure 2.6.

The lattice of $NC(6)^F$ is decomposed into four sub-lattices. The first lattice

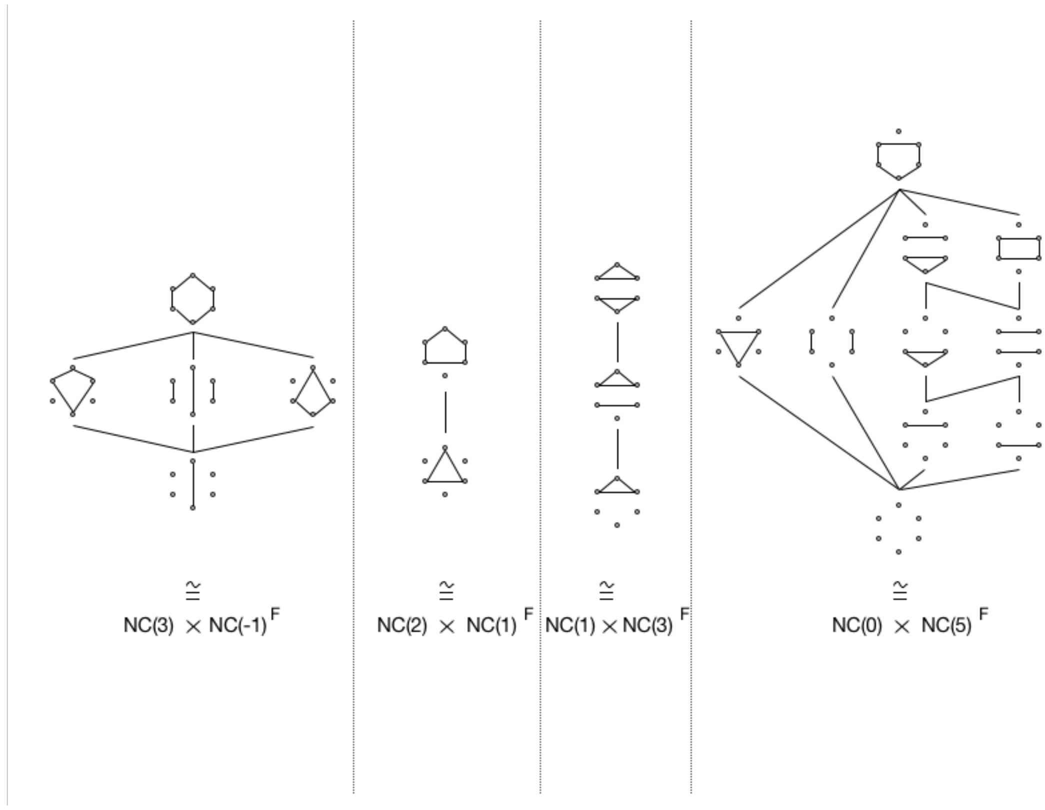


Figure 2.6: Structural decomposition of $NC(6)^F$

consists of all non-crossing partitions fixed by reflection with vertex 1 connected with vertex 6. In this case, our maximal height connected to vertex 1 is 3 and we only need to make the partition of the set [3] (all left images of mirror pairs together with vertices 1 and 6) non-crossing. This sub-lattice is clearly isomorphic to $NC(3)$ and hence isomorphic to $NC(3) \times NC(-1)^F$ by defining $NC(-1)^F$ to be a poset with one element. The middle two sub-lattices are isomorphic to $NC(2) \times NC(1)^F$ and $NC(1) \times NC(3)^F$ respectively. The rightmost sub-lattice, by ignoring the vertex 1, is just the lattice of $NC(0) \times NC(5)^F$, where $NC(0)$ is defined as a poset with one element.

Since we know that $NC(n)$ is counted by the Catalan numbers and $NC(n)^F$ is counted by the central binomial coefficients, we obtain the corollary as follows:

Corollary 2.2.4.

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} = \sum_{i=0}^{\lfloor n/2 \rfloor} \mathrm{Cat}(i) \cdot \binom{n-1-2i}{\lfloor \frac{n-1-2i}{2} \rfloor}.$$

This is a very nice recursive formula which illustrates the relation between the central binomial coefficients and the Catalan numbers.

Definition 2.2.5. A *generating function* is a formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

whose coefficients a_n encode information about a sequence of numbers that is indexed by the natural numbers.

Example 2.2.6. The generating function for the Catalan numbers [15] is

$$C(x) = \sum_{n=0}^{\infty} \mathrm{Cat}(n)x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

The generating function for the central binomial coefficients [11, A001405] is

$$G(x) = \sum_{n=1}^{\infty} \binom{n}{\lfloor n/2 \rfloor} x^n = \frac{1-4x^2-\sqrt{1-4x^2}}{4x^3-2x^2}.$$

We can also approach this formula in Corollary 2.2.4 from the perspective of generating functions of posets.

Let $ncf(n) = \#NC(n)^F$, and $F(x) = 1 + x + x^2G(x)$. It is easy to see from Theorem 2.1.5 that

$$F(x) = \sum_{n=-1}^{\infty} ncf(n)x^n,$$

with the convention that $ncf(-1) = ncf(0) = 1$.

Define two generating functions $F_e(x)$ and $F_o(x)$ by

$$F_e(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n,$$
$$F_o(x) = \sum_{n=0}^{\infty} \binom{2n+1}{n} x^n.$$

It is a well known result [11, A000984] that $F_e(x) = \dfrac{1}{\sqrt{1-4x}}$, and we can also obtain that $F_o(x) = \dfrac{F(x)+F(-x)}{2}$.

One may establish Corollary 2.2.4 by check the equalities

$$\begin{cases} F_e(x) = C(x)F_o(x), \\ \frac{F_o(x)-1}{x} = C(x)F_e(x). \end{cases}$$

2.3 Pruned Sublattice of $NC(n)^F$

We have seen in the section 2.1 that $NC(n)^F$ is counted by the central binomial coefficient. In this section we will look into a very special subposet of $NC(n)^F$, which is the "pruned sublattice".

Before we investigate "pruned sublattice" of $NC(n)^F$, recall that a lattice is a poset L for which every pair of elements has a least upper bound and greatest lower bound.

From theorem 2.0.2, we know that $NC(n)^F$ is a lattice.

Recall from Chapter 1 that the lattice of $NC(n)$ is graded with rank function as

$$rank(\pi) := n - |\pi|,$$

where $|\pi|$ is the number of blocks in π.

However, the lattice $NC(n)^F$ is not graded. In Figure 2.2, we see that there is a maximal chain in $NC(4)^F$ of only length 2. Nevertheless, we can get sublattice of $NC(n)^F$ which is graded by deleting some bad elements, which leads to the idea of "Pruned Sublattice of $NC(n)^F$".

Definition 2.3.1. The *pruned poset of* $NC(n)^F$ is a subposet of $NC(n)^F$ where all elements lie in a chain of maximal length in $NC(n)^F$ with length n.

Recall the labelling of the $n-$gon we developed in Figure 2.1. We can specify two special types of pairs of blocks in an element $\pi \in NC(n)^F$.

Definition 2.3.2. Let $\pi \in NC(n)^F$. A pair of type A π is a pair of symmetric blocks with respect to the reflection axis where each block consists of more than one vertex from only either the left column of vertices $\{2, 3, \ldots, m\}$ or the right column $\{m + 1, m + 2, \ldots, 2m - 1\}$.

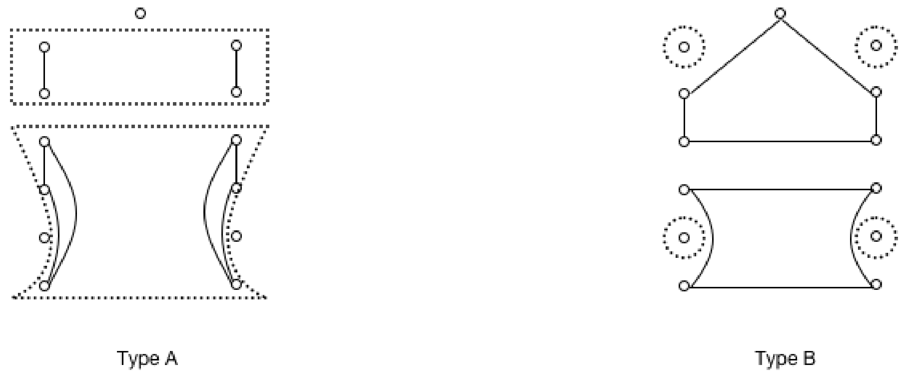


Figure 2.7: Examples of pairs of type A and type B

A pair of type B is pair of two symmetric vertices with respect to the reflection axis, by connecting which we will obtain a crossing partition.

Example 2.3.3. Figure 2.7 shows examples of pairs of type A and B. on the left picture, the pairs in the dotted boxes are two pairs of type A. On the right picture, there are two pairs of singletons (each pair with the same height), by crossing which we cannot avoid crossing. Hence, these two pairs are of type B.

Note that type A and type B are related by Kreweras Complement. If we have a pair is of type A in $\pi \in NC(n)^F$, then we have a corresponding pair of type B in $K(\pi)$.

Theorem 2.3.4. *The pruned poset of $NC(n)^F$ contains all elements avoiding symmetric pairs of type A and type B.*

Proof. For any $\pi \in NC(n)^F$, in order to refine or coarsen π, we have to either break bridges connecting mirror vertices or break pairs of vertical edges in the left column of vertices and the right column of vertices simultaneously.

Since all elements in the pruned poset of $NC(n)^F$ must lie in a chain of $NC(n)^F$ with length n. It must be possible to get $\hat{0}$ and $\hat{1}$ by breaks and connections that only add or subtract one block. If π has a pair of type A, by breaking vertical mirror edges we will get at least two more blocks and there will be no way to break this par without adding two blocks. Similarly, if π has a pair of type B, there will be no way to merge these without subtracting two blocks. In either case, π cannot lie in a chain with length n.

Conversely, if $\pi \in NC(n)^F$ avoids pairs of type A and type B, then very block of π has the form of either a big bulk in the middle including mirror vertices from the left column of vertices and the right column, or a singleton by connecting whose mirror vertex we get a bridge (except for vertex 1 and $2m$ if it exists). For the mirror vertices in a big bulk, say vertices i and $i+m-1$ $(2 \leq i \leq m)$, if i is connected to some other vertex from A (or even vertex 1 or $2m$ if it exists), we may cut this edge and its mirror in B to add a block (actually we obtain a block with only one bridge). Otherwise, we simply cut the bridge connecting them. Similar thing happens when we want to subtract one block.

□

Note that the obtained pruned subposet of $NC(n)^F$ is also a sublattice, since it is a graded subposet of $NC(n)$ and the properties of lattice are easy to see.

Denote the pruned sublattice of $NC(n)^F$ as $NC(n)^F_{pr}$. Then we have the following:

Theorem 2.3.5. *$NC(n)^F_{pr}$ is counted by the Fibonacci Number F_{n-1}, where $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}, \forall\ n \geq 2$.*

Proof. We will use induction to prove the theorem:

(1) It is easy to check $\#NC(1)^F_{pr} = 1, \#NC(2)^F_{pr} = 2$.

(2) Suppose the result holds for all $n \leq k-1$, if we can show $\#NC(k)_{pr}^{F} = F_{k-1}$, then we are done.

- When k is even.

  If $k = 2m$, by the labelling in Figure 2.1, we notice that either vertex $2m$ is a singleton or it is in the block with the m and $2m-1$.

  All $\pi \in NC(k)_{pr}^{F}$ with $2m$ a singleton form a lattice which is isomorphic to the lattice of $NC(k-1)_{pr}^{F}$ since we can just simply ignore $2m$.

  All $\tau \in NC(k)_{pr}^{F}$ with $2m$ connected to m and $2m-1$ form a lattice which is isomorphic to the lattice of $NC(k-2)_{pr}^{F}$, since we can regard $2m$, m and $2m-1$ as a single vertex in the a $k-2$-gon which plays the role of $2m-2$ in the labelling of a $k-2$-gon.

- When k is odd. If $k = 2m-1$, by the labelling, the vertex 1 is either a singleton or it is in the block with 2 and $m+1$.

  Consider the lattice of $NC(k-1)_{pr}^{F}$. The Kreweras Complement will map $NC(k-1)_{pr}^{F}$ to a anti-isomorphic graded lattice, where we can label the left column of vertices from 2 to m and right column from $m+1$ to $2m-1$.

  All $\pi \in NC(k)_{pr}^{F}$ with 1 a singleton form a lattice which is isomorphic to the lattice of $NC(k-1)_{pr}^{F}$ by the anti-isomorphism above and ignoring 1.

  All $\tau \in NC(k)_{pr}^{F}$ with 1 connected to 2 and $m+1$ form a lattice which is isomorphic to the lattice of $NC(k-2)_{pr}^{F}$, since we can regard 1, 2 and $m+1$ as a single vertex in the a $k-2$-gon which plays the role of 1 in the labelling of a $k-2$-gon.

  In either case, $\#NC(k)_{pr}^{F} = \#NC(k-1)_{pr}^{F} + \#NC(k-2)_{pr}^{F} = F_{k-2} + F_{k-3} = F_{k-1}$. The result follows.

□

Definition 2.3.6. An *order ideal* of a poset P is a subset I of P such that if $t \in I$ and $u \leq v$ then $u \in I$.

The set of all order ideals of P, ordered by inclusion forms a poset denoted by $J(P)$.

Definition 2.3.7. Given a lattice L, L is said *distributive* if for any $s, t, u \in L$, the following two equalities hold:

1. $s \vee (t \wedge u) = (s \vee t) \wedge (s \vee u)$;
2. $s \wedge (t \vee u) = (s \wedge t) \vee (s \wedge u)$.

Theorem 2.3.8. *[16, Fundamental Theorem For Finite Distributive Lattices] Let L be a finite lattice. Then L is a distributive lattice if and only if there is a unique poset P (up to isomorphism) for which $L \cong J(P)$.*

Remark. Indeed, since the union and intersection of two order ideals is still an order ideal, we know that $J(P)$ is a distributive lattice for any poset P by the distributivity of set union and intersection.

Definition 2.3.9. A *Zigzag poset* of $[n]$, which is also called a $fence$, is a poset in which the order relations form a path with alternating orientations: $1 > 2 < 3 > 4 < 5 > \cdots n$, denoted as $Z_{[n]}$.

$NC(n)^F_{pr}$ has a nice poset structure which is shown by the following theorem:

Theorem 2.3.10. *The pruned sublattice $NC(n)^F_{pr}$ is isomorphic to the lattice of order ideals of Zigzag poset of [n-1].*

Proof. For a regular n-gon labelled as before, by connecting all the bridges and all the symmetric edges, we may label all the bridges up and down with even number

$\{2,4,6,\cdots\}$ and label all the other pairs of symmetric edges up and down with odd numbers $\{1,3,5,\cdots\}$. Such a labelling will give us a bijection between the pruned lattice and the lattice of order ideals of Zigzag poset.

Indeed, we have seen in Theorem 2.3.4 that if $\pi \in NC(n)^F_{pr}$, it must avoid pairs of type A and type B. Hence all the blocks of π has the form of either a big block with no curved edges in the middle including mirror vertices from both the left and right columns of vertices, or a mirror singletons by connecting which we get a bridge. Then every $\pi \in NC(n)^F_{pr}$ gives an order ideal with the labelling of existing edges and bridges in the order ideal. Conversely, for any order ideal, by joining the corresponding paths, we get an element of $NC(n)^F$ avoiding type of A and B, which tells it is in $NC(n)^F_{pr}$.

□

Example 2.3.11. In Figure 2.8, we see that the non-crossing partition $\{\{1,2,3,6,7\},\{4\},\{8\},\{5,9\}\} \in NC(9)^F_{pr}$ corresponds to the order ideal $\{1,2,3,4,8\} \in J(Z_{[8]})$ under the isomorphism in the preceding proof. Each bold line on the left corresponds to a solid point on the right with the same label. Note that even though the bridge connecting vertices of height 2, i.e. bridge with label 2, is not actually seen from the non-crossing partition, we still mark it because vertex 2 and 6 are in the same block.

Corollary 2.3.12. *The pruned lattice $NC(n)^F_{pr}$ is a distributive lattice.*

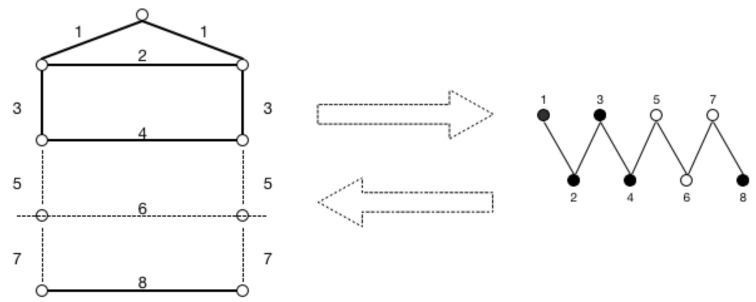


Figure 2.8: Isomorphism between $NC(n)^F_{pr}$ and $J(Z_{[n-1]})$

Chapter 3

Characters of the Dihedral Group Acting on Non-crossing Partition Lattices

We have seen from the first chapter that each element g in the Dihedral group D_{2n} acts as an automorphism of the non-crossing partition lattice $NC(n)$. In this chapter, we will first recall Montenegro's work in his unpublished manuscript [8] and compute the characters of a reflection acting on $NC(n)$, and extend the results to $NC(n)^F$ to finish the computation of $\beta_{[n-2]}$. In the next two sections, we will take a look at the α characters of rank selected subposets of $NC(n)$.

3.1 The Action of a Reflection on $NC(n)$

Definition 3.1.1. A finite poset P with $\hat{0}$ and $\hat{1}$ is called *Cohen-Macaulay* over $\mathbb{C}$ if for every $s < t$ in P, the order complex $\Delta(s,t)$ of the open interval (s,t) satisfies

$$\tilde{H}_i(\Delta(s,t),\mathbb{C}) = 0, \forall i < \dim\Delta(s,t),$$

where $\tilde{H}_i(\Delta(s,t),\mathbb{C})$ denotes the reduced simplicial homology with coefficients in $\mathbb{C}$.

Theorem 3.1.2. *[16] If P is Cohen-Macaulay, then the Möbius function of P* *alternates in sign.*

It is known that the lattice of $NC(n)$ is Cohen-Macaulay [13]. Hence the action of the Dihedral group D_{2n} on the top homology of the order complex has its character $\beta_{[n-1]}$ differing from the Möbius invariant $\mu_{NC(n)^F}(\hat{0}, \hat{1})$ at most by a sign.

Definition 3.1.3. A *closure operation* on P is a map $x \to \bar{x}$ satisfying:

1. $x \leq \bar{x}$,
2. $x \leq y$ then $\bar{x} \leq \bar{y}$,
3. $\bar{\bar{x}} = \bar{x}$.

Theorem 3.1.4. *[4]*

$$\sum_{z \in P, \bar{z} = \bar{y}} \mu_P(x, z) = \begin{cases} \mu_{\bar{P}}(\bar{x}, \bar{y}) & \textit{if } x = \bar{x}, \\ 0 & \textit{otherwise.} \end{cases}$$

where $\bar{P} = \{\bar{x} : x \in P\}$.

Definition 3.1.5. Let L be a finite lattice. An *atom* of L is an element which covers $\hat{0}$. An *coatom* of L is an element which is covered by $\hat{1}$.

Theorem 3.1.6. *[1] For a lattice L, if $\hat{1}$ is not the join of atoms, then the Möbius invariant $\mu_L = 0$. Similarly, if $\hat{0}$ is not the meet of coatoms, then $\mu_L = 0$.*

The following theorem and proof are from the unpublished manuscript of Montenegro, here we did some organization and summarize as follows:

Theorem 3.1.7. *[8]*

$$\mu_{NC(n)^F} = \begin{cases} 0 & \textit{if } n \textit{ is odd,} \\ -\mu_{NC(n/2)} & \textit{if } n \textit{ is even.} \end{cases}$$

Remark. Since the Kreweras Complement maps $NC(n)^F$ to $NC(n)^{RF}$, which is actually an anti-isomorphism, we have that $\mu_{NC(n)^{RF}} = \mu_{NC(n)^F}$.

Proof. **Case 1**: When $n = 2m + 1$.

Labelling vertices as before in Figure 2.1, we have that the atoms of $NC(n)^F$ are those non-crossing partitions with exactly one mirror pair connected, i.e. with exactly one block {i, m+i-1} of size greater than one. The join of all the atoms is the non-crossing partition with exactly two blocks, one of which is the vertex 1 as singleton, which is not $\hat{1}$. By the theorem above, $\mu_{NC(n)^F} = 0$.

Case 2: When $n = 2m$.

We may consider the Möbius function $\mu_{NC(n)^{F*}}$ on the dual lattice $NC(n)^{F*}$, since $\mu_{NC(n)^{F*}} = \mu_{NC(n)^F}$. Let π_0 be the partition containing exactly two blocks with vertex 1 as a singleton. Define a closure operation on $NC(n)^{F*}$ by sending $\hat{0}^* \to \hat{0}^*$ and $\pi \to \pi \wedge \pi_0$ for $\pi \neq \hat{0}^*$, where $\wedge$ is the regular meet in $NC(n)$. Then clearly the quotient lattice $Q = \{\pi : \bar{\pi} = \pi\}$ has the unique atom π_0 and hence $\mu_Q = 0$.

Consider a special element $\tau \in NC(n)^{F*}$ with all vertices singletons expect the block $\{1, 2m\}$. We claim that for $\pi \neq \hat{1}^*$, $\pi \wedge \pi_0 = \hat{1}^*$ if and only if $\pi = \tau$.

Indeed, $\pi_0 \wedge \tau = \hat{1}$. For the converse, suppose that $\pi \neq \hat{1}^*$ and $\pi \neq \tau$. There is a block B of π of size greater than one and a member in $\{1, 2, \ldots, m, 2m\}$. If $\{1, 2m\} \cap B = \emptyset$, then the non-crossing partition with B and $F(B)$ as the only blocks of size greater than one is an upperbound of $\pi \wedge \pi_0$ in $\mu_{NC(n)^{F*}}$. If If $\{1, 2m\} \cap B \neq \emptyset$ for some $i \in \{2, \ldots, m\}$, then the partition with $\{i, m+i-1\}$ as the only block of size greater than one is an upperbound of $\pi \wedge \pi_0$ in $\mu_{NC(n)^{F*}}$.

Hence we obtain that

$$0 = \mu_Q = \sum_{\bar{z}=\hat{1}^*} \mu_{NC(n)^{F^*}}(\hat{0}^*, z) = \mu_{NC(n)^{F^*}}(\hat{0}^*, \tau) + \mu_{NC(n)^{F^*}}(\hat{0}^*, \hat{1}^*).$$

Hence $\mu_{NC(n)^F} = -\mu_{NC(n/2)}$, since it is clear that the subposet $\{\pi \in NC(n)^{F^*} : \hat{0}^* \leq \pi \leq \tau\}$ is isomorphic to the lattice $NC(n/2)$.

□

Corollary 3.1.8.

$$\beta_{[n-2]}(F) = \beta_{[n-2]}(RF) = \begin{cases} 0 & \textit{if } n \textit{ is odd,} \\ (-1)^{\frac{n}{2}}\mathrm{Cat}(\frac{n}{2}-1) & \textit{if } n \textit{ is even.} \end{cases}$$

Montenegro and Reiner also computed the $\beta_{[n-2]}(R^d)$ independently, which is summarized in the following theorem:

Theorem 3.1.9. *[8, 9]*

$$\beta_{[n-2]}(R^d) = \begin{cases} \mathrm{Cat}_{n-1} & \textit{if } d = n, \\ (-1)^{gcd(d,n)+n}(1-2gcd(d,n))\mathrm{Cat}_{gcd(d,n)-1} & \textit{if } d \neq n. \end{cases}$$

Remark. $gcd(d, n)$ is the greatest common divisor of d and n.

3.2 Non-crossing Partitions with a Certain Number of Blocks Fixed by a Reflection

In Chapter 1, Theorem 1.1.12 tells us that the number of $NC(n)$ with k blocks is counted by the classic Narayana Number, i.e. $\#\{\pi \in NC(n) : \pi \text{ has k blocks}\} =$

$Nar(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. We will consider the cyclic sieving phenomenon (CSP) on $NC(n)$ which is introduced in section 1.3 and try to extend this to a "dihedral sieving phenomenon". In addition to the usual cyclic sieving phenomenon, we will show that in the lattice of $NC(n)^F$, the number of $NC(n)^F$ with k blocks is counted by the $q-$Narayana Number evaluated at $q = -1$.

Consider the Dihedral group D_{2n} acting on $NC(n)$. Recall from section 1.4 that $\alpha_S(g)$ is actually the number of chains of maximal length (i.e. with $|S| + 2$ elements) in the lattice of $NC(n)^g$, for $g \in D_{2n}$. If we restrict our rank selected set S to be a single element set, that is $S = \{k\}$, for some $k \in [n-1]$, then $\alpha_k(g)$ counts the number of elements in the lattice of $NC(n)^g$ with $n-k$ blocks.

Note that the rank function in $NC(n)$ is defined to be $rank(\pi) := n - |\pi|$. Hence all the $NC(n)$ with the same rank have the same number of blocks. And therefore, Theorem 1.1.12 can be rewritten as :

Theorem 3.2.1. *Let D_{2n} act on the lattice of $NC(n)$, then*

$$\alpha_{\{n-k\}}(1) = \mathrm{Nar}(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}, \forall\ k \in [n-1].$$

It is known from Reiner, Stanton and White [10, Theorem 7.2] that the number of $NC(n)$ fixed by rotation R^d with k blocks is counted by the $q-$Narayana Number evaluated at $q = e^{2\pi i d/n}$, that is

Theorem 3.2.2. *[10] For all $k \in [n-1]$, we have*

$$\alpha_{\{n-k\}}(R^d) = \mathrm{Nar}_q(n,k)|_{q=e^{e\pi i d/n}} = \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \Bigg|_{q=e^{2\pi i d/n}}.$$

To finish the computation of the character $\alpha_{\{n-k\}}$, we need to know $\alpha_{\{n-k\}}(F)$ and $\alpha_{\{n-k\}}(RF)$.

Now we restrict to the lattice of $NC(n)^F$. We are interested in the number of $NC(n)^F$ with exactly k blocks. Before we compute $\alpha_{\{n-k\}}(F)$, we will first introduce an interesting recurrence relationship of $NC(n)^F$ with certain blocks.

Definition 3.2.3. Denote the set of elements in $NC(n)^F$ with k blocks as $NC(n,k)^F$ and $\#NC(n,k)^F$ is denoted $ncf(n,k)$, $\#NC(n)$ with k blocks is denoted $nc(n,k)$.

Theorem 3.2.4. *If n is even, then $ncf(n,k) = ncf(n-1,k) + ncf(n-1,k-1)$.*

Proof. Label all the vertices of $n-$gon as in Figure 2.1. Then the vertex n is either a block itself as singleton or in the block connected to the last bridge. In the first case, the number of $NC(n,k)^F$ with vertex n as a singleton is just $ncf(n-1,k-1)$ by ignoring the last vertex n added to $NC(n-1,k-1)^F$. In the latter case, vertex n does not contribute to the number of blocks and hence the number is equal to $ncf(n-1,k)$ as in the set of $NC(n-1,k)^F$. The recursive formula follows. □

Theorem 3.2.5. *If n is odd, then*

$$ncf(n,k) = \begin{cases} ncf(n-1,k) + ncf(n-1,k-1), & \textit{if } k \textit{ is odd,} \\ ncf(n-1,k) + ncf(n-1,k-1) - nc(\frac{n-1}{2}, \frac{k}{2}), & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. For any element in $NC(n,k)^F$, consider deleting the vertex 1.

Case 1: When k is odd,

If vertex 1 is a block itself as a singleton, by deleting it, we obtain a non-crossing partition of $[n-1]$ with $k-1$ blocks. If vertex 1 is connected to some vertices below, after deleting vertex 1, we get a non-crossing partition of $[n-1]$ with exactly k blocks. The result follows easily.

Case 2: When k is even,

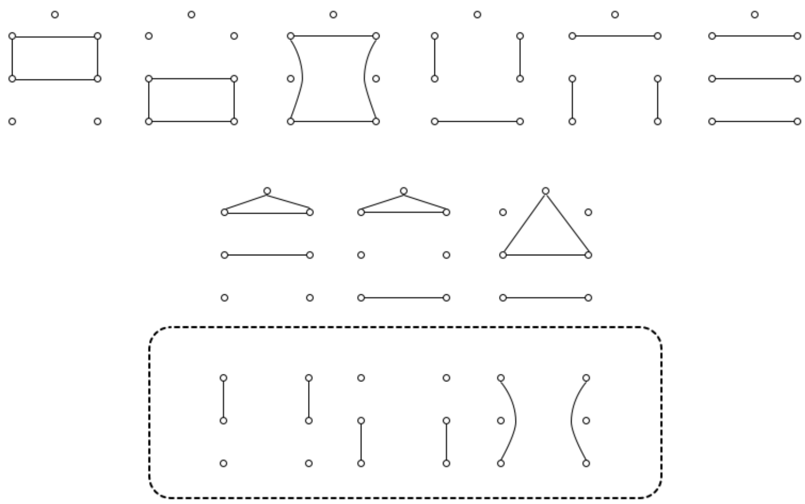


Figure 3.1: Example of $NC(7,4)^F$

If vertex 1 is a block itself as a singleton, by deleting it, we obtain a non-crossing partition of $[n-1]$ with $k-1$ blocks. If vertex 1 is connected to some vertex below, by deleting it, we get a non-crossing partition of $[n-1]$ with k blocks. However, we need to avoid the case that there is no bridge underneath, since otherwise, the number of blocks under vertex 1 is an even number (the numbers of blocks coming from left column of vertices and right column are the same) and we have no way to put vertex 1 in another block without building a bridge.

Notice that the subposet of $NC(n-1,k-1)^F$ with no bridge is isomorphic to the lattice of $NC(\frac{n-1}{2}, \frac{k}{2})$. The recursive formula follows.

□

It is easier to understand this theorem if we take a look at an example.

Example 3.2.6. Consider the set of $NC(7,4)^F$. In Figure 3.1, all the non-crossing partitions on the first row are those for which vertex 1 is a singleton. By deleting 1 we obtain six elements from $NC(6,4)^F$ (i.e. the first row by ignoring vertex 1 is just the set $NC(6,4)^F$). If vertex 1 is not a singleton, we will miss

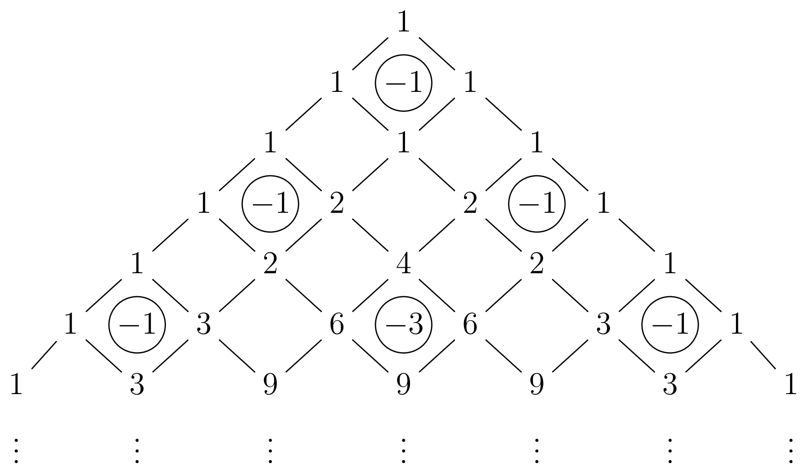


Figure 3.2: Jumping Triangle

three elements from $NC(6,4)^F$ which are in the dotted box on the third row, because otherwise we cannot have an even number of blocks. It is easy to see those three elements in the dotted box form a set isomorphic to $NC(3,2)$ by putting an imaginary symmetric axis in the middle..

We can write down this recursive relationship by a triangle, whose rows consist of $ncf(n,k)$ for k from 1 to n. The circled numbers are exactly the negatives of the classic Narayana numbers.

Now we turn back to compute $ncf(n,k)$.

We want to find an explicit formula for $ncf(n,k)$.

Recall that the $q-$analogue of n, which is denoted as $[n]_q$, is the polynomial $1+q+q^2+\ldots+q^{n-1}$.

And

$$[n]_q\big|_{q=-1} = \begin{cases} 1, & n \text{ is odd,} \\ \\ 0, & n \text{ is even.} \end{cases}$$

We may use the Lemma 2.1.3 above to compute the explicit formula for $q-$Narayana numbers evaluated at $q = -1$. Since $q-$Narayana number $\text{Nar}_q(n,k) = \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$ is always a polynomial [6], hence the limit as q approaches -1 is exactly its evaluation at -1.

First, assume that n is odd, then by definition we have $\frac{1}{[n]_q}|_{q=-1} = 1$.

Case 1. If k is odd, then $k-1$ is even.

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q[k-1]_q \cdots [1]_q}, \\ \begin{bmatrix} n \\ k-1 \end{bmatrix}_q &= \frac{[n]_q[n-1]_q \cdots [n-k+2]_q}{[k-1]_q[k-2]_q \cdots [1]_q}. \end{aligned}$$

Then using the Lemma 2.1.3, we obtain at $q = -1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \Bigg|_{q=-1} = \frac{1}{1} \cdot \frac{n-1}{k-1} \cdot \frac{1}{1} \cdot \frac{n-3}{k-3} \cdots \frac{n-k+2}{2} = \frac{(n-1)\cdot(n-3)\cdots(n-k+2)}{(k-1)\cdot(k-3)\cdots(2)}.$$

Similarly we have,

$$\begin{aligned} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \Bigg|_{q=-1} &= \frac{[n]_q[n-1]_q[n-2]_q[n-3]_q \cdots [n-k+2]_q}{[1]_q[k-1]_q[k-2]_q[k-3]_q \cdots [2]_q}\Bigg|_{q=-1} \\ &= \frac{1}{1} \cdot \frac{n-1}{k-1} \cdot \frac{1}{1} \cdot \frac{n-3}{k-3} \cdots \frac{n-k+2}{2} \\ &= \frac{(n-1)\cdot(n-3)t\cdots(n-k+2)}{(k-1)\cdot(k-3)\cdots(2)}. \end{aligned}$$

Recall the definition and the following properties of the double factorial:

$$\begin{aligned} n!! &= n \cdot (n-2) \cdots 3 \cdot 1, \\ 2^n n! &= 2n \cdot (2n-2) \cdot (2n-4) \cdots = (2n)!!, \\ \frac{(2n)!}{2^n n!} &= (2n-1) \cdot (2n-3) \cdots = (2n-1)!! \end{aligned}$$

Then we have

$$\begin{aligned} (n-1)!! &= 2^{\frac{n-1}{2}} \cdot (\frac{n-1}{2})!, \\ (k-1)!! &= 2^{\frac{k-1}{2}} \cdot (\frac{k-1}{2})!, \\ (n-k)!! &= 2^{\frac{n-k}{2}} \cdot (\frac{n-k}{2})!. \end{aligned}$$

This implies that

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_q \Bigg|_{q=-1} &= \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \Bigg|_{q=-1} = \frac{(n-1) \cdots (n-3) \cdots (n-k+2)}{(k-1) \cdot (k-3) \cdots (2)} \\ &= \frac{(n-1)!!}{(k-1)!!(n-k)!!} = \frac{2^{\frac{n-1}{2}} \cdot (\frac{n-1}{2})!}{2^{\frac{k-1}{2}} \cdot (\frac{k-1}{2})! 2^{\frac{n-k}{2}} \cdot (\frac{n-k}{2})!} \\ &= \frac{(\frac{n-1}{2})!}{(\frac{k-1}{2})!(\frac{n-k}{2})!} \\ &= \binom{\frac{n-1}{2}}{\frac{k-1}{2}}. \end{aligned}$$

Hence,

$$\mathrm{Nar}_q(n,k)|_{q=-1} = \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \Bigg|_{q=-1} = \binom{\frac{n-1}{2}}{\frac{k-1}{2}}^2.$$

Case 2. If k is even, then $k-1$ is odd.

Using a method similar to case 1 above, one can compute that

$$\left.\begin{bmatrix} n \\ k \end{bmatrix}_q\right|_{q=-1} = \binom{\frac{n-1}{2}}{\frac{k}{2}},$$
$$\left.\begin{bmatrix} n \\ k-1 \end{bmatrix}_q\right|_{q=-1} = \binom{\frac{n-1}{2}}{\frac{k-2}{2}},$$

and hence,

$$\mathrm{Nar}_q(n,k)|_{q=-1} = \left.\frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q\right|_{q=-1} = \binom{\frac{n-1}{2}}{\frac{k}{2}} \cdot \binom{\frac{n-1}{2}}{\frac{k-2}{2}}.$$

Next assume that n is even. In this case we have $\frac{1}{[n]_q}|_{q=-1} = 0$, so we need to use some trick here.

Case 1. If k is odd, then $k-1$ is even. At $q=-1$,

$$\begin{aligned}
&\left.\frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q\right|_{q=-1} \\
&= \left.\frac{[n]_q}{[n]_q}\frac{[n-1]_q[n-2]_q\cdots[n-k+1]_q}{k]_q[k-1]_1\cdots[1]_q} \cdot \frac{[n]_q[n-1]_1\cdots[n-k+2]_q}{[k-1]_q[k-2]_1\cdots[1]_q}\right|_{q=-1} \\
&= (\frac{n-2}{k-1}\cdot\frac{n-4}{k-3}\cdots\frac{n-k+1}{2})\cdot(\frac{n}{k-1}\cdot\frac{n-2}{k-3}\cdots\frac{n-k+3}{2}) \\
&= \frac{(n-2)!!}{(k-1)!!(n-k-1)!!}\cdot\frac{n!!}{(k-1)!!(n-k+1)!!} \\
&= \binom{\frac{n-2}{2}}{\frac{k-1}{2}}\cdot\binom{\frac{n}{2}}{\frac{k-1}{2}}.
\end{aligned}$$

Case 2. If k is even, then $k-1$ is odd.

Similarly we may compute that

$$
\begin{aligned}
&\frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \Bigg|_{q=-1} \\
&= \frac{[n]_q}{[n]_q}\frac{[n-1]_q[n-2]_q\cdots[n-k+1]_q}{k]_q[k-1]_1\cdots[1]_q} \cdot \frac{[n]_q[n-1]_1\cdots[n-k+2]_q}{[k-1]_q[k-2]_1\cdots[1]_q}\Bigg|_{q=-1} \\
&= (\frac{n-2}{k}\cdot\frac{n-4}{k-2}\cdots\frac{n-k+2}{2})\cdot(\frac{n}{k-2}\cdot\frac{n-2}{k-4}\cdots\frac{n-k+2}{2}) \\
&= \frac{(n-2)!!}{(k-2)!!(n-k)!!}\cdot\frac{n!!}{k!!(n-k)!!} \\
&= \binom{\frac{n-2}{2}}{\frac{k-2}{2}}\cdot\binom{\frac{n}{2}}{\frac{k}{2}}.
\end{aligned}
$$

In summary, we computed the explicit formulae for the $q-$Narayana numbers evaluated at $q =\!-1$:

$$
\frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \Bigg|_{q=-1} = \begin{cases} \binom{\frac{n-1}{2}}{\frac{k-1}{2}}^2 & n \text{ odd, } k \text{ odd,} \\ \binom{\frac{n-1}{2}}{\frac{k}{2}}\cdot\binom{\frac{n-1}{2}}{\frac{k-2}{2}} & n \text{ odd, } k \text{ even,} \\ \binom{\frac{n-2}{2}}{\frac{k-1}{2}}\cdot\binom{\frac{n}{2}}{\frac{k-1}{2}} & n \text{ even, } k \text{ odd,} \\ \binom{\frac{n-2}{2}}{\frac{k-2}{2}}\cdot\binom{\frac{n}{2}}{\frac{k}{2}} & n \text{ even, } k \text{ even.} \end{cases}
$$

Remark. We can rewrite the equations above as

$$
\frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \Bigg|_{q=-1} = \binom{\lfloor\frac{n-1}{2}\rfloor}{\lfloor\frac{k-1}{2}\rfloor}\cdot\binom{\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{k}{2}\rfloor},
$$

for simplicity.

Theorem 3.2.7. $\alpha_{\{n-k\}}(F) = ncf(n,k) = \mathrm{Nar}_q(n,k)|_{q=-1} = \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \Big|_{q=-1}$.

Proof. If we can prove that the $q-$Narayana numbers evaluated at $q = -1$ satisfy all the recursive formulae in Theorem 3.2.4 and Theorem 3.2.5, with the initial condition that

$$\mathrm{Nar}_{q=-1}(1,1) = 1,$$
$$\mathrm{Nar}_{q=-1}(2,1) = 1,$$
$$\mathrm{Nar}_{q=-1}(2,2) = 1,$$

then we are done.

Here we only check the most difficult recursive formula:

$$ncf(n-1,k) + ncf(n-1,k-1) - nc(\frac{n-1}{2}, \frac{k}{2}), \text{ if } n \text{ is odd and } k \text{ is even.}$$

All the others can be justified easily by the reader.

When n is odd and k is even, our goal is to show that at $q = -1$,

$$\binom{\frac{n-1}{2}}{\frac{k}{2}} \cdot \binom{\frac{n-1}{2}}{\frac{k-2}{2}} = \binom{\frac{n-3}{2}}{\frac{k-2}{2}} \cdot \binom{\frac{n-1}{2}}{\frac{k}{2}} + \binom{\frac{n-3}{2}}{\frac{k-2}{2}} \cdot \binom{\frac{n-1}{2}}{\frac{k-2}{2}} - \frac{1}{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{\frac{k}{2}} \cdot \binom{\frac{n-1}{2}}{\frac{k-2}{2}}.$$

Notice that

$$\begin{aligned}
\text{RHS} &= \binom{\frac{n-3}{2}}{\frac{k-2}{2}} \cdot \binom{\frac{n-1}{2}}{\frac{k}{2}} + \binom{\frac{n-3}{2}}{\frac{k-2}{2}} \cdot \binom{\frac{n-1}{2}}{\frac{k-2}{2}} - \frac{1}{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{\frac{k}{2}} \cdot \binom{\frac{n-1}{2}}{\frac{k-2}{2}} \\
&= \frac{(\frac{n-3}{2})(\frac{n-5}{2})\cdots(\frac{n-k-1}{2}+1)}{(\frac{k-2}{2})!} \cdot \frac{(\frac{n-1}{2})(\frac{n-3}{2})\cdots(\frac{n-k-1}{2}+1)}{(\frac{k}{2})!} \\
&+ \binom{\frac{n-3}{2}}{\frac{k-2}{2}} \cdot \binom{\frac{n-1}{2}}{\frac{k-2}{2}} - \frac{1}{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{\frac{k}{2}} \cdot \binom{\frac{n-1}{2}}{\frac{k-2}{2}} \\
&= \frac{(\frac{n-1}{2})(\frac{n-3}{2})\cdots(\frac{n-k+1}{2}+1)}{(\frac{k-2}{2})!} \cdot \frac{(\frac{n-3}{2})(\frac{n-5}{2})\cdots(\frac{n-k-1}{2}+1)}{(\frac{k}{2})!} \cdot (\frac{n-k-1}{2}+1) \\
&+ \binom{\frac{n-3}{2}}{\frac{k-2}{2}} \cdot \binom{\frac{n-1}{2}}{\frac{k-2}{2}} - \frac{1}{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{\frac{k}{2}} \cdot \binom{\frac{n-1}{2}}{\frac{k-2}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \binom{\frac{n-1}{2}}{\frac{k-2}{2}} \left[\frac{(\frac{n-3}{2})(\frac{n-5}{2})\cdots(\frac{n-k-1}{2}+1)}{(\frac{k}{2})!} \cdot (\frac{n-k-1}{2}+1) + \binom{\frac{n-3}{2}}{\frac{k-2}{2}} - \frac{1}{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{\frac{k}{2}} \right] \\
&= \binom{\frac{n-1}{2}}{\frac{k-2}{2}} \left[\frac{1}{(\frac{k}{2})!} (\frac{n-3}{2}) \cdots (\frac{n-k-1}{2}+1)(\frac{n-k-1}{2}+1) \right. \\
&\left. + \frac{1}{(\frac{k}{2})!} (\frac{n-3}{2}) \cdots (\frac{n-k-1}{2}+1) \cdot \frac{k}{2} - (\frac{n-1}{2}) \cdots (\frac{n-k-1}{2}+1) \cdot \frac{1}{\frac{n-1}{2}} \right] \\
&= \binom{\frac{n-1}{2}}{\frac{k-2}{2}} \frac{1}{(\frac{k}{2})!} \left[(\frac{n-3}{2}) \cdots (\frac{n-k-1}{2}+1) \left(\frac{n-k-1}{2} + 1 + \frac{k}{2} - 1 \right) \right] \\
&= \binom{\frac{n-1}{2}}{\frac{k-2}{2}} \frac{1}{(\frac{k}{2})!} (\frac{n-1}{2}) \cdots (\frac{n-k-1}{2}+1) \\
&= \binom{\frac{n-1}{2}}{\frac{k-2}{2}} \binom{\frac{n-1}{2}}{\frac{k}{2}} \\
&= \text{LHS}.
\end{aligned}
$$

□

Under the Kreweras Complement, using the fact that

$$\mathrm{Nar}_q(n, n-k+1) = \mathrm{Nar}_q(n, k),$$

it is easy to see that $\alpha_{\{n-k\}}(RF) = \mathrm{Nar}_q(n,k)|_{q=-1} = \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q |_{q=-1}$ as well.

In summary, we have completed the calculation of the character α_S when $|S| = 1$.

1. When n is odd,

$g \in D_{2n}$	1	R^d $(1 \leq d \leq n)$	F
$\alpha_{\{n-k\}}$	$\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$	$\frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \Big\vert_{q=e^{2\pi i/d}}$	$\binom{\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{k-1}{2} \rfloor} \cdot \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor}$

2. When n is even,

$g \in D_{2n}$	1	R^d $(1 \le d \le n)$	F	RF
$\alpha_{\{n-k\}}$	$\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$	$\frac{1}{[n]_q}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n\\k-1\end{bmatrix}_q\Big\vert_{q=e^{2\pi i/d}}$	$\binom{\lfloor\frac{n-1}{2}\rfloor}{\lfloor\frac{k-1}{2}\rfloor}\cdot\binom{\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{k}{2}\rfloor}$	$\binom{\lfloor\frac{n-1}{2}\rfloor}{\lfloor\frac{k-1}{2}\rfloor}\cdot\binom{\lfloor\frac{n}{2}\rfloor}{\lfloor\frac{k}{2}\rfloor}$

3.3 Maximal Chains of $NC(n)$

In the previous section, we investigated the number of non-crossing partitions of $[n]$ of the certain rank, which is $\alpha_{[i]}$ for $\forall i \in [n]$. Next, our goal is to compute the character α_S when S is as large as possible, i.e. $S = [n-2]$.

Recall from section 1.4 that $\alpha_S(1)$ actually counts the number of chains of length $|S|+1$ of the rank-selected subposet of $NC(n)$, where $S \subset [n-2]$. In this section, we will look into a special S when $S = [n-2]$, that is, to study $\alpha_{[n-2]}$ evaluated at any element g in the dihedral group D_{2n}.

First we need to consider the number of maximal chains of length $n-1$ in the total poset $NC(n)$, which is counted by the number of $\alpha_{[n-2]}(1)$.

Definition 3.3.1. Let P be a finite poset. If $\#P = k \geq 2$, then define $Z(P,k)$ to be the number of multi-chains $t_1 \leq t_2 \leq \ldots \leq t_{k-1}$ in P. We call $Z(P,k)$ (regarded as a function of k) the *zeta polynomial* of P.

Theorem 3.3.2. *[16] Let b_i be the number of chains with $i-1$ elements in P. Then $b_{i+2} = \Delta^i Z(P,2), i \geq 0$, where Δ is the finite difference operator. In other words, $Z(P,n) = \sum_{i\geq 2} b_i \binom{k-2}{i-2}$.*

In particular, $Z(P,k)$ is a polynomial function of k whose degree d is equal to the length of the longest chain of P, and whose leading coefficient is $b_{d+2}/d!$. Moreover, we have $Z(P,2) = \#P$.

The following theorem is from Edelman:

Theorem 3.3.3. *[5]*

$$Z(NC(n),k) = \frac{1}{n!}\prod_{i=1}^{n-1}((k-1)n+i+1).$$

Note that the zeta polynomial allows us to compute the number of maximal chains. We will illustrate this by reproving the following theorem of Kreweras, which he proved by a different method.

Theorem 3.3.4. *[7]* $\alpha_{[n-1]}(1) = n^{n-2}$.

Proof. What we need is only to compute b_n.

By Theorem 3.3.2, $Z(NC(n),n) = \frac{b_n}{(n-1)!}k^{n-1} + \dots$, where all the terms following $\frac{b_n}{(n-1)!}k^{n-1}$ are those whose degrees are lower than $n-1$. Hence $\lim_{k\to\infty}\frac{Z(NC(n),k)}{k^{n-1}} = \frac{b_n}{(n-1)!}$. According to Theorem 3.3.3,

$$\begin{aligned}\lim_{k\to\infty} Z(NC(n),k) &= \frac{1}{n!}\cdot\frac{kn+2}{k}\cdot\frac{kn+3}{k}\cdot\ldots\cdot\frac{kn+n}{k}\\ &= \frac{1}{n!}\cdot n\cdot n\cdot\ldots\cdot n\\ &= \frac{n^{n-1}}{n!}.\end{aligned}$$

which implies

$$b_n = \frac{(n-1)!n^{n-1}}{n!} = \frac{n^{n-1}}{n} = n^{n-2}.$$

□

Now we consider $NC(n)^g$, for $g \in D_{2n}$.

Recall we established an isomorphism in Theorem 2.2.9 that the lattice of $NC(n)^F_{pr}$ is isomorphic the the lattice of order ideals of Zigzag poset of $[n-1]$.

Definition 3.3.5. The number of alternating permutations [14] $\omega \in \mathfrak{S}_n$ is denoted E_n (with $E_0 = 1$). Such a number is called an Euler Number.

Recall that the Zigzag poset of $[n]$ is denoted as $Z_{[n]}$.

Theorem 3.3.6. *# maximal chains in* $J(Z_{[n]}) = E_n$.

Proof. The number of maximal chains of order ideals in $J(Z_{[n]})$ is equal to the number of linear extensions of $Z_{[n]}$, which is equal to the number of alternating permutations of $[n]$. □

Note that by the argument of the Theorem 2.3.4 and 2.3.10 the number of chains with length $n-1$ in $NC(n)^F$ all lie in the pruned sublattice of $NC(n)^F_{pr}$. Hence we obtain:

Corollary 3.3.7. $\alpha_{[n-2]}(F) = E_{n-1}$.

For n is even, we have seen that through Kreweras complement, $\alpha_{[n-1]}(RF) = E_{n-1}$ as well.

It remains only to copute $\alpha_{[n-2]}(R^d)$ for $d \in [\lfloor \frac{n}{2} \rfloor]$.

Theorem 3.3.8. $\alpha_{[n-2]}(R^d) = 0$*, for all* $d \in [\lfloor \frac{n}{2} \rfloor]$.

Proof. For a fixed $d \in [\lfloor \frac{n}{2} \rfloor]$, suppose there exists a chain of length $n-1$ which is fixed by R^d, consider the coatom on this chain. Since such a coatom is also fixed by R^d, it has d symmetric parts. It is impossible to get a non-crossing partition with $d \geq 3$ blocks which is fixed by R^d because we need to do the same refinement to those d symmetric parts. Hence such a coatom cannot lie on a chain of length $n-1$, which is a contradiction the existence of such a chain of length $n-1$.

□

Now we fully understand $\alpha_{[n-2]}(g), \forall g \in D_{2n}$, and we summarize our results in the following tables:

1. When n is odd,

$g \in D_{2n}$	1	R	$\dots$	$R^{\frac{n-1}{2}}$	F
$\alpha_{[n-2]}$	n^{n-2}	0	0	0	E_{n-1}

2. When n is even,

$g \in D_{2n}$	1	R	$\dots$	$R^{\frac{n}{2}}$	F	RF
$\alpha_{[n-2]}$	n^{n-2}	0	0	0	E_{n-1}	E_{n-1}

3.4 Multiplicities of Irreducible Characters in α_S and β_S

In the previous sections, we computed the α_S and β_S for some rank-subsets $S \subset [n-2]$. We may then compute the multiplicities of irreducible characters in α_S and β_S.

Definition 3.4.1. Let χ and ψ be two characters, then the inner product of χ and ψ is

$$\langle \chi, \psi \rangle = \frac{1}{G} \sum_{g \in G} \chi(g)\overline{\psi(g)}.$$

Theorem 3.4.2. *[12] Let χ be an irreducible character and ψ be any character. Then the multiplicity of χ in ψ is equal to $\langle \chi, \psi \rangle$.*

In section 1.2, we established the character tables of D_{2n} in Figure 1.4 and Figure 1.5. Let χ^{ϕ} be the character of an irreducible representation ϕ of the dihedral group D_{2n}. We can use the theorem above to compute the multiplicities of irreducible characters in α_S and β_S for some rank-sets S explicitly.

Case 1. When n is odd.

In Figure 1.4, we see that there are $\frac{n+3}{2}$ irreducible characters. α_S can be interpreted by the following formula (similar for β_S):

$$\alpha_S = \langle \chi^{Triv}, \alpha_S \rangle \cdot \chi^{Triv} + \langle \chi^{Det}, \alpha_S \rangle \cdot \chi^{Det} + \sum_{i=1}^{(n-1)/2} \langle \chi^{\phi_i}, \alpha_S \rangle \cdot \chi^{\phi_i}.$$

If S is a subset of $[n-2]$ with only one element, from section 3.2 and the formula for inner product in definition 3.4.1 (note that α_S is real-valued), we may compute the multiplicities of irreducible characters in α_{n-k} as follows:

$\langle \chi^{Triv}, \alpha_{\{n-k\}} \rangle = \frac{1}{2n}\{1 \cdot \frac{1}{n}\binom{n}{k}\binom{n}{k-1} + n \cdot 1 \cdot \binom{\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{k-1}{2} \rfloor} \cdot \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} + 2 \cdot \sum_{d=1}^{(n-1)/2} 1 \cdot \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q |_{q=e^{2\pi i/d}}\}$

$\langle \chi^{Det}, \alpha_{\{n-k\}} \rangle = \frac{1}{2n}\{1 \cdot \frac{1}{n}\binom{n}{k}\binom{n}{k-1} + n \cdot (-1) \cdot \binom{\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{k-1}{2} \rfloor} \cdot \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} + 2 \cdot \sum_{d=1}^{(n-1)/2} 1 \cdot \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q |_{q=e^{2\pi i/d}}\}$

$\langle \chi^{\phi_i}, \alpha_{\{n-k\}} \rangle = \frac{1}{2n}\{2 \cdot \frac{1}{n}\binom{n}{k}\binom{n}{k-1} + n \cdot 0 \cdot \binom{\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{k-1}{2} \rfloor} \cdot \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} + 2 \cdot \sum_{d=1}^{(n-1)/2} 2\cos\frac{2id\pi}{n} \cdot \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q |_{q=e^{2\pi i/d}}\}$, $\forall i \in [\frac{n-1}{2}]$.

If S is $[n-2]$, from section 3.3, we may get the multiplicities of irreducible characters in $\alpha_{[n-2]}$ as follows:

$\langle \chi^{Triv}, \alpha_{[n-2]} \rangle = \frac{1}{2n}\{1 \cdot n^{n-2} + n \cdot 1 \cdot E_{n-1} + 2 \cdot \sum_{d=1}^{(n-1)/2} 1 \cdot 0\} = \frac{1}{2n}\{n^{n-2} + nE_{n-1}\}$

$\langle \chi^{Det}, \alpha_{[n-2]} \rangle = \frac{1}{2n}\{1 \cdot n^{n-2} + n \cdot (-1) \cdot E_{n-1} + 2 \cdot \sum_{d=1}^{(n-1)/2} 1 \cdot 0\} = \frac{1}{2n}\{n^{n-2} - nE_{n-1}\}$

$\langle \chi^{\phi_i}, \alpha_{[n-2]} \rangle = \frac{1}{2n}\{2 \cdot n^{n-2} + 0 \cdot E_{n-1} + 2 \cdot \sum_{d=1}^{(n-1)/2} 2\cos\frac{2id\pi}{n} \cdot 0\} = \frac{1}{2n} n^{n-2}$, $\forall i \in [\frac{n-1}{2}]$.

Note that $\beta_{[n-2]}$ is also real-valued, hence by section 3.1 we obtain:

$\langle \chi^{Triv}, \beta_{[n-2]} \rangle = \frac{1}{2n}\{1 \cdot \text{Cat}_{n-1} + n \cdot 1 \cdot 0 + 2 \cdot \sum_{d=1}^{(n-1)/2} 1 \cdot (-1)^{gcd(d,n)+n}(1 - 2gcd(d,n))\text{Cat}_{gcd(d,n)-1}\}$

$\langle \chi^{Det}, \beta_{[n-2]} \rangle = \frac{1}{2n}\{1 \cdot \text{Cat}_{n-1} + n \cdot (-1) \cdot 0 + 2 \cdot \sum_{d=1}^{(n-1)/2} 1 \cdot (-1)^{gcd(d,n)+n}(1 - 2gcd(d,n))\text{Cat}_{gcd(d,n)-1}\}$

$\langle \chi^{\phi_i}, \beta_{[n-2]} \rangle = \frac{1}{2n}\{2 \cdot \mathrm{Cat}_{n-1} + n \cdot 0 \cdot 0 + 2 \cdot \sum_{d=1}^{(n-1)/2} 2\cos\frac{2id\pi}{n} \cdot (-1)^{gcd(d,n)+n}(1 - 2gcd(d,n))\mathrm{Cat}_{gcd(d,n)-1}\}$, $\forall i \in [\frac{n-1}{2}]$.

Case 2. When n is even.

In Figure 1.5, we see that there are $\frac{n+6}{2}$ irreducible characters. And α_S and β_S can be interpreted similarly as in the case 1.

We may compute $\alpha_{[n-2]}$ and get the following results:

$\langle \chi^{Triv}, \alpha_{[n-2]} \rangle = \frac{1}{2n}\{1 \cdot n^{n-2} + n \cdot 1 \cdot E_{n-1} + 2 \cdot \sum_{d=1}^{(n-2)/2} 1 \cdot 0 + 1 \cdot 0\} = \frac{1}{2n}\{n^{n-2} + nE_{n-1}\}$

$\langle \chi^{Det}, \alpha_{[n-2]} \rangle = \frac{1}{2n}\{1 \cdot n^{n-2} + n \cdot (-1) \cdot E_{n-1} + 2 \cdot \sum_{d=1}^{(n-2)/2} 1 \cdot 0 + 1 \cdot 0\} = \frac{1}{2n}\{n^{n-2} - nE_{n-1}\}$

$\langle \chi^{Lin1}, \alpha_{[n-2]} \rangle = \frac{1}{2n}\{1 \cdot n^{n-2} + \frac{n}{2} \cdot 1 \cdot E_{n-1} + \frac{n}{2} \cdot (-1) \cdot E_{n-1} + 2 \cdot \sum_{d=1}^{(n-2)/2} (-1)^d \cdot 0 + (-1)^{n/2} \cdot 0\} = \frac{1}{2n}\{n^{n-2}\}$

$\langle \chi^{Lin2}, \alpha_{[n-2]} \rangle = \frac{1}{2n}\{1 \cdot n^{n-2} + n \cdot 1 \cdot E_{n-1} + 2 \cdot \sum_{d=1}^{(n-2)/2} (-1)^d \cdot 0 + (-1)^{n/2} \cdot 0\} = \frac{1}{2n}\{n^{n-2} + nE_{n-1}\}$

$\langle \chi^{\phi_i}, \alpha_{[n-2]} \rangle = \frac{1}{2n}\{2 \cdot n^{n-2} + n \cdot 0 \cdot E_{n-1} + 2 \cdot \sum_{d=1}^{(n-2)/2} 2\cos\frac{2id\pi}{n} \cdot 0 + 2\cos\frac{2in\pi}{n} \cdot 0\} = \frac{1}{2n} n^{n-2}$, $\forall i \in [n/2]$.

The multiplicities of irreducible characters in $\alpha_{\{n-k\}}$ and $\beta_{[n-2]}$ can be checked by the reader easily.

3.5 Directions for Future Research and Some Open Problems

In this thesis, we investigated poset structure on $NC(n)^F$ and computed the characters of α_S and β_S for some rank-selected subsets $S \subset [n-2]$. There are still some open problems which we may work with in the future.

1. Combinatorial interpretation of coefficients of α_S and β_S in terms of irreducible characters for D_{2n}.

   We have already seen that both α and β are actual representations, which means characters α_S and β_S can be expressed as linear combination of irreducible characters with positive coefficients. We just know that the coefficient of the trivial representation is the number of orbits. Is there any way we can figure out the meaning of all the other coefficients?

2. α_S and β_S for some other rank-sets $S \subset [n-2]$.

   In Chapter 3, we computed $\beta_{[n-2]}$, α_i for $i \in [n-2]$ and $\alpha_{[n-2]}$. We are also interested in other rank-sets $S \subset [n-2]$. Are we able to compute those α_S and β_S?

3. Zeta polynomial of $NC(n)^F$.

   In Kreweras' paper [7], he computed the zeta polynomial of $NC(n)$. When we investigated the zeta polynomial of $NC(n)^F$, we did not get a nice formula or conjecture. Maybe we could use some techniques to compute the zeta polynomial for $NC(n)^F$ explicitly?

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