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We study a hydrodynamic limit for a system of N diffusions moving in an open domain $D \subseteq \mathbb{R}^d$ undergoing branching when one particle reaches a certain subset of the boundary. The particle at the boundary and another random neighbor are eliminated and replaced with two new particles created instantaneously at a random point with distribution $\gamma(dx)$ in D. This thesis proves the $d = 1$ case with $D = (0, 1)$, $\gamma(dx) = \delta_c(dx)$, $c \in (0, 1)$ while the general case is done in an upcoming paper. The mechanism represents a hybrid between the Fleming-Viot branching and a mean-field version of the Bak-Sneppen fitness model where the absorbing boundary represents the not viable, or *minimal* configuration. The limiting profile is the normalization of the solution of a heat equation with mass creation, which is studied using its representation via an auxiliary measure valued supercritical process. Self-criticality is manifested by the presence of the quasi-stationary distributions emerging as profiles under equilibrium.

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Table of Contents

Chapter 1

Introduction

1.1 Self Organized Criticality and Bak-Sneppen Model

The underlying reason for why certain natural phenomena, such as avalanches, take place, and how large-scale ecosystems such as water webs and dunes in desert have got their current appearances has long caught the attentions of scientists and sparked numerous inspiring ideas. Questions like this can't be easily answered by Newtonian mechanics, whose most classical theories are all reductive, as is discussed in [5]. One promising approach is to look at equilibrium states, a familiar concept in dynamical systems, present in stochastic systems as well, and possible critical behavior in the proximity of such equilibria, or instability associated with them. Mathematical models tend to be simplified as far as possible, retaining some essential aspects. In their attempts to unveil the underlying reason for natural phenomena like self-organizing criticality and behavior near criticality, in the spirit as mentioned above, in 1993, two physicists, Bak and Sneppen [6] proposed a model describing an ecosystem of N interacting species that evolve by mutation, natural selection, and recombination.

Suppose $\eta_1^N(t) \in (0,1)$ is the typical height of a fitness column at time $t > 0$

in such a system, represented as heights attached to the periodic lattice (in a circle) of sites $i = 1, 2, \ldots, N$ with $N + 1 \equiv 1$. At time zero, all are independent and uniformly distributed. At discrete times $t = 1, 2, \ldots$, the minimum height column, together with its two nearest neighbors , are replaced by three iid uniform random variables. The time indexed process is Markovian, recurrent and has an invariant measure. We notice that due to the fact taht the uniform distribution is absolutelely continuous, almost sureley, there are no ties. The main conjecture in their paper is the following.

Does there exist a critical value c^* such that the following is true?

$$\lim_{N\to\infty} \lim_{t\to\infty} P(\eta_1^N(t) \leq c^*) = 0 \quad \text{and, moreover} \tag{1.1.1}$$

above c^*, the profile is uniform.

Simulation and studies of closely related models support the conjecture with $c^* \approx .66$ (but not equal necessarily to 2/3). Of course this value in itself is related to the ratio between two chosen neighbors and one minimal *trigger* column.

Naturally, one can ask what is *self-organizing criticality* to begin with. The simple answer in this example is that there is no explicit parameter in the setup, yet critical behavior emerges naturally. Power laws, avalanches are shown to be present near c^*.

Thereafter various mathematical attempts had been made to study this question, among which the most representative works include: Kenyon et al [7] studied a Bernoulli version of the original Bak-Sneppen model in the sense that the fitness of each species follows a Bernoulli distribution, instead of the continuous uniform distribution on $[0, 1]$; Derrida et al [9] studied a mean-field version of the original model in the sense that the law of how a particle moves depends on the average of all the other particles. Rigorous proofs that the conjecture proposed in [6] do not exist for the full statement, but for instance, the existence of a critical value is done by Meester and Znamenski [26, 27].

Essentially (1.1.1) is a statement about the stationary distribution in the equilibrium state (as $t \to \infty$).

In this thesis, we shall investigate an interacting model bearing similarities with (1.1.1). The columns will not be static between re-sampling times, diffusing as Brownian particles on $x \in [0, 1]$ with reflection at $x = 1$. The minimum column will be replaced by the particle hitting $x = 0$, which will generate a re-sampling. Besides the"killed" particle, only one is chosen at random, and they both start over at a new point $c \in (0, 1)$. Of course, the uniform distributin has been replaced by the delta measure at c. However, this limitation is not essential. Since reaching the boundary is not sensitive to order (even though that aspect is of interest in itself in the original problem), multidimensional versions of the dynamics are straightforward. These aspects are justified by our genelarization in [17].

Due to the random choice of a neighbor (also known as *mean-field* in physics) and the fact that re-sampling can be regarded as killing at the boundary and birth at c, the dynamics is equally related to the well known Fleming-Viot family of branching mechanism [16], and closer to this model [19].

This thesis answers, among other results, the analogous question for the proposed particle system. The equilibrium profile will turn out to be explicitly computable and the analogous quantity to c^* is the value λ^*, the rate at which the *auxiliary branching process* ζ_t constructed in Section 2.3. The profile, under equilibrium, will be given by a quasi-stationary distribution for a super-critical branching system.

The next theorems will prove the existence and properties of the solution ν_t

to heat equation with mass creation (3.2.5). The proof is probabilistic, using a representation of the solution as the expected value of the empirical measure of a super-critical branching process defined below.

1.2 Previous Work

The motivation of this work is mainly determined by the Fleming-Viot branching dynamics, originally introduced in [16], and a series of papers on the corresponding jump-diffusion particle model [10], [15], [2], [3], [31]. We are particularly using methods similar to [19] and [20] and the particle model under present consideration, *Bak-Sneppen Branching Diffusions* is introduced in [21].

Non-conservative dynamical systems have been studied especially for the dissipative case starting with [33] and a vast literature on quasi-stationarity exists, see [29] for a comprehensive bibliography. The connection between Fleming-Viot particle systems and quasi-stationarity is that the particle systems are ergodic (have an invariant distribution) and, under mean-field scaling, approach the quasi-stationary distribution (qsd) profile of the underlying non-conservative Markov process, that seen in isolation, has only a trivial invariant measure at the cemetery state (the dissipative case). For more details, current methods and applications to mathematical biology, [11], [25] and the references herein are main resources.

It is remarkable that our model is *supercritical*, so quasi-stationarity is the result not of extinction (dissipation), but of *unlimited growth* (exponential). We believe there is much less literature on this aspect of non-conservative branching systems, an exception being [28].

Problems related to the actual convergence under scaling of the equilibrium measure are not tackled in our present work. Nor is the *minimal conjecture* (see

[4]). For supercritical non-conservative systems, where the newly born particles exceed the killed particles in the preceding generation, there is a unique qsd, given by the Green function of the underlying process (here Brownian motion killed at the boundary), in our case corresponding to a value $\lambda_* > 0$ in the resolvent set. This value becomes the equivalent of a spectral gap, but now modeling growth, not decay, and being positive, accordingly.

The model presented here is one dimensional, a case that in itself is not necessarily trivial (see [12], even though there an unbounded domain is considered); the redistribution measure is nonrandom, but can be substantially generalized and will be presented in two follow up papers [17] and [18].

To elaborate on the main feature, the comment is that in the F-V case, the hydrodynamic limit is the normalization of the solution to heat equation with Dirichlet boundary conditions, which is dissipative, mass vanishing exponentially fast at rate $e^{\lambda_1 t}$, with $\lambda_1 < 0$. This is exactly the first eigenvalue for the Dirichlet Laplacian when $(L, \mathcal{D}(L))$ is the killed BM. In the BSB case, the hydrodynamic limit is the normalization of ν_t, a process accruing mass exponentially fast at rate $e^{\lambda_* t}$, where λ_* and $\log E[K]$ have the same sign, implying supercriticality in our case $K = 2$ cf. [18]. Essentially, we need a non-conservative process in either case. While the dissipative case allows a representation with a single particle, the mass creation can be modeled stochastically using a Markov semigroup only as a measure-valued process. The Yaglom limit and quasi-stationarity are also discussed in this case. This is explained in more detail in Section 2.2.

Chapter 2

Motivation and Model Setup

A key characteristic of the Bak-Sneppen model is the selection, mutation and recombination step that essentially drives the system. Analyzing the model from the perspective of Brownian diffusions in an open domain $D \subset \mathbb{R}^d$, Grigorescu and Kang [21] proposed the *diffusive Bak-Sneppen model*. We present the problem in a general setup. However, in this thesis, we study the special case where $D = (0, 1) \subseteq \mathbb{R}$. A general case in higher dimensions is presented in [17].

2.1 The Bak-Sneppen Branching Diffusion

Let $D \subseteq \mathbb{R}^d$ an open domain with smooth boundary ∂D and a diffusion on D generated by $(L, \mathcal{D}(L))$, where L is strongly elliptic with smooth coefficients up to the boundary and $\mathcal{D}(L) \subseteq C_b^1(\bar{D}) \cap C^2(D)$ is given by boundary conditions obtained by partitioning $\partial D = (\partial D)_r \cup (\partial D)_a$ in a component $(\partial D)_r$, the regular component, and $(\partial D)_a$, the absorbing component. To fix ideas, we assume reflecting boundary conditions on $(\partial D)_r$. While D may be unbounded, and the regular component may be taken empty, it will be assumed that the hitting time τ^D of the absorbing component will have an exponential moment, i.e. there exits

$\beta_1 > 0$ such that

$$E_x[e^{\beta_1 \tau^D}] < +\infty\,, \qquad \forall x \in D\,. \tag{2.1.1}$$

The construction of the BSB branching process can be done on more general sets, such as a smooth connected domain $D \subseteq \mathbb{R}^d$, in the d - dimensional Euclidean space. Even though in most of our work we shall work with $D = (0,1)$, this setup can be done in geneal. See also [17].

For a test function $\phi \in C^1([0,1]) \cap C^2(0,1)$, we shall denote $\phi \in (BC)_r$ if the function satisfies the conditions $\nabla\phi(x) = 0$ on $x \in (\partial D)_r$, the *reflecting boundary* which is $\{1\}$ in our case, and $\phi \in (BC)_a$ if it vanishes (or is constant) on the *absorbing boundary* $(\partial D)_a$, which is $\{0\}$. The boundary $(\partial D)_a$ can be assimilated to the cemetery state $\mathfrak{b}$ and a function $\phi \in (BC)_a$ will take values $\phi(\mathfrak{b})$ on $(\partial D)_a$.

The diffusion described solves the martingale problem $(L, \mathcal{D}(L))$ with

$$\mathcal{D}(L) = \{\phi \in C^1(\bar{D}) \cap C^2(D) \,|\, \phi \in (BC)_r \cap (BC)_a\} \tag{2.1.2}$$

(this is not necessarily equal to the generator of the Feller semigroup). When the boundary condition on $(\partial D)_a$ is replaced by $(BC)_{ac}$ we denote the set $\mathcal{D}_c(L)$,

$$\mathcal{D}_c(L) = \{\phi \in C^1(\bar{D}) \cap C^2(D) \,|\, \phi \in (BC)_r \cap (BC)_{ac}\}\,. \tag{2.1.3}$$

Now consider N Brownian particles moving in the interval $D = (0,1)$, denoted as $X_t(\omega) = (x_t^1(\omega), x_t^2(\omega), \ldots, x_t^N(\omega))$ where the subscript t represents the time. This particle system evolves according to the following mechanism.

1. Starting at an initial position $X_0 = (x_0^1, x_0^2, \ldots, x_0^N) \in [0,1]^N$, the particles

move independently with reflection at $(\partial D)_r$, until the first one, let's say with index i, $1 \le i \le N$, hits $(\partial D)_a$.

2. Then x^i, along with another particle, uniformly chosen with probability $1/(N-1)$ among the other particles, denoted by x^j, $j \neq i$, jump instantaneously to a fixed point $c \in D$ (resampling) and the system resumes as is described in 1) until another particle hits $(\partial D)_a$, triggering another redistribution described in 2). The whole particle system evolves according to these iterations.

We remark that (2.1.1) implies that the jump times are almost surely finite, and no two particles can reach the point zero simultaneously due to the absolute continuity of the law of independent Brownian motions.

Moreover, the sequence of the jump times $0 \le \tau_1 \le \tau_2 < \ldots$ is strictly increasing. This implies that $\tau^* = \lim_{i \to \infty} \tau_i$ exists. The case when $\tau^* < \infty$ with positive probability is said *explosive*.

Since our case is $D = (0,1)$, the next theorem shows that this is not possible. The result is true in higher dimensions and with a more general redistribution measure.

As mentioned before, the Bak-Sneppen Branching diffusion is an example of a non-exploding jump-diffusion process introduced in Section 4.2 of [21].

Theorem 2.1.1. *[Theorem 1 from [21], applied to Example 4.2 in the same paper] Assume that Conditions 1 and 2 are satisfied for the same* $\delta > 0$. *Then for any* $x \in D$, *we have* $P_x(\tau^* = \infty) = 1$.

Condition 1 pertains exclusively to the driving diffusion; in our case, this is immediately satisfied for Brownian motion on the unit interval with reflection at one. Condition 2 is saying that with probability one, the process enters an interior set, as would be, in our case $[\delta, 1)^N$, i.e. away from the boundary $\{0\}$ by at least $\delta > 0$. This is essentially a Doeblin type condition. Our process returns in the interior set with positive probability, bounded below independently of N, in not more than N steps. This implies Condition 2, by a geometric random variable argument.

More formally, we define the resulting process, denoted as $(X_t^N(\omega))_{t\geq 0}$, componentwise

$$X_t^N(\omega) = (x_t^{N,1}(\omega), \ldots, x_t^{N,N}(\omega)), \qquad t \geq 0$$

on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $\omega \in \Omega$, where $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. We name it the *Bak-Sneppen branching diffusion*, or BSB - process. By construction, $(x_t^N(\omega))_{t\geq 0}$ is a jump-diffusion on the Skorokhod space $\mathbb{D}^N([0,\infty),[0,1]^N)$ of right continuous with left limits paths.

The jump part of the diffusion comes from the boundary hits by particles, which is the driving force of the process. We denote the number of times that particle x^i hits 0 up to time $t \geq 0$, respectively the average of boundary hits at 0 by

$$A_t^{N,i}(\omega) := \int_0^t \mathbf{1}_{\{0\}}(x_s^{N,i}(\omega))\, ds\,, \quad t \geq 0 \tag{2.1.4}$$

$$A_t^N(\omega) := \frac{1}{N-1} \sum_{i=1}^N A_t^{N,i}(\omega)\,, \tag{2.1.5}$$

where the normalization constant $(N-1)^{-1}$ is chosen for convenience, but is naturally of order N^{-1}.

Because a trip from any interior point to the boundary takes a strictly positive time almost surely, again from (2.1.1) and also see [23], there are only finitely many boundary hits in a finite time interval, that is, locally finite. Such processes are said *counting processes* [13].

An important element in the study of this interacting particle system is its empirical measure process, defined as, for any fixed $t \geq 0$,

$$\mu_t^N(dy, \omega) := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_t^{N,i}(\omega)}(dy) \in M_1([0,1])$$

where $M_1([0,1])$ is the set of probability measures over $[0,1]$.

Notice that by construction, N remains constant, making the empirical measure not just a finite measure on the domain $D = [0,1]$, but a proper probability measure. We contrast this case to the empirical process ζ_t corresponding to the auxiliary branching diffusion in section 2.3, which is naturally *supercritical*.

The Law of Large Numbers (Theorem 3.1.3) will be proven at the level of the full trajectories of the particle system. Such a limit is called *hydrodynamic limit*.

The system is interacting, because the dependence is built in at branching/jump, when particle i picks a specific neighbor j to pair with. From that point on, even though the paths between jumps are independent, the starting position is not, being coordinated, and in the simplified model we work on, is actually the same, equal to the point $c \in (0,1)$. Intuitively, a law of large numbers can be proven because the correlations between the depenedent particles are weak, that is, of order at most N^{-1}. This is heuristically built in the uniform choice of the "other" particle j.

More precisely, we look at the map

$$t \to \mu_t^N(dy, \omega) \in \mathbb{D}([0,\infty), M_1([0,1])), \quad t \geq 0, \tag{2.1.6}$$

which defines a jump-diffusion stochastic process, in the sense that it is continuous between jumps triggered by the boundary hits at 0. Formally it is a random variable with values in the metric Skorokhod space $\mathbb{D}([0,\infty), M_1([0,1]))$ with J_1 topology covered in section 10.2.

This space will allow us to identify the limit of $\mu_t^N(dy,\omega)$, when $N\to\infty$, as a deterministic path $\mu_\cdot(dy)\in M_1([0,1])$, formalized in the following chapter, our main result, Theorem 3.1.3. In a nutshell, the measure $t\to\mu_t$ is conservative with unit total mass, and can be represented as the normalization of a finite measure with exponentially growing total mass $t\to\nu_t$

$$\mu_t(dy)=\frac{\nu_t(dy)}{\langle\nu_t,1\rangle}\,, \tag{2.1.7}$$

precisely described in Theorem 3.2.2.

2.2 Relation to the Fleming-Viot Process and the Dissipative Heat Equation

A general class of branching processes, the Fleming-Viot dynamics relates closely to our problem. We focus on a special case, when N particles diffuse independently in a connected open domain $D\subseteq\mathbb{R}^d$ and when one reaches the boundary, it picks one of the other $N-1$ at random and jumps on its location. The process continues indefinitely patching together episodes between consecutive jumps. One can see the analogy with the interaction in the Bak-Sneppen branching diffusion. The original construction interprets the jump as branching in the sense that the particle reaching the boundary is killed and then one new particle is born afresh at the location of one of the remining individulas. Since we deal with the empirical measure, statistically (in probability distribution) there is no difference between the jump vs. branching constructions.

The hydrodynamic limit for the Fleming-Viot process satisfies a similar representation as (2.1.7), only that ν_t is simply the solution to the forward heat equation with Dirichlet boundary conditions. In other words, the one-particle process (x_t) of the diffusion that drives each particle, considered with killing at the boundary, has a probability density $p^D(t,x,dy)$, where, for τ^D the hitting time of the boundary we have the *dissipative heat kernel*

$$p^D(t,x,dy) = P_x(x_t \in dy, \tau^D > t)\,.$$

Assuming the process starts with profile $\mu_0(dx)$, the analogue of the non-conservative measure is

$$\nu_t(dy) = \int_D p^D(t,x,dy)\mu_0(dx)\,,$$

with $\lim_{t\to\infty} \nu_t(B) = 0$ (dissipation of mass) for any B Borel set in D. It is well known that the dissipation is exponential at rate $\lambda_1 > 0$, where $-\lambda_1$ is, for example in the case of Brownian motion, the first eigenvalue of the Dirichlet half-Laplacian

$$\lim_{t\to\infty} \frac{1}{t} \ln \nu_t(D) = -\lambda_1\,. \tag{2.2.8}$$

In an intuitive sense, because the F-V dynamics "revives" the particle at a "typical" point of the mass distribution, it essentially recycles the dissipation and preserves the mass.

In our case, the representation (2.1.7) cannot be achieved with a one-particle process. The only way to produce a mass profile ν_t is to look at the opposite type of branching than simple killing at the boundary, which is a super-critical process (ζ_t) described in the next section.

2.3 The Auxiliary Processes Z_t and ζ_t

In the development of the main results in the following chapter, an auxiliary process naturally arises and proves to be significant in the formulation of the main theorem. We define the auxiliary process as a pure birth process $(Z_t(\omega))_{t\geq 0}$ that evolves as follows:

1. The whole system starts with a single Brownian particle z_t on $[0,1]$, starting at x. The particle is reflected at 1 and upon hitting 0:
2. a new particle z'_t is born and resampled, along with z_t itself, to c, where the two particles resume independent Brownian motions.
3. when either of these two particles hit 0, the whole system repeat (inductively) the mechanism descried in part 1 and 2.

It is clear that the numner of particles is non-decreasing and as long as a bound like (2.1.1) holds, in must count the number of visits to the boundary and goes to infinity almost surely.

Let $\tilde{D} = (0,1) \cup \{\mathfrak{d}\}$ be the state space augmented by a cemetery state $\mathfrak{d}$ corresponding to the boundary hits at 0 and $N_t(\omega)$ be the number of particles in the system at time $t \geq 0$. In this special case, when $N_t(\omega)$ is almost surely non-decreasing, can define the process $Z_t(\omega)$

$$Z_t(\omega) := \left(z_t^1(\omega), z_t^2(\omega), \ldots, z_t^{N_t}(\omega)\right), \quad t \geq 0 \tag{2.3.9}$$

on $\tilde{D}_0^\infty$, the subspace of $\tilde{D}^\infty$ with only finitely many components outside $\mathfrak{d}$.

In this work we follow the measure-valued process construction approach which is standard to branching processes [13]. Similar to the idea of defining the empirical measure process μ_t^N in the previous section, we can define a measure-valued process ζ_t as follows:

$$\zeta_t(\omega) = \sum_{i=1}^{N_t} \delta_{z_t^i(\omega)} \in M_F([0,1]), \quad \text{if } N_t > 0\,, \tag{2.3.10}$$

where $M_F([0,1])$ is the space of finite measures on the set $[0,1]$. By construction, its law is a probability measure on the Skorokhod space $\mathbb{D}([0,\infty), M_F([0,1])$.

We denote $\langle \phi, \alpha \rangle$ as the integral $\int \phi(x)\, \alpha(dx)$, where ϕ is a smooth test function (in an appropriate sense) and $\alpha \in M_F([0,1])$. Under this notation, notice that $N_t = \langle \zeta_t, 1 \rangle$. More importantly, assuming the process is non-explosive, which we shall prove shortly in Proposition 7.1.1, we can define

$$\nu_t^x = E[\zeta_t^x(\omega)]$$

in the sense that

$$E(\langle \zeta_t^x, \phi \rangle) = \langle \nu_t^x, \phi \rangle \tag{2.3.11}$$

for any test function $\phi \in C^1[0,1] \cap C^2(0,1)$. Here we use the superscript x to denote the starting point. The process can be started at a random point X with distribution $\nu_0(dx)$ and in that case

$$\langle \nu_t, \phi \rangle = \int_{[0,1]} \langle \nu_t^x, \phi \rangle \nu_0(dx)\,. \tag{2.3.12}$$

The deterministic measure, the measure-valued process $t \to \nu_t(dy)$ is the weak solution of the heat equation with mass creation (3.2.4) and will fully determine the hydrodynamic limit (2.1.7) of the process $(X_t^N)_{t\geq 0}$, which satisfies the weak equation (3.1.3).

Chapter 3

Main Results

3.1 Hydrodynamic Limit of the Bak-Sneppen Branching Diffusion

Definition 3.1.1. [Tightness] Given a probability space $(S, \mathcal{S})$. A family of probability measures, Π, is *tight* if for every ϵ there exists a compact set K such that $P(K) > 1 - \epsilon$ for every P in Π.

The key point of the main theorem below is that the empirical measure process, $\mu_t^N(\omega, dy)$, whose paths are viewed as functions in the Skorokhod space $\mathbb{D}([0, \infty), M_1([0, 1]))$, is tight and thus has a deterministic limit point in the space $\mathbb{D}([0, \infty), M_1([0, 1]))$. The limit is obtained in the sense of convergence in probability, uniformly in time, on every finite time interval $[0, T]$. This is a Law of Large Numbers at the full path level for an interacting particle system.

To characterize the limit, we need the solution of a pde which we call *the heat equation with mass creation* (3.2.4). This admits a representation as the expected value of the empirical mass of a super critical branching process ζ_t.

Definition 3.1.2. The process $(\mu_\cdot^N)_{N>0}$ *converges weakly in probability* to $(\mu_\cdot)$ if

(i) For any $t \geq 0$, $(\mu_t^N)_{N>0}$ is a tight family and

(ii) For any $T > 0$ and any test function $\phi \in \mathcal{D}_b$, the process $t \to \langle \mu_t^N, \phi(t, \cdot) \rangle_{t \geq 0}$ satisfies

$$\forall \epsilon > 0 \qquad \lim_{N \to \infty} P\Big(\sup_{t \in [0,T]} |\langle \mu_t^N, \phi(t, \cdot)\rangle - \langle \mu_t, \phi(t, \cdot) \rangle| > \epsilon \Big) = 0 \,. \tag{3.1.1}$$

We shall say that $\mu_0^N(dy, \omega)$ converges *weakly in probability* to the deterministic $\mu_0(dy)$ if for any test function ϕ

$$\forall \epsilon > 0 \qquad \lim_{N \to \infty} P(|\langle \mu_0^N, \phi \rangle - \langle \mu_0, \phi \rangle| > \epsilon) = 0 \,. \tag{3.1.2}$$

The following is our main result.

Theorem 3.1.3. *Assume (3.1.2) holds and there exists $\rho_0 \in C([0,1])$ such that $\mu_0(dy) = \rho_0(y)dy$. Then*

1) the empirical measure process $(\mu_t^N(dy, \omega))_{t \geq 0}$ converges weakly in probability, as $N \to \infty$, to a deterministic path $(\mu_t(dy))_{t \geq 0} \in \mathbf{C}([0, \infty), M_1([0,1]))$;

2) the average number of jumps $(A_t^N)_{t \geq 0}$ converges in probability, uniformly on any time interval $[0, T]$, to the non-decreasing deterministic function $(A_t)_{t \geq 0}$, which can be identified as $\ln\langle \nu_t, 1\rangle = A_t = \int_0^t a_s \, ds$, where a_t is a non-negative continuous function and ν_t is the solution of (3.2.4),

3) the limiting profile $(\mu_t)_{t \geq 0}$ satisfies

$$\langle \mu_t, \phi(x) \rangle = \langle \mu_0, \phi(x) \rangle + \int_0^t \langle \mu_s, \frac{1}{2}\phi''(x) - a_s \phi(x) \rangle \, ds \tag{3.1.3}$$

for all $\phi \in C^1[0,1] \cap C^2(0,1)$, with boundary condition (3.2.5). Moreover, $\mu_t(dy)$ has a unique density function $\rho(t, y)$ and can be identified by normalizing the solution $\nu_t(dy) = v(t, y)dy$ given in Theorem 3.2.2. More precisely, $\mu_t = \nu_t / \langle \nu_t, 1 \rangle$

and satisfies (3.3.8).

Remark 3.1.4. 1) Theorem 3.1.3 is a Law of Large Numbers at the path level for interacting (dependent) particles.
2) We notice that $\mu_t(dy)$ is a probability measure, so the normalization of the exponentially growing $\nu_t(dy)$ arises naturally. In other particle systems, such as the Fleming-Viot process, the role played by $\nu_t(dy)$ is the usual dissipative solution (with Dirichlet boundary conditions) to the heat equation on the domain.
3) By letting $t \to \infty$ we can see in (3.3.8) how the equilibrium profile emerges as a left-eigenfunction of the Laplacian with boundary conditions (3.2.5).

Remark 3.1.5. Later on in (3.3.9), and in more detail in Section 9.2 we look at

$$\lim_{N\to\infty}\lim_{t\to\infty} E[\phi(x_t^1)] = \langle \rho(x), \phi(x) \rangle \,,$$

where ρ is the quasi-invariant measure of the auxiliary process, which gives an estimate on marginal distribution and thus sheds light on the appearance of the particle system at equilibrium. The relevance of this formula lies with its similarity with (1.1.1), identifying the qsd as a left-side eigenvalue corresponding to the eigenvalue $\lambda_* > 0$ associated to the evolution semigroup of the auxiliary process. The constant λ_* can be considered intrinsic to the dynamics, as such a self-organizing feature.

The main result is proved by analyzing the martingale representation (5.2.4) via Ito's Formula for general semi-martingales, of the empirical measure process for the particle system. First of all, we write the martingale representation of the particle system. This will help us make clear that the true problem is the tightness of A_t^N, which is the hardest part of the entire proof. In a word, we plan to mathematically justify that the majority of the particles near 0 are moving relatively slowly while the fast-moving particles are far away from 0. The net effect is 0 doesn't get hit so frequently that the resampling step gets out of control and finally explode. Lastly, through Theorem 3.2.2 below, we prove equation (3.1.3) as well as the corresponding identifications.

3.2 Representation of the Solution of the Heat Equation with Mass Creation

As is mentioned earlier, the identification of the weak limit $\mu_t(dy)$ is given with the help of $\nu_t(dy)$ defined in (2.3.11), which turns out to be the *weak solution* of the *heat equation with mass creation* formally presented in Theorem 3.2.2.

Definition 3.2.1. For any $\nu_0(dy) \in M_1([0,1])$, $\nu_t(dy) \in C([0,\infty), M_F([0,1]))$ is said a weak solution of *the heat equation with mass creation at* c with initial value ν_0 if,

$$\langle \nu_t, \phi \rangle = \langle \nu_0, \phi \rangle + \int_0^t \langle \nu_s, \frac{1}{2}\phi'' \rangle \, ds \tag{3.2.4}$$

with boundary conditions for any $\phi \in C^1[0,1] \cap C^2(0,1)$, with boundary conditions

$$\phi'(1) = 0 \qquad \text{reflection at } x = 1\,, \tag{3.2.5}$$

$$2\phi(c) = \phi(0) \qquad \text{mass creation at } x = c\,. \tag{3.2.6}$$

Let ν_t, ν_0 as in (2.3.12). The theorem gives a representation of the solution of the partial differential equation (3.2.1).

Theorem 3.2.2. *For any* $\nu_0(dy) \in M_1([0,1])$, $\nu_t(dy) \in C([0,\infty), M_F([0,1]))$ *is a deterministic measure-valued path, which is the weak solution of* the heat equation with mass creation at c *in the sense of (3.2.1). Moreover,* $\nu_t(dy)$ *has a density* *function* $v(t,y) \in C^{1,2}((0,\infty) \times (0,1))$ *and extends all the way to zero if* ν_0 *is absolutely continuous with a continuous bounded density function.*

Remark 3.2.3. The interesting part is the boundary condition $2\phi(c) = \phi(0)$, which represent a mass creation at the point c upon each redistribution. In principle, we can solve this using purely analytic methods but probabilistic methods turn out to be easier and provides insight into the real phenomena occurring in the process. This is one of the motivations for our construction of the auxiliary process Z_t.

Remark 3.2.4. We check that $v(t,x)$ satisfies the conjugate boundary conditions (9.2.17). When $D = (0,1)$, $(\partial D)_r = \{1\}$, $(\partial D)_a = \{0\}$, $\gamma = \delta_c$, $c \in (0,1)$ and $L = \frac{1}{2}\frac{d^2}{dy^2}$ with $\nu_0(dx) = v_0(x)dx$. Then $L = L^*$, $\nu_t(dy) = v(t,y)dy$ with $v(0+,\cdot) = v_0(\cdot)$ and v has continuous time derivative. In addition, one can verify directly that for any $t > 0$, v is smooth in $(0,c) \cup (c,1)$ and satisfies the boundary conditions

$$\begin{gathered} v(t,c-) = v(t,c+)\,, \quad v'(t,1) = 0\,, \quad v(t,0) = 0 \\ (v'(t,c+) - v'(t,c-)) + 2v'(0) = 0\,. \end{gathered} \tag{3.2.7}$$

3.3 Quasi-stationarity and Equilibrium Profile

After fully identifying the hydrodynamic limit $\mu_t(dy)$, we turn to the question of status of the particle system at equilibrium. On the other hand, we know from Theorem 3.2.2 that there exists a density function $v(t,y)$ so that $\int_0^1 v(t,y)\,dy = e^{A_t}$. Denoting $\rho(t,y) = v(t,y)/e^{A_t}$, equation (3.1.3) of Theorem 3.1.3 will read:

$$\frac{\partial\rho(t,x)}{\partial t} = \frac{1}{2}\Delta\rho(t,x) - a(t)\rho(t,x)\,. \tag{3.3.8}$$

Notice that $a(t)$ here also depends on ρ, making (3.3.8) a reaction-diffusion type equation. Heuristically, at equilibrium, the density function ρ doesn't depend on t, therefore $\frac{\partial\rho}{\partial t} = 0$. Also at equilibrium $a(t)$ will change into a constant number, which will be given by Proposition 7.1.1, that is, $a(t) \to \lambda_*$ when $t \to \infty$, making (3.3.8) as:

$$\frac{1}{2}\Delta\rho(x) - \lambda_*\rho(x) = 0 \tag{3.3.9}$$

The solution, which is the left eigenfunction of (3.3.9) with boundary conditions (3.2.5) and (3.2.6), will identify the quasi-stationary distribution for the original model. We summarize this in the following theorem.

The following theorem is proven in Section 9.2.

Theorem 3.3.1. *The quasi-stationary distribution has a density function*

$$\rho(x)dx = \lim_{t\to\infty}\frac{\nu_t(dx)}{\langle\nu_t,1\rangle}\,, \quad \textit{in distribution} \tag{3.3.10}$$

which is equal to the left-eigenfunction of the semigroup of the auxiliary process corresponding to the critical value λ_*.

This result is generalized in [18]. The number $K = 2$ is the number of particles in the next generation after branching, and can be generalized to be a random variable as in [18]. This influences the nature of the quasi-stationary measure ρ, including its existence and multiplicity, adding a degree of freedon to the problem.

In order to identify $\rho(x)$, done in (9.2.18), seen as a left-eigenvalue, we integrate it against a test function $\phi(x)$ and proceed by integration by parts in Section 9.2.

We'll find in Chapter 9.2 that the boundary condition corresponding to the mass creation now becomes (9.2.17) with solution of the form

$$g(x) = \begin{cases} e^{\beta x} + e^{2\beta} e^{-\beta x}\,, & c \le x \le 1 \\ q(e^{\beta x} - e^{-\beta x})\,, & 0 \le x \le c \end{cases}\,, \tag{3.3.11}$$

for a specific $\beta_* > 0$ with

$$q = \frac{e^{\beta c} + e^{2\beta} e^{-\beta c}}{e^{\beta c} - e^{-\beta c}}\,.$$

The constant β_* solves the equation

$$[(e^{\beta c} - e^{2\beta} e^{-\beta c}) - q(e^{\beta c} + e^{-\beta c})] + 2Kq = 0\,, \tag{3.3.12}$$

where the number of particles born is, in our case, $K = 2$. Proposition 9.2.1 relates this constant to the eigenvalue λ_* from (7.1.2) by the formula

$$\beta_* = \sqrt{2\lambda_*}\,.$$

Chapter 4

Uniform Bounds in N and T

Let $\mathcal{C}^N = C^{1,2}((0,\infty) \times D^N, \mathbb{R}) \cap C^{0,1}([0,\infty) \times \bar{D}^N, \mathbb{R})$ be the class of N - dimensional time-space test functions $F(t,x)$ *continuous up to the boundary* and, by analogy to (2.1.3)

$$\mathcal{D}_c^N = \{F \mid F \in \mathcal{C}\,, F_{|x_i} \in (BC)_r \cap (BC)_{ac}\,, 1 \leq i \leq N\}\,, \tag{4.0.1}$$

where $F_{|x_i}$ is the marginal function when all but component x_i are fixed and the boundary conditions are described in the paragraph containing the definition (2.1.2).

Denote $L^{\otimes N}$ the direct sum of the one variable operator L, by F^{ij} (defined precisely below) the configuration under F after redistribution of the particle i, which has reached ∂D, and has chosen particle j and both are created anew at the same random point with distribution $\gamma(dx)$

$$L^{\otimes N} F(s,X) = \sum_{i=1}^{N} L_{x_i} F(s,\ldots,x_i,\ldots) \tag{4.0.2}$$

$$F^{ij}(s,X) = 2\int_{\{x_i = x_j\}} F(s,\ldots,x_i,\ldots,x_j\ldots)\gamma(dx_i)\gamma(dx_j) \tag{4.0.3}$$

$$= 2\int_D F(s,\ldots,x,\ldots,x\ldots)\gamma(dx)\,, \tag{4.0.4}$$

where the identical entries are on position i and j.

Let $A_t^{N,i}$ be the number of hits of particle i to the absorbing boundary $(\partial D)_a$ from (2.1.4). Notice that $X_{t-}^{N,i} = \mathfrak{b}$ if and only if the counting process $A_t^{N,i}$ has a discontinuity, with probability one.

The joint set of interacting processes $(X_t^{N,i}, A_t^{N,i})_{t\geq 0}$, for $1 \leq i \leq N$, was defined constructively in Section 2.1, based on the strong Markov property, the fact that there are no simultaneous boundary hits, and the non-explosion result.

We also denote by $\mathcal{M}_t^F$, for each $F \in \mathcal{C}^N$ the processes

$$\mathcal{M}_t^F = F(t, X_t^N) - F(0, X_0^N) - \int_0^t L^{\otimes N} F(s, X_s^N)\, ds \tag{4.0.5}$$

$$- \sum_{i=1}^N \int_0^t \Big(\frac{1}{N-1} \sum_{j\neq i} F^{i,j}(s, X_{s-}^N) - F(s, X_{s-}^N) \Big)\, dA_s^{N,i}\,. \tag{4.0.6}$$

We have explicit expressions for the quadratic variations for both the jump martingale and the continuous martingale, through which we can show A_t^N is bounded, for fixed $t \geq 0$.

Proposition 4.0.1. *The processes* $(\mathcal{M}_t^F)$ *are* $\mathcal{F}_t$ *- martingales with continuous and jump components* $\mathcal{M}_t^F = \mathcal{M}_t^{F,c} + \mathcal{M}_t^{F,J}$*, such that* $\mathcal{N}_t^{F,c}$*, respectively* $\mathcal{N}_t^{F,J}$ *are also* $\mathcal{F}_t$ *- martingales, where*

$$\mathcal{N}_t^{F,c} = (\mathcal{M}_t^{F,c})^2 - \sum_{i=1}^N \int_0^t (L_{x_i} F^2 - 2\langle F, L_{x_i} F \rangle)(s, X_s^N) ds \tag{4.0.7}$$

$$\mathcal{N}_t^{F,J} = (\mathcal{M}_t^{F,J})^2 - \sum_{i=1}^N \int_0^t \frac{1}{N-1} \sum_{j\neq i} (F^{i,j}(s, X_s^N) - F(s, X_{s-}^N))^2\, dA_s^{N,i}\,. \tag{4.0.8}$$

Proposition 4.0.2. *Theres exists a constant* $C(\gamma)$*, independent of* t *and* N *but* *dependent on the initial configuration, such that, for all $t \geq 0$ and $N \in \mathbb{Z}_+$,*

$$E[\sum_{i=1}^{N} A_t^{N,i}] \leq C(\gamma)Nt\,. \tag{4.0.9}$$

Remark 4.0.3. As *Step 1* below shows it, it is not hard to see that the processes in the statement are local martingales. In fact, all the processes in Proposition 4.0.1 are proper martingales, which is equivalent to showing that $E[A_t^{N,i}] < \infty$ for all components $1 \leq i \leq N$ and $t \geq 0$.

Proof. Step 1. The process (X_t^N) is non-explosive (cf. [21]) and stated here in Theorem 2.1.1, so $\lim_{t\to\infty} A_t^{N,i} = +\infty$ a.s., which implies, due to the boundedness of all integrand terms in the martingales, that setting T_m, $m \geq 1$ the first hitting time of the positive integer m by the sum $\sum_{i=1}^{N} A_t^{N,i}$, the processes $\mathcal{M}$, (4.0.7), (4.0.8) are local martingales by setting $t \to t \wedge T_m$, in other words with localization sequence T_m.

Step 2. We prove the processes are martingales. Set $F(t, X) = \frac{1}{N}\sum_{i=1}^{N} \phi(x_i)$ for a function $\phi \in (BC)_a$, $0 \leq \phi \leq 1$ with $c_\gamma = 2\langle\gamma, \phi\rangle - 1 > 0$. Such a function exists since γ has integral one and ϕ can be taken as a smooth function approximating the indicator function of a compact set in D. In that case, the integrand of the dA_t^N term is greater or equal to c_γ, so we obtain, almost surely,

$$\begin{aligned} c_\gamma A_{t\wedge T_m}^N \leq -\mathcal{M}_{t\wedge T_m}^F &+ F(t \wedge T_m, X_{t\wedge T_m}^N) \\ &- F(0, X_0^N) - \int_0^{t\wedge T_m} L^{\otimes N} F(s, X_s^N)\, ds\,. \end{aligned} \tag{4.0.10}$$

Taking the expected value, we see that there exists a constant $C(\gamma)$, independent of t and N because it is simply a uniform bound on the function ϕ and its derivatives, such that $E[A_{t\wedge T_m}^N] \leq C(\gamma)t$. Since $\lim_{m\to\infty} T_m = +\infty$ a.s. we obtain by dominated convergence the same bound for $E[A_t^N]$, proving the proposition. $\square$

Chapter 5

Sketch of the Proof

5.1 Martingale Representation and Ito's Formula

As we mentioned in the chapter on the main results, tightness of (A_t^N) is the most important step of the entire proof. The importance is manifested in the martingale representation of the particle system.

Let $f(t, X) = f(t, x_1, x_2, \ldots, x_N)$ be a function in $C^{1,2}([0, \infty) \times (0, 1)^N, \mathbb{R}) \cap C^{1,1}([0, \infty) \times [0, 1]^N, \mathbb{R})$. The evolution of the particle system can be characterized, using (4.0.5) for this test function, with the particularization of the driving motion being Brownian motion with reflection at 1,

$$\text{particle } i \text{ moves according to} \quad L_{x_i} = \frac{1}{2}\Delta_{x_i} \quad \text{reflected at one} \,,$$

by the generalized Ito formula

$$
\begin{aligned}
f(t, X_t^N) = f(0, X_0^N) &+ \int_0^t \partial_s f(s, X_s^N) + \frac{1}{2}\Delta_N f(s, X_s^N)\, ds \qquad (5.1.1)\\
&- \sum_{i=1}^N \partial_i f(s, X_s^N) d\ell_s^i \\
&+ \int_0^t \sum_{i=1}^N \frac{1}{N-1} \sum_{i \neq j} \int_0^1 \int_0^1 \left(f^{ij}(s, X_s^N) - f(s, X_{s-}^N) \right) dA_s^{N,i} \\
&+ \sum_{i=1}^N \int_0^t \partial_i f(s, X_s^N)\, dB_s^{N,i} + \mathcal{M}_t^{J,f,N}\,, \qquad (5.1.2)
\end{aligned}
$$

where the continuous martingale part is now given explicitly in (5.1.2).

There are several things to clarify. First of all, the notation

$$\Delta_N f(s, X) = \sum_{i=1}^N \partial_i^2 f(s, X)$$

is the N - dimensional Laplacian. Next, $d\ell_t^i$ represents the *local time* of particle i spent at the point $x = 1$, indexed by the time $t \geq 0$.

We start with the general definition of local time at a point x, for a one dimensional Brownian motion W_t starting at x_0, along the lines of [23].

Definition 5.1.1 (local time)**.** Let $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a Brownian Motion where $P(W_0 = x_0) = 1$ and $\{\mathcal{F}_t\}$ satisfies the usual conditions. Denote $\ell_t(x)$ as the *local time* at point x:

$$\ell_t(x) = \lim_{\epsilon \to 0} \frac{1}{4\epsilon} meas\{0 \leq s \leq t; |W_s - x| \leq \epsilon\}$$

where $meas(ds)$ is the Lebesgue Measure. In our case, $x = 1$ because each particle in the system is reflected at 1 and we omit the redunadant parenthesis $\ell_t := \ell_t(1)$.

In our case, each particle $x_t^{N,i}$ follows a standard Brownian motion W_t^i part until reaching the boundary. If τ^D is the hitting time of the point $x = 0$, we write

$$x_t^{N,i} = 1 - |1 - W_t^i|\,, \qquad t \leq \tau^D\,, \qquad 1 \leq i \leq N \tag{5.1.3}$$

and notice that $S_t = |1 - W_t^i|$, and $S_{t\wedge\tau^D}$ are continuous sub-martingales. It is a standard construction that

$$S_{t\wedge\tau^D} - \ell_{t\wedge\tau^D} \qquad \text{is a martingale with respect to } \mathcal{F}_{t\wedge\tau^D}$$

from the Doob-Meyer decomposition of $S_{t\wedge\tau^D}$.

In our derivations, the local time terms will be omitted because the boundary conditions will always be chosen such that $\partial_i f(X) = 0$ whenever $x_i = 1$.

Secondly, the function $f^{ij}(s, X)$ is defined as $f(s, x_1, \ldots, \underbrace{c}_{\#i}, \ldots, \underbrace{c}_{\#j}, \ldots, x_N)$, meaning the resampling of particles x^i and x^j to c. Lastly, $\mathcal{M}_t^{J,f,N}$ represents the jump martingale part of the process.

5.2 From Ito Formula to a PDE in Weak Form

We shall take

$$f(t, x_1, \ldots, x_N) = \frac{1}{N}(\phi(x_1) + \ldots, +\phi(x_N))\,,$$

$$\phi(x) \in C^1([0,1]) \cap C^2(0,1)$$

and satisfies $\phi'(1) = 0$. With this $f(x_1, \ldots, x_N)$ so defined, the above formula can be written as:

$$\langle \phi, \mu_t^N \rangle - \langle \phi, \mu_s^N \rangle = \int_s^t \langle \frac{1}{2}\phi'', \mu_r^N \rangle \, dr + \sum_{i=1}^N \int_s^t \phi'(x_r^{N,i}) dB_r^i + \tag{5.2.4}$$

$$+ \int_s^t \sum_{i=1}^N \frac{1}{N-1} \sum_{i \neq j} \Big(\frac{1}{N} (\phi(x_r^{N,1}) \ldots + \underbrace{\phi(c)}_{\# i} + \ldots + \underbrace{\phi(c)}_{\# j} + \ldots \phi(x_r^{N,N}))$$

$$- \frac{1}{N} \sum_{i=1}^N \phi(x_r^{N,i}) \Big) \, dA_r^{N,i} + \mathcal{M}_{t-s}^{J,\phi,N}$$

To simplify the expression, we consider a typical term from the second line of formula 5.2.4:

$$\int_s^t \frac{2}{N}\phi(c) - \frac{1}{N}\phi(0) - \frac{1}{N(N-1)} \sum_{j \neq i} \phi(x_r^{N,j}) \, dA_r^{N,i}$$

Hence,

$$\begin{aligned} \langle \phi, \mu_t^N \rangle - \langle \phi, \mu_s^N \rangle = & \int_s^t \langle \frac{1}{2}\phi'', \mu_r^N \rangle \, dr + \sum_{i=1}^N \int_s^t \phi'(x_r^{N,i}) dB_r^i \ + \mathcal{M}_{t-s}^{J,\phi,N} \\ & + \int_s^t 2\phi(c) - \phi(0) - \langle \phi, \mu_r^N \rangle - \Big(\frac{\langle \mu_r^N, \phi \rangle - \phi(0)}{N-1} \Big) \, dA_r^N \end{aligned} \tag{5.2.5}$$

which is,

$$\begin{aligned} \langle \phi, \mu_t^N \rangle - \langle \phi, \mu_s^N \rangle = & \int_s^t \langle \frac{1}{2}\phi'', \mu_r^N \rangle \, dr + \sum_{i=1}^N \int_s^t \phi'(x_r^{N,i}) dB_r^i \\ & + \mathcal{M}_{t-s}^{J,\phi,N} + \int_s^t 2\phi(c) - \phi(0) - \langle \phi, \mu_r^N \rangle - \frac{2\phi(0)}{N} \, dA_r^N \end{aligned} \tag{5.2.6}$$

Remark 5.2.1. Since $2\phi(c) - \phi(0) - \langle \phi, \mu_s^N \rangle > 0$, $\langle \phi, \mu_t^N \rangle$ is a submartingale, which implies that particles won't cluster near 0. This is largely due to the "mass creation" that takes place at point c.

Since the limit we aim at is a deterministic path, as a result of the Law of Large Numbers, in principle, we plan to prove that the martingale part goes to 0. The first three terms on the right hand side of formula (5.2.6) would goes to 0 as $N \to \infty$ due to the smoothness of test function $\phi(x)$ and Doob's maximal inequality. In particular,

$$
\begin{aligned}
& P\big(\sup_{0\le t\le T}\sum_{i=1}^{N}\int_0^T \phi'(x_r^{N,i})\,dB_r^i \ge N\epsilon\big) \quad \text{(from Doob's inequality)} \\
& \le (N\epsilon)^{-2}E\big[\langle\sum_{i=1}^{N}\int_0^T \phi'(x_r^{N,i})\,dB_r^i\rangle\big] \quad \text{(where } \langle\cdot\rangle \text{ here is the quadratic variation)} \\
& \le N^{-1}\epsilon^{-2}E\big[\int_0^T \langle(\phi')^2,\mu_r^N\rangle dr\big] \le N^{-1}\epsilon^{-2}(\sup_{x\in[0,1]}|\phi'(x)|)^2 = O(N^{-1})\,.
\end{aligned}
$$

However difficulties remain: the first one is the quadratic variation of the jump martingale, which will be also shown to be of order N^{-1} in (6.2.19). The next is to observe that to close the formula we need to remove the term $2\phi(c) - \phi(0)$ in the dA_r^N term

$$\int_0^t 2\phi(c) - \phi(0) - \langle\phi,\mu_s^N\rangle - \frac{2\phi(0)}{N}\,dA_s^N\,.$$

Finally, one has to prove the tightness, and identify the limit of A_t^N. The most crucial step of proving this tightness is a Wald type (renewal) theorem for the expected value of the number of boundary hits at 0 by each particle in a given time interval, based on a coupling with a process that increases its "distance" to 0 with each jump, as is shown below.

If all these steps are proven, we assume all objects in the formula have a limit, and supressing the superscript N, equation (5.2.6) becomes

$$\langle\phi,\mu_t\rangle - \langle\phi,\mu_s\rangle = \int_s^t \langle\frac{1}{2}\phi'',\mu_r\rangle\,dr - \int_s^t \langle\phi,\mu_r\rangle\,dA_r \tag{5.2.7}$$

which is (3.1.3), the partial differential equation in weak form.

Chapter 6

Tightness

To guarantee that the weak limit μ_t is a continuous path, we need A_t^N to be C-tight, which means the weak limit A_t not only belongs to $\mathbb{D}([0,\infty),[0,1])$, but is continuous as well, that is, $A_t \in \mathbb{C}([0,\infty),[0,1])$.

C-tightness is usually proved through the Aldous Criterion, originally from [1], with an updated presentation in the context of branching particle systems in [32]. Let X be a Polish space with the norm $||\cdot||$ ad let $\mathbb{D}([0,T],X)$ be the Skorohod space of RCLL functions on $[0,T]$. The following are the sufficient conditions for tightness in $\mathbb{D}([0,T],X)$ of the family of the processes $\{y^N(\cdot)\}_{N>0} \in \mathbb{D}([0,T],X)$, adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

(i) for every $t \in [0,T]$ and every $\epsilon > 0$, there exists an $M > 0$ such that

$$P\Big(|y^N(t)| > M\Big) \leq \epsilon \tag{6.0.1}$$

(ii) For any $\epsilon > 0$ and any stopping time τ,

$$\lim_{\eta\to 0}\limsup_{N\to\infty}\sup_{\tau\in[0,T]} P\Big(|y^N(\tau+\eta) - y^N(\tau)| > \epsilon\Big) = 0 \tag{6.0.2}$$

In the context, we want to prove that 1) A_t^N is uniformly bounded by N and t and 2) modulus of continuity, which we summarize in the following proposition.

Proposition 6.0.1. *Assume $\mu_0^N \Rightarrow \mu_0$ and $\mu_0 \in M_1(D)$. Then, for any arbitrary but fixed $T > 0$,*

$$\limsup_{N\to\infty} E[A_T^N] < +\infty \tag{6.0.3}$$

$$\lim_{\eta\to 0}\limsup_{N\to\infty}\sup_{t\in[0,T]} P(A_{t+\eta}^N - A_t^N > \epsilon) = 0\,. \tag{6.0.4}$$

Remark 6.0.2. Notice that modulus of continuity (6.0.4) is stronger than Aldous Criterion in the sense that we take supremum over each t in $[0,T]$, not only stopping times. The above proposition is essentially adapted from Theorem 10.2.8.

The proposition is proven in Section 6.2, after we prove some key estimates on the process in the following lemmas.

6.1 Lemma on the Lifetime of a Given Particle

Next, we turn to modulus of continuity (6.0.4). We begin by splitting the even A_η^N where η is a small positive number.

Fix $T > 0$ and let τ be a stopping time for the entire particle system in $[0,T]$, we have

$$P\big(|A_{\tau+\eta}^N - A_\tau^N| > \epsilon\big) \le \frac{1}{\epsilon}E[|A_{\tau+\eta}^N - A_\tau^N|] \le \frac{1}{\epsilon}\sup_{\tau\in[0,T]} E[E_{X_\tau^N}[A_\eta^N]]$$

The first inequality is justified by Markov Inequality and the second by strong Markov property applied at X_τ^N. Notice that for any particle x^i,

$$A_\eta^{N,i} = \mathbf{1}_{\tau_{x^i}\le\eta}\big(1 + (A_\eta^{N,i} - A_{\tau_x^i}^{N,i})\big)$$

where τ_{x^i} is the hitting time of 0 for x^i . This rewriting is natural in the sense that it counts the number of "episodes" that take place during $[0,\eta]$ where "episode" means x^i finish a trip starting at c and ending at 0. Hence,

$$A_\eta^N = \frac{1}{N}\sum_{i=1}^N A_\eta^{N,i} = \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\tau_{x^i}\le\eta} + \frac{1}{N}\sum_{i=1}^N (A_\eta^{N,i} - A_{\tau_{x^i}}^{N,i})\mathbf{1}_{\tau_{x^i}\le\eta}$$

Take expectation both sides, we get the upper bound

$$E_{X_\tau^N}[A_\eta^N] \le \frac{1}{N}\sum_{i=1}^N P_{X_\tau^N}(\tau_{x^i}\le\eta) + \frac{1}{N}\sum_{i=1}^N E_{X_\tau^N}[A_\eta^{N,i}] \tag{6.1.5}$$

$$\frac{1}{\epsilon}\sup_{\tau\in[0,T]} E[E_{X_\tau^N}[A_\eta^N]] \le \frac{1}{N\epsilon}\sup_{\tau\in[0,T]} E[P_{X_\tau^N}(\tau_{x^i}\le\eta)] + \frac{1}{N\epsilon}\sup_{\tau\in[0,T]} E[E_{X_\tau^N}[A_\eta^{N,i}]]$$

where the second summand on the right hand side comes from *Strong Markov Property*. We'll need to bound both $\sup_{\tau\in[0,T]} E[P_{X_\tau^N}(\tau_{x^i}\le\eta)]$ and $\sup_{\tau\in[0,T]} E[E_{X_\tau^N}[A_\eta^{N,i}]]$ above. As is previously mentioned in Section 3.1, the proof comes down to analysis on the lifetime of particles that move near 0 and particles that are relatively far away from 0.

Remark 6.1.1. Later a split on time and τ becomes t...

Lemma 6.1.2. [*Particles that move close to* 0]. Let $|\cdot|$ be the cardinal number of a finite set. Then

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{t\in[0,T]} E\Big[\frac{|\{i|x_t^i\le\delta\}|}{N}\Big] = 0\,.$$

Proof. Consider a nonnegative test function $\rho(t,x)\in C^{1,2}([0,T]\times[0,1])$ that satisfies the following conditions: 1) $2\rho(t,c)=\rho(t,0)$; 2) $\rho(0,x)=\psi_\delta(x)$ where $\psi_\delta(x)$ is a decreasing $C^2([0,1])$ function that is identically equal to 1 over $[0,\delta]$ and vanishes to 0 at 1. Apply Ito Formula to $\langle\mu_t^N,\rho(T-t,x)\rangle$. For simplicity, write $\rho(T-t,x)$ as $\phi(t,x)$.

$$\langle \mu_t^N, \phi(t,x)\rangle - \langle \mu_0^N, \phi(0,x)\rangle - \int_0^t \langle \mu_s^N, \frac{\partial \phi}{\partial s}(s,x) + \frac{1}{2}\phi''(s,x)\rangle \, ds -$$
$$\int_0^t (2\phi(s,c) - \phi(s,0)) - \langle \mu_s^N, \phi(s,x)\rangle - \frac{\phi(s,0)}{N} \, dA_s^N = \mathcal{M}_t^c + \mathcal{M}_t^j$$

which reduces to the fact that the following difference is a martingale

$$\langle \mu_t^N, \phi(t,x)\rangle - \langle \mu_0^N, \phi(0,x)\rangle + \int_0^t \langle \mu_s^N, \phi(s,x)\rangle + \frac{2\phi(s,0)}{N} dA_s^N \, .$$

Taking expectation for both sides, we get:

$$E\big[\langle \mu_T^N, \phi(T,x)\rangle\big] \leq E\big[\mu_0^N, \phi(0,x)\big]$$

which implies, for any $t \in [0,T]$,

$$E\Big[\frac{|\{i|x_t^i \leq \delta\}|}{N}\Big] \leq E\big[\langle \mu_t^N, \psi_\delta\rangle\big] \leq E\big[\langle \mu_0^N, \rho(T,x)\rangle\big] \, .$$

Hence,

$$\begin{aligned} &\limsup_{N\to\infty} \sup_{t\in[0,T]} E\Big[\frac{|\{i|x_t^i \leq \delta\}|}{N}\Big] \\ &\leq \sup_{t\in[0,T]} E\big[\langle \mu_0, \rho(t,x)\rangle\big] \leq c(\mu_0) e^{\lambda_* T} \int_0^1 \psi_\delta(y)\, dy \, . \end{aligned} \tag{6.1.6}$$

Here, the term $e^{\lambda_* T}$ comes from Proposition 7.1.1. Now let $\delta \to 0$, $\int_0^1 \psi_\delta(y)\, dy \to 0$ and the lemma is proved. $\square$

Remark 6.1.3. $\rho(t,x)$ above is the solution to heat equation in auxiliary process.

Lemma 6.1.4. For some positive constant c,

$$\sup_{x_0^{N,i}=c} E_{x_0^{N,i}=c}[A_\eta^{N,i}] \le c\eta$$

Proof. Let ϕ be a test function in $C^2([0,1])$ that satisfies the following conditions: 1) $0 \le \phi' \le c_2$; 2)$|\phi''| \le 2c_1$; 3)$\phi(0) = 0$ and $\phi(x) \equiv 1$ over $[c, 1]$. Apply Ito formula to $y_t = \phi(x_t^i)$, which is a semimartingale, and get:

$$dy_t = \frac{1}{2}\phi''(x_t^i)dt + \phi'(x_t^i)dB_t^i + dJ^y(t)$$

where $J^y(t)$ is the jump part of y_t. Formally let $z_t = y_t$ without jumps, meaning

$$dz_t = \frac{1}{2}\phi''(x_t^i)dt + \phi'(x_t^i)dB_t^i\,.$$

Then we have

$$\begin{aligned} P_{x_0^i=c}\big(\tau_{x^i} \le \eta\big) \le P_{x_0^i=c}\big(\inf_{t\in[0,\eta]} y_t = 0\big) &\le P_{x_0^i=c}\big(\inf_{t\in[0,\eta]} z_t = 0\big) \\ &= P_{x_0^i=c}\big(\sup_{t\in[0,\eta]} |z_t - 1| \ge 1\,|z_0 = 1\big) \end{aligned}$$

Denote $\tilde{z}_t := |z_t - 1|$. We have

$$d\tilde{z}_t = \frac{1}{2}\phi''(x_t^i)dt + \phi'(x_t^i)dB_t^i$$

and $\tilde{z}_0 = 0$. So for $t \in [0, \eta]$,

$$\tilde{z}_t = \int_0^t \frac{1}{2}\phi''(x_s^i)ds + \int_0^t \phi'(x_s^i)dB_s^i \le c_1\eta + \int_0^t \phi'(x_s^i)dB_s^i\,.$$

If we choose $\eta \le \frac{1}{2c_1}$, then we ll have:

$$P_{x_0^i=c}\big(\sup_{t\in[0,\eta]} |z_t - 1| \geq 1 \,|z_0 = 1\big)$$

$$\leq P_{x_0^i=c}\big(\sup_{t\in[0,\eta]} |\int_0^t \phi'(x_s^i) dB_s^i| \geq \frac{1}{2}\big) \leq 4\eta c_2^2$$

Thus, for inductively defined hitting times of 0, $\{\tau_m^{x^i}\}_{m=1}$, by particle x^i, we have:

$$\begin{aligned}
E_{x_0^i=c}[A_\eta^i] &= \sum_k P_{x_0^i=c}\big(A_\eta^i > k\big) \\
&= \sum_k P_{x_0^i=c}\big(\tau_1^{x^i} + (\tau_2^{x^i} - \tau_1^{x^i}) + \ldots + (\tau_k^{x^i} - \tau_{k-1}^{x^i})\big) \\
&\leq \sum_k P_{x_0^i=c}\big(\max_{1\le l\le k}(\tau_l^{x^i} - \tau_l^{x^{i-1}}) \leq \eta\big) = \sum_k (4c_2^2\eta)^k \leq 8c_2^2\eta
\end{aligned}$$

where the first inequality is justified by Strong Markov Property at $\mathcal{F}_{\tau_{m-1}^{x^i}}$ to each summand. $\square$

Next, we show that two consecutive visits to 0 by a given particle takes relatively long time.

Condition 1. Let $q(dx) \in M_1(D)$. We shall say that the absorbing boundary $(\partial D)_a$ and $q(dx)$ are *separated* if there exists $d_a > 0$ such that

$(C0)$ The operator L will have bounded coefficients on $D \setminus D_{d_a}$, (6.1.7)

$(C1)$ $dist((\partial D)_a, supp(q) \cup (\partial D)_r) \geq d_a$, (6.1.8)

where $supp(q)$ is the topological support of $q(dx)$.

Lemma 6.1.5 (Particles don't move too fast)**.** Assume conditions $(C0)$ and $(C1)$ are satisfied for a probability measure $q(dx)$. Let i, $1 \leq i \leq N$ be a fixed index of one of the particles. We assume $X_t^{N,i}$ starts at a finite stopping time τ from a configuration with marginal distribution $q(dx) \in M_1(D)$. Then there exists a constant $c(q)$, dependent only on $q(dx)$ only, such that, for any $\eta > 0$,

$$P_{X_\tau^N}(\tau_X^D \le \tau + \eta) \le c(q)\eta\,. \tag{6.1.9}$$

Remarks. 1) This lemma will be applied twice, once for $\tau = 0$ and the distribution of $X_0^{N,i}$, in order to prove tigtness for the tagged particle, and another time with τ a time when $X_{\tau-}^{N,i} \in (\partial D)_a$ and $b = \gamma$. In the second case it will be essential that $b(dx)$, and consequently $q(b)$, do not depend on τ, N or the index i.

2) Lemma 6.1.5 is the only place where the condition that *supp*(γ) (the topological support of the redistribution measure) is at a positive distance from the absorbing boundary.

Proof. We construct a coupling between two processes, one without jumps, and then use a small ball estimate based on Doob's maximal inequality.

Step 1. Let $\psi \in C^2(\bar{D}, \mathbb{R})$ be a test function with the properties

1) $0 \le \psi(x) \le 1$,

2) $\psi(x) = 1$ on *supp*(γ) and $\psi(x) = 0$ if and only if $x \in (\partial D)_a$,

3) There exists $0 < \delta < \frac{d_a}{2} \wedge 1$, such that $\psi(x) = dist(x, (\partial D)_a)$ on $D \setminus D_\delta$.

4) $\psi \in (BC)_r$

Define $y_t = \psi(X_t^{N,i})$, $t \ge \tau$. Notice that by construction, at any τ', a jump time of $X_t^{N,i}$, y_t jumps $y_{\tau'} - y_{\tau'-} \ge 0$, a non-negative jump. This is because the values on the support of γ, where it jumps, are guaranteed to equal the maximum value of ψ over the full set $\bar{D}$. We notice that $(y_t) \in [0, 1]$ is a semi-martingale, adapted to $(\mathcal{F}_{t\wedge\tau})$, deriven by the *full process* (X_t^N), not just the particle i, due to the jumps it undergoes at times when $X_t^{N,i}$ is chosen randomly by another particle hitting the absorbing boundary, in addition to its own jumps triggered by hitting the absorbing boundary. This process will be coupled with a new process denoted $(z_t)_{t\geq\tau}$, with the same initial value, driven by the same equations between jumps, only *with all jumps suppressed.* Then

$$0 \leq z_t \leq y_t \leq 1 \qquad a.s.$$

and $(z_t)_{t\geq\tau}$ is an Ito process $dz_t = \alpha_t dt + \beta_t dw_t$, with coefficients given by

$$dz_t = L\psi(X_t^{N,i})dt + (\nabla\psi)(X_t^{N,i}) \cdot [\sigma(s, X_t^{N,i})dB_t] \,, \quad z_0 = y_0 = \psi(X_\tau^{N,i}) \,,$$

if the driving diffusion is given by $L\phi = \sum b_k \partial_k \phi + \frac{1}{2}\sum(\sigma^*\sigma)_{kl}\partial_{kl}\phi$ and B_t is the d - dimensional Brownian motion used in the construction of (X_t^N). We can see that the times to hit zero are ordered a.s. for the three processes $\tau_z^0 \leq \tau_y^0 \leq \tau_X^D$, where τ_X^D is the hitting time of the absorbing boundary by the process $X_t^{N,i}$.

Let $\alpha_0 \geq 0$ and $\beta_0 \geq 0$ be bounds for the coefficients

$$\alpha_0 = \sup_{x\in D} |L\psi(x)| \,, \quad \beta_0^2 = \sup_{x\in D} ||\sigma^*\sigma|| \, ||D^2\psi(x)||$$

where the norms are the sum of the maximum of all elements of a matrix/vector, depending on ψ and its derivatives, and L.

It remains to evaluate, for an initial value $X_\tau^{N,i}$ as prescribed in the lemma, the sequence of upper bounds

$$P(\tau_X^D \leq \tau + \eta | X_\tau^{N,i}) \leq P(\tau_z^0 \leq \tau + \eta | X_\tau^{N,i}) \leq P(\inf_{t\in[\tau,\tau+\eta]} z_t \leq 1 - d_a | z_\tau = 1)$$

$$\leq P(\sup_{t\in[\tau,\tau+\eta]} |z_t - 1| \geq d_a | z_\tau = 1)$$

$$\leq P(\sup_{t\in[\tau,\tau+\eta]} |\int_\tau^t \beta_s dw_s| \geq d_a - \alpha_0\eta) \leq \left(\frac{\beta_0}{d_a - \alpha_0\eta}\right)^2 \eta \leq \frac{4\beta_0^2}{d_a^2}\eta$$

as soon as $0 < \eta < \frac{\beta_0}{2\alpha 0}$. Taking $c(q) = \frac{2\alpha_0}{\beta 0} \vee \frac{4\beta_0^2}{d_a^2}$ we conclude the proof. $\square$

6.2 Proof of C-tightness

We now proceed to the proof of Proposition 6.0.1.

Remarks. 1) Evaluating (6.0.4) is based on the argument from line (6.2.11), which is a form of Wald's theorem for non-iid random variables $(\tau_X^D)_i$, $i \geq 1$, the waiting times between visits to the absorbing boundary. Independence is replaced by the condition in Lemma 6.1.5 and the strong Markov property.

2) Condition (6.0.4) is stronger than Aldous's criterion. It says cf. [22] that $(A_\cdot^N)$ is C - tight in the Skorokhod space, i.e. tight and that any limit point is continuous in time. Alternatively, if tightness is shown in the Skorokhod space, we recall that the *maximum jump size* $J_T(\omega(\cdot))$ of a path in D is a continuous functional in the J_1 norm. Since the jumps of $A_\cdot^N$ are at most of size $1/N$, it follows that a limit point $A_\cdot$ is continuous. This approach would prove immediately that $\mu_\cdot(dx)$ is also continuous in time.

Proof. Let $t \in [0,T]$, $\eta > 0$ and $J_1 < J_2 < \dots$ be the first jump times after t.

$$A_{t+\eta}^{N,i} - A_t^{N,i} = [1 + m^\gamma(J_1, t+\eta)]\mathbf{1}_{\{J_1 \leq t+\eta\}}\,, \tag{6.2.10}$$

with $m^\gamma(s,t)$ denoting the number of episodes when $X_\cdot^i$ travels from the redistribution point with distribution γ to the absorbing boundary, observed in the time interval $(s,t]$, $0 \leq s \leq t$. Recall that τ^D is the hitting time of the boundary at $x = 0$ by the driving diffusion process. Applying the Markov property, we can start at X_t^N.

$$E[A^{N,i}_{t+\eta} - A^{N,i}_t] = \sum_{k=1}^{\infty} P(A^{N,i}_{t+\eta} - A^{N,i}_t \geq k)$$

$$\leq E[P_{X^N_t}(\tau^D_X \leq \eta)] + \sum_{k=1}^{\infty} E[P_{X^N_{J_1}}(m^\gamma(J_1, t+\eta) \geq k)]\,.$$

The general term of the infinite sum can be bounded

$$P_{X^N_t}(m^c(J_1, t+\eta) \geq k \,|\, X^{N,i}_{J_1} \sim \gamma) \tag{6.2.11}$$
$$\leq P_{X^N_t}((\tau^D_X)_1 + \ldots (\tau^D_X)_k \leq \eta \,|\, X^{N,i}_{J_1} \sim \gamma)$$
$$\leq P_{X^N_t}(\max_{1\leq l\leq k}(\tau^D_X)_l \leq \eta \,|\, X^{N,i}_{J_1} \sim \gamma) \leq P_{X^N_t}((\tau^D_X)_k \leq \eta \,|\, \mathcal{A}_{k-1}) P_{X^N_t}(\mathcal{A}_{k-1})\,,$$

where $\mathcal{A}_{k-1} = \{\max_{1\leq l\leq k-1}(\tau^D_X)_l \leq \eta\}$. In our count, $J_2 - J_1 = (\tau^D_X)^1$, ending with $J_{k+1} - J_k = (\tau^D_X)^k$. Using the strong Markov property recursively, we get the further bound

$$E[\Pi^k_{l=1} P_{X^N_{J_l}}((\tau^D_X)_l \leq \eta)] \leq [c(\gamma)\eta]^k \tag{6.2.12}$$

due to the fact that $X^{N,i}_{J_l}$, $l \geq 1$ starts with distribution γ, and applying Lemma 6.1.5.

We obtained

$$E[A^{N,i}_{t+\eta} - A^{N,i}_t] \leq E[P_{X^i_t}(\tau^D \leq \eta)] + \frac{c(\gamma)\eta}{1 - c(\gamma)\eta} \tag{6.2.13}$$

and after summation and division by $N-1$,

$$E[A^N_{t+\eta} - A^N_t] \leq \frac{1}{N-1} \sum_{i=1}^{N} E[P_{X^i_t}(\tau^D \leq \eta)] + (\frac{N}{N-1}) \frac{c(\gamma)\eta}{1 - c(\gamma)\eta} \tag{6.2.14}$$

Let $\delta > 0$ be an arbitrary number not exceeding $d_a/2$. Working on the first term

$$\frac{1}{N}\sum_{i=1}^{N} E[P_{X_t^i}(\tau^D \le \eta)] \tag{6.2.15}$$
$$\le \left[\sup_{dist(X_t^{N,i},(\partial D)_a)\ge\frac{\delta}{2}} P_{X_t^N}(\tau_X^D \le \eta)\right] E[1-\frac{U_t(\frac{\delta}{2})}{N}] + E[\frac{U_t(\frac{\delta}{2})}{N}]$$
$$\le c(\delta)\eta + E[\frac{U_t(\frac{\delta}{2})}{N}]\,,$$

where $c(\delta)$ refers to the constant corresponding to an initial value away from the absorbing bounday ny at least δ.

To finalize the proof, we turn to (6.0.4). Let $0 < \eta_0 < \eta$, momentarily fixed. We split the interval $[0, T]$, to calculate

$$\sup_{t\in[0,\eta_0]} E[A_{t+\eta}^{N,i} - A_t^{N,i}] \le E[A_{2\eta_0}^{N,i} - A_0^{N,i}] = E[A_{2\eta_0}^{N,i}] \tag{6.2.16}$$
$$\le \frac{1}{N-1}\sum_{i=1}^{N} E[P_{X_0^i}(\tau^D \le 2\eta_0)] + (\frac{N}{N-1})\frac{c(\gamma)(2\eta_0)}{1-c(\gamma)(2\eta_0)}$$

and

$$\sup_{t\in[\eta_0,T]} E[A_{t+\eta}^{N,i} - A_t^{N,i}] \tag{6.2.17}$$
$$\le \sup_{t\in[\eta_0,T]}\left(\frac{1}{N-1}\sum_{i=1}^{N} E[P_{X_t^i}(\tau^D \le \eta)]\right) + (\frac{N}{N-1})\frac{c(\gamma)\eta}{1-c(\gamma)\eta}$$

The first term on the right-hand side of these inequalities is reduced to a bound on the number of particles within $\delta > 0$, for (6.2.16), respectively $\delta' > 0$

for (6.2.17), as we did in (6.2.15). Taking $\eta c(\gamma) < \frac{1}{2}$ and $N \geq 2$, we obtain

$$\begin{aligned} \sup_{t\in[0,T]} E[A^{N,i}_{t+\eta} - A^{N,i}_t] &\leq \sup_{t\in[0,\eta_0]} E[A^{N,i}_{t+\eta} - A^{N,i}_t] + \sup_{t\in[\eta_0,T]} E[A^{N,i}_{t+\eta} - A^{N,i}_t] \qquad (6.2.18) \\ &\leq [4c(\gamma) + 2c(\delta')](2\eta_0) + 2E[\frac{U_0(\frac{\delta'}{2})}{N}]\,, \\ &+ [4c(\gamma) + 2c(\delta)]\eta + 2\sup_{t\in[\eta_0,T]} E[\frac{U_t(\frac{\delta}{2})}{N}]\,. \end{aligned}$$

Lemma 6.1.2 concludes the proof, by having the limits over $N \to \infty$, $\eta \to 0$, $\delta \to 0$, $\eta_0 \to 0$, and finally $\delta' \to 0$, in this order. □

This concludes the proof of the tightness of A^N_t, whose weak limit is denoted as A_t. The term $\int_s^t \langle \frac{1}{2}\phi'', \mu^N_r \rangle \, dr$ in the martingale representation 5.2.6 is essentially Lipschitz because $\phi(x) \in C^2([0,1])$. For this same reason,

$$\int_s^t \big(2\phi(c) - \phi(0) - \langle \phi, \mu^N_r \rangle\big)\, dA^N_r \leq B|A^N_t - A^N_s|$$

for some positive number B. The continuous martingale part can be bounded using Doob Maximal Inequality, as is shown in Section 5.2 and similarly, the jump martingale part can be bounded by Doob's inequality and calculation of quadratic variations,

$$E[\sup_{0\leq t\leq T} |\mathcal{M}^{N,J,\phi}_t|] \leq N^{-1}C(\phi)E[A^N_T] \sim O(N^{-1}) \qquad (6.2.19)$$

So, in this way we verified the tightness of the family of processes $(\mu^N_t)_{t\geq 0}$, indexed by $N > 0$. The next step is to prove that there exists only one limit point $(\mu_t)_{t\geq 0}$, identified in Theorem 3.2.2. The first two claims of Theorem 3.1.3 are proved.

Theorem 6.2.1. *Assume $\mu^N_0 \Rightarrow \mu_0$, $\mu_0 \in M_1(D)$. Then the pair $(\mu^N_\cdot, A^N_\cdot)_{N>1}$ is*

C - tight on $D([0,\infty), M_1(D) \times \mathbb{R}_+)$, i.e. is tight and the limit is continuous in time.

Proof. We can apply (5.2.6) for $\phi \in \mathcal{D}_b$ for two times t, t' in $[0,T]$ with $0 < t' - t < \eta$. There exist constants $K(c,\phi)$, $K(J,\phi)$, independent of t, N such that the squares of the martingales are bounded by $N^{-1}K(c,\phi)T$ for the continuous part and $N^{-1}K(J,\phi)A_T^N$ for the jump part. In similar facsion, the integrands of dt and dA_t^N parts are bounded by $K(c,\phi)\eta$, respectively $K(J,\phi)(A_{t'}^N - A_t^N)$. Due to Proposition 6.0.1, part (ii) of Definition 3.1.2 is satisfied. To obtain (i) we turn to (6.1.6) for g a smooth approximation of the indicator function of the complement of a compact set in D. The bound we need to prove is pointwise in t, due to the rcll property and the compactness of $[0,T]$; in that sense, less than (6.1.6) is needed. More precisely, let $g(x)$ be a smooth function and $v(t,x) = E_x[\langle \zeta_t, g(x)\rangle]$. Based on (4.0.5), we observe that $s \to \langle \mu_s^N, v(t-s,x)\rangle$ $s \in [0,t]$ is a supermartingale. The expected values at $s=0$ and $s=t$ give the inequality

$$E[\langle \mu_t^N, g\rangle] \leq E[\langle \mu_0^N, u(t,\cdot)\rangle] = \int_D v(t,x)\mu_0^N(dx) = E_{\mu_0^N}[\langle \zeta_t, g\rangle] \tag{6.2.20}$$

which is what we want to show. All measures are concentrated, within $\epsilon > 0$ error, on a compact set, if the same is true at time $t = 0$. This is true simply because μ_0 charges D and not the boundary. The C - tightness is true because the criterion we used (i), (ii) in Proposition 6.0.1 implies C - tightness. $\square$

Chapter 7

Properties of the Auxiliary Process

7.1 Bound on N_t

Before we continue, we have to verify the non-explosiveness of the process $(Z_t(\omega))_{t\geq 0}$:

Proposition 7.1.1. *The total number of particles of the auxiliary process grows exponentially. More precisely,* $\lim\limits_{t\to\infty} e^{-\lambda_* t} E_x[N_t] = C_*(x)$ *for* $\lambda_* > 0$*, which can be calculated precisely as the solution of (7.1.3).*

Proof. Let $g(t,x) = E_x[N_t]$ and $f(t,x)$ be the density function of the first exit time τ at 0 for the process, when the initial particle is released at $x \in (0,1)$. The existence of $f(t,x)$ is guaranteed by [23].

Assume $g(t,x)$ doesn't explode to infinity at some t. Condition on the first branching time τ, we have the Renewal-Type Equation:

$$g(t,x) = P_x(\tau > t) + 2\int_0^t g(t-s,c) f(s,c)\, ds \tag{7.1.1}$$

In order to apply a result from [14], which will give us a bound on $g(t,x)$ when t is large, we need $g(t,c)$ to be convoluted with a distribution function.

To do this, consider the equation:

$$\widehat{f}(\lambda_*, x) = E_x[e^{-\lambda_* \tau}] = \int_0^\infty e^{-\lambda_* t} f(t,x)\, dt = \frac{1}{2} \tag{7.1.2}$$

$\widehat{f}(\lambda_*, x)$ has an explicit expression by solving a Feynman-Kac formula type ODE:

$$\widehat{f}(\lambda_*, x) = \frac{e^{\sqrt{2\lambda_*}x} + e^{2\sqrt{2\lambda_*} - \sqrt{2\lambda_*}x}}{1 + e^{2\sqrt{2\lambda_*}}} = \frac{1}{2} \tag{7.1.3}$$

Now multiply the equation by $e^{-\lambda_* t}$ where λ_* is the solution to the above equation.:

$$e^{-\lambda_* t} g(t,x) = e^{-\lambda_* t} P_x(\tau > t) + 2\int_0^t e^{-\lambda_* t} g(t-s, c) f(s,c)\, ds$$

By a change of variable, we get

$$e^{-\lambda_* t} g(t,x) = e^{-\lambda_* t} P_x(\tau > t) + \int_0^t \Big(e^{-\lambda_* s} g(s,c)\Big)\Big(2e^{-\lambda_*(t-s)} f(t-s,c)\Big)\, ds$$

so that $\int_0^\infty 2e^{-\lambda_* t} f(t,c)\, dt = 1$.

So now we can apply the estimate in [2]:

$$\lim_{t\to\infty} e^{-\lambda_* t} g(t,x) = \frac{\int_0^\infty e^{-\lambda_* t} P_x(\tau > t)\, dt}{\int_0^\infty \int_s^\infty 2e^{-\lambda_* u} f(u)\, du\, ds} = C_*(x) \in (0,\infty) \tag{7.1.4}$$

Note that $\int_0^\infty e^{-\lambda_* t} P_x(\tau > t)\, dt = \frac{1}{\lambda_*} - \frac{\widehat{f}(\lambda_*)}{\lambda_*} = \frac{1}{2\lambda_*} > 0$ and

$$0 < \int_0^\infty \int_s^\infty 2e^{-\lambda_* u} f(u)\, du\, ds \leq$$

$$\int_0^\infty \int_s^\infty 2e^{-\lambda_* s} f(u)\, du\, ds \leq \int_0^\infty 2e^{-\lambda_* s}\, ds < \infty$$

Now we need to address the issue of the non-explosiveness of $g(t,x)$.

Let τ_m be the hitting time when the particle system hits 0 for m^{th} time and consider $E_x[N_{t\wedge\tau_m}] := g_m(t)$. Let $\widetilde{N}^c_{t\wedge\tau_m}$ and $\widetilde{\widetilde{N}^c}_{t\wedge\tau_m}$ be the number of particles generated by the two independent copies of particles after first branching. We have the following path-by-path inequality:

$$\mathbf{1}_{\tau\leq t}N^x_{t\wedge\tau_m} \leq \mathbf{1}_{\tau\leq t}(\widetilde{N}^c_{t\wedge\tau_m} + \widetilde{\widetilde{N}^c}_{t\wedge\tau_m}) \leq \mathbf{1}_{\tau\leq t}(\widetilde{N}^c_{t\wedge\tau_{m-1}(\theta_\tau\omega)} + \widetilde{\widetilde{N}^c}_{t\wedge\tau_{m-1}(\theta_\tau\omega)})$$

By conditioning on τ_1, we get the renewal equation:

$$\begin{aligned} g_m(t) &= P_x(\tau_1 > t) + \int_0^t E_x[N_{t\wedge\tau_m}, t \geq \tau_1 | \tau_1 = s] P_x(\tau_1 \in ds) \\ &\leq P_x(\tau_1 > t) + \int_0^t 2E_c[N_{(t-s)\wedge(\tau_{m-1})}] f(s,x)\, ds \qquad \text{(Strong Markov Property)} \\ &= P_x(\tau_1 > t) + \int_0^t 2g_{m-1}(t-s) f(s,x)\, ds \end{aligned}$$

Now multiply both sides of the inequality by $e^{-\widetilde{\lambda}t}$, where $\widetilde{\lambda}$ is chosen based on (3.3) so that $\int_0^t f(s,x)e^{-\widetilde{\lambda}s}\, ds \leq r(t,x) < 1$. ($r(t,x)$ being some positive number.)

$$e^{-\widetilde{\lambda}t}g_m(t) \leq e^{-\widetilde{\lambda}t}P_x(\tau_1 > t) + 2\int_0^t g_{m-1}(t-s)e^{-\widetilde{\lambda}(t-s)}f(s,x)e^{-\widetilde{\lambda}s}\, ds$$

Let $M_m = \sup_{0<s<t}\{e^{-\lambda_* s}g_m(s)\}$. We have:

$$M_m \leq e^{-\widetilde{\lambda}t}P(\tau_1 > t) + 2M_m \int_0^t f(s,x)e^{-\widetilde{\lambda}s}\, ds \leq e^{-\lambda_* t}P(\tau_1 > t) + 2r(t,x)M_m$$

Thus, $M_m \leq \dfrac{e^{-\lambda_* t}P(\tau_1 > t)}{1 - 2r(t,x)}$.

Now that $E_x[N_{t\wedge\tau_m}]$ is uniformly bounded in m, we can appeal to monotone convergence theorem to carry out the estimate on $g(t,x)$. $\square$

7.2 The Semigroup Property

Proposition 7.2.1. *(Dependence on marginals only) Let a non-random initial finite point measure be $\mu = \sum_{k=1}^{N} \delta_{x_k} \in M_{FP}(D)$, with N a nonrandom positive integer. Then, the process (ζ_t) is a* pure branching process, *in the sense that*

$$E_\mu[F(\zeta_t)] = \sum_{i=1}^{N} E_{x_k}[F(\zeta_t)] = \langle E_\cdot[F(\zeta_t)], \mu\rangle = \int_D E_x[F(\zeta_t)]\mu(dx)\,. \tag{7.2.5}$$

Proof. The relation is a consequence of the construction of the process. Particles independent at time $s \geq 0$ remain independent forever. The only dependence is through the ancestry tree. Particles distributed deterministically at time $t = 0$ are independent. Hence the result. □

Proposition 7.2.2. *The mapping $t \to E_x[\langle \psi, \zeta_t\rangle] = S_t\phi(x)$ defines a continuous semigroup on $C_b(D)$.*

Proof. Using (7.2.5) and the Markov property for $\zeta_\cdot$,

$$E_\mu[F(\zeta_{s+t})] = E_\mu[E_{\zeta_s}[F(\zeta_t)]] = E_\mu[\langle E_\cdot[F(\zeta_t)], \zeta_s\rangle]\,. \tag{7.2.6}$$

When F is linear, i.e. $F(\mu) = \langle \phi, \mu\rangle$, we have

$$\begin{aligned} E_\mu[\langle E_\cdot[F(\zeta_t)], \zeta_s\rangle] &= E_\mu[\langle E_\cdot[\langle \phi, \zeta_t\rangle], \zeta_s\rangle] = E_\mu[\langle S_t\phi(\cdot), \zeta_s\rangle] \\ &= E_\mu[S_s(S_t\phi(\cdot))] = \int_D S_sS_t\phi(x)\mu(dx)\,. \end{aligned} \tag{7.2.7}$$

□

The continuity in t derives from the renewal equation and the continuity of the underlying semigroup killed at the boundary.

7.3 Proof of Theorem 3.2.2

The steps concluding the proof of Theorem 3.1.3 are put together in the following proposition.

Proposition 7.3.1. *Let $\nu_0(dx) = m(dx)$ in the notation of Definition 3.2.1. Then, equation (3.2.4) has a unique solution equal to*

$$\langle \phi(t,\cdot), \nu_t^m \rangle := E_m[\sum_{j=1}^{N_t} \phi(t, z_t^j)] = E_m[\langle \zeta_t, \phi(t,\cdot)\rangle]\,,$$

where m represents the initial configuration.

Proof. Existence. We prove a little more, by taking a time dependent test function. By construction, the process Z_t and its measure valued formulation ζ_t introduced in Section 2.3 satisfy the following martingale problem. Let ϕ be a test function of class C^1 in time and $\phi(t,\cdot)$ exactly as in Definition 3.2.1, i.e. satisfying the regularity conditions from Theorem 3.2.2 and boundary conditions (3.2.5)-(3.2.6). Then

$$\langle \zeta_t, \phi(t,\cdot)\rangle - \langle \zeta_0, \phi(0,\cdot)\rangle - \int_0^t \langle \zeta_s, \partial_s \phi(s,\cdot) + \frac{1}{2}\phi''(s,\cdot)\rangle\, ds\,, \qquad t \geq 0\,,$$

is a $\mathcal{F}_t$ - martingale. Taking the expected value, provided we start with distribution $m(dx)$, we obtain that ν_t^m satisfies the weak equation from Definition 3.2.1.

Regularity. In (9.3.20), and independently of the results in this section, we calculate explicitly the renewal equation for the semigroup of Brownian motion with particle creation, i.e. the semigroup from Proposition 7.2.2. The regularity properties result from noticing that u is the convolution of the density kernel $p^D(t,x,y)dy$ of the Brownian motion with reflection at $x = 1$ and killed at $x = 0$, $D = (0,1)$, and the density function of the hitting time τ^D of the boundary point $x = 0$. These are explicit C^∞ functions, and as soon as we integrate gainst a continuous initial profile $g(x)$, they immediately have all regularity properties required in Theorem 3.2.2.

Uniqueness. With the the same notations as in Theorem 2 of [18], fix $T > 0$ and let $\phi(t,x) := u(T-t,x)$, $t \in [0,T]$. More precisely, $u(t,x)$ is the density function of $\nu_t(dx)$ from above with $m(dx) = g(x)dx$, $g(x)$ a continuous function on $[0,1]$. Due to the regularity of the solution just proven in the preceding paragraph, we can use ϕ as test function in (3.1.3). We obtain that for any weak solution $m_t(dx)$

$$\langle \phi(t,\cdot), m_t \rangle = \langle \phi(0,\cdot), m \rangle \,, \qquad 0 \le t \le T \,. \tag{7.3.8}$$

As $t \uparrow T$, this implies

$$\begin{aligned} \langle g, m_T \rangle &= \langle u(T,\cdot), m \rangle = \langle \langle g(\cdot), \nu_T \rangle, m \rangle = \int_D \int_D \nu_T^x(dy) g(y) m(dx) \\ &= \int_D g(y) \int_D \nu_T^x(dy) m(dx) = \int_D g(y) \nu_T^m(dy) = \langle g, \nu_T^m \rangle \end{aligned} \tag{7.3.9}$$

which implies that $m_T = \nu_T^m$. This is true for arbitrary $T > 0$, concluding the proof. $\square$

Chapter 8

Identification of the Limit

8.1 The Rescaled Process ν_t^N

This section will study the scaling limit of the derived process (8.1.4).

According to [22], let $\mathbf{X}(t) = (X_1(t), \cdots, X_m(t))$ be an $m-$dimensional semimartingale and F a smooth function on $\mathbb{R}^m$. Denote

$$\tilde{\Delta}X(t) = \sum_{0 \le s \le t} \Big(X(s) - X(s-) \Big)$$

and $\langle (X_k)^c, (X_l)^c \rangle(s)$ the cross variation of the continuous martingale parts of $X_k(t)$ and $X_l(t)$. Then we have:

$$F(\mathbf{X}(t)) - F(\mathbf{X}(0)) = \sum_{l=1}^{m} \int_0^t \partial_l F(\mathbf{X}(s-)) dX_l(s) \tag{8.1.1}$$

$$+\frac{1}{2} \sum_{k,l=1}^{m} \int_0^t \partial_{kl} F(\mathbf{X}(s-)) d\langle (X_k)^c, (X_l)^c \rangle(s) \tag{8.1.2}$$

$$+ \sum_{0 \le s \le t} \Big[F(\mathbf{X}(s)) - F(\mathbf{X}(s-)) - \sum_{k=1}^{m} \partial_k F(\mathbf{X}(s-)) \tilde{\Delta} X_k(s) \Big] \tag{8.1.3}$$

Now fix N and let $\phi \in C^2([0,1])$ with conditions $\phi'(1) = 0$. Then apply the Ito Formula above with $m = 2$ and $\mathbf{X}(t) = (X_1(t), X_2(t)) = (A_t^N, \langle \phi(x), \mu_t^N \rangle)$. The function F we are going to use is defined as $F(X_1, X_2) = e^{X_1}X_2$, for the purpose of eliminating dA_t^N term. Denote

$$\nu_t^N = F(X_1(t), X_2(t)) \tag{8.1.4}$$

and we expect, via Ito formula and the boundary condition, the following, as $N \to \infty$:

$$\langle \nu_t, \phi \rangle - \langle \nu_0, \phi \rangle - \int_0^t \langle \nu_s, \frac{1}{2}\phi'' \rangle \, ds = 0 \tag{8.1.5}$$

By Ito Formula, for any $t > 0$:

$$\langle \phi(x), \nu_t^N \rangle - \langle \phi(x), \nu_0^N \rangle = \int_0^t e^{A_s^N} \langle \phi(x), \mu_s^N \rangle dA_s^N + \int_0^t e^{A_s^N} d\langle \phi(x), \mu_s^N \rangle + \mathcal{E}^N(t) \tag{8.1.6}$$

$$= \int_0^t e^{A_s^N} \langle \frac{1}{2}\Delta\phi, \mu_s^N \rangle ds + \int_0^t e^{A_s^N} d\mathcal{M}_s^c + \int_0^t e^{A_s^N} d\mathcal{M}_s^j + \mathcal{E}^N(t)$$

where (2.6) follows from (2.1) and the boundary condition that $2\phi(c) - \phi(0) = 0$. The error term $\mathcal{E}^N(t)$ comes from (2.4) and equals

$$\sum_{0 \le s \le t} \Big[e^{A_s^N} \langle \phi(x), \mu_s^N \rangle - e^{A_{s-}^N} \langle \phi(x), \mu_{s-}^N \rangle - \Big(e^{A_{s-}^N} \langle \phi, \mu_{s-}^N \rangle \tilde{\Delta} A_s^N + e^{A_{s-}^N} \tilde{\Delta}\big(\langle \phi(x), \mu_s^N \rangle \big) \Big) \Big]$$

Nontrivial terms in this formula are those with time $s = \tau$ where τ marks the time when the system hit 0. Denote J as the set of all jump times τ of the system, up to time t. Then the above formula equals:

$$\sum_{\tau \in J} \Big[e^{A_{\tau-}^N + \frac{1}{N}} \big(\langle \phi, \mu_{\tau-}^N \rangle - \frac{1}{N}\phi(x_{\tau-}^j) - \frac{1}{N}\phi(0) + \frac{2}{N}\phi(c) \big) \Big] \tag{8.1.7}$$

$$-e^{A_{\tau-}^N} \langle \mu_{\tau-}^N, \phi \rangle - \frac{1}{N} e^{A_{\tau-}^N} \langle \phi, \mu_{\tau-}^N \rangle + \frac{e^{A_{\tau-}^N} \phi(x_j)}{N} \tag{8.1.8}$$

Because of the boundary condition $2\phi(c) - \phi(0) = 0$, the formula can thus be simplified as:

$$\sum_{\tau\in J}\Big[e^{A^N_{\tau-}+\frac{1}{N}}\big(\langle\phi,\mu^N_{\tau-}\rangle - \frac{1}{N}\phi(x^j_{\tau-})\big)\Big] - e^{A^N_{\tau-}}\langle\mu^N_{\tau-},\phi\rangle - \frac{1}{N}e^{A^N_{\tau-}}\langle\phi,\mu^N_{\tau-}\rangle + \frac{e^{A^N_{\tau-}}\phi(x_j)}{N}$$

$$= \sum_{\tau\in J} F(X_1+\frac{1}{N}, X_2-\frac{\phi(x_j)}{N}) - F(X_1,X_2) - \big(\partial_1 F(X_1,X_2)\frac{1}{N} - \partial_2 F(X_1,X_2)\cdot\frac{\phi(x_j)}{N}\big) \tag{8.1.9}$$

which, by Taylor Formula, equals:

$$\sum_{\tau\in J}\frac{1}{2}\Big(\frac{\overline{e^{A^N_{\tau-}}}\langle\phi,\mu^N_{\tau-}\rangle}{N^2} + \frac{2e^{A^N_{\tau-}}\phi(x_j)}{N^2}\Big) + o(\frac{1}{N^2}) \tag{8.1.10}$$

$$\leq \int_0^t \frac{Me^{A^N_s}}{N^2}d(NA^N_s) + o(\frac{1}{N^2}) = \int_0^t \frac{Me^{A^N_s}}{N}dA^N_s + o(\frac{1}{N^2}) \tag{8.1.11}$$

where M is the upper bound for terms involving ϕ. So now we notice that the integrand is $1/N$ multiplied by a term not larger than $E[e^{2A^N_t}] \leq M(2,T)$, as shown in Proposition 8.2.1.

Remark 8.1.1. The choice of $F(X_1, X_2) = e^{X_1}X_2$ is crucial since it will eventually allows us to cancel out the term $\langle\phi,\mu^N_r\rangle$ in (3).

Remark 8.1.2. The process $F(A^N_t, \mu^N_t) = e^{A^N_t}\mu^N_t$ defined in the proof will give us the identification of μ_t when $N \to \infty$, namely, $\nu_t = e^{A_t}\mu_t$. At the same time, the normalization $\mu_t = \nu_t/\langle\nu_t, 1\rangle$ mentioned in Theorem 3.1.3 can be proved By uniqueness, this implies that $e^{A_t} \sim \langle\nu_t, 1\rangle$.

8.2 Uniform Exponential Bound For A_t^N

Proposition 8.2.1. *For any $T > 0$, $\beta > 0$*

$$M(\beta, T) = \limsup_{N\to\infty} E[e^{\beta A_T^N}] < \infty \,. \tag{8.2.12}$$

Proof. From Hölder's inequality we see that it is suficient to prove the exponential bound for each tagged particle, where i fixed, $N \geq i$

$$M_i(\beta, T) = \limsup_{N\to\infty} E[e^{\beta A_T^{N,i}}] < \infty \,. \tag{8.2.13}$$

Let $\eta > 0$ be such that $\eta < (c(\gamma)e^{\beta})^{-1}$. Assume, for a moment, that there exists a number $\bar{M}(\beta, \eta) > 0$

$$E_{X_t^N}[e^{\beta A_\eta^{N,i}}] \leq \bar{M}(\beta, \eta) \qquad a.s. \tag{8.2.14}$$

The Markov property shows that

$$E[e^{\beta A_T^{N,i}}] = E[E[e^{\beta(A_T^{N,i} - A_{T-\eta}^{N,i})} \,|\, \mathcal{F}_{T-\eta}]e^{\beta A_{T-\eta}^{N,i}}]$$

$$= E[E_{X_{T-\eta}^N}[e^{\beta(A_\eta^{N,i}}]e^{\beta A_{T-\eta}^{N,i}}] \leq \bar{M}(\beta, \eta)^{[\frac{T}{\eta}]+1} < \infty \,,$$

an upper bound independent of N, proving that $M_i(\beta, T) < \infty$. It remains to show (8.2.14). Using (6.2.10) and the calculations in (6.2.11) we see, since

$$P_{X_t^N}\left(A_\eta^{N,i} > \frac{\ln s}{\beta}\right) \leq (c(\gamma)\eta)^{[\frac{\ln s}{\beta}]} \leq (c(\gamma)\eta)^{\frac{\ln s}{\beta}-1} \leq (c(\gamma)\eta)^{-1} s^{\beta^{-1}\ln(c(\gamma)\eta)}$$

that

$$E_{X_t^N}[e^{\beta A_\eta^{N,i}}] = \int_1^\infty P_{X_t^N}\left(A_\eta^{N,i} > \frac{\ln s}{\beta}\right) ds \le (c(\gamma)\eta)^{-1} \int_1^\infty s^{-\beta^{-1}\ln(\frac{1}{c(\gamma)\eta})}\, ds < +\infty\,,$$

due to the choice of η. $\square$

8.3 A Continuous, Bounded Functional on the Skorokhod Space

For a smooth test function $\phi \in C_c^\infty([0,\infty) \times D, \mathbb{R})$ define the functional $\Phi : \mathbb{D}([0,\infty), M_F(D)) \to \mathbb{R}$

$$\Phi(\sigma.) := \sup_{t\in[0,T]} \Big| \langle \sigma_t, \phi(t,\cdot)\rangle - \langle \sigma_0, \phi(0,\cdot)\rangle - \int_0^t \langle \sigma_s, \frac{\partial}{\partial s}\phi(s,\cdot) + L\phi(s,\cdot)\rangle\, ds \Big|\,, \tag{8.3.15}$$

which can be shown to be bounded and continuous, practically following the steps of [19], and also in [34], Proposition 6.5, p. 84. Here $\sigma_s \in M_F(D)$ is the value at time $s \in [0,\infty)$. As a technical point, the functional may not be automatically bounded, but due to the bound (8.2.12), which is a bound for $\langle \sigma_t, \phi(t,\cdot)\rangle$ when $t \in [0,T]$, the proof can proceed as usual, by truncation.

8.4 Proof of the Hydrodynamic Limit For (ν_t^N, A_t^N)

Proposition 8.4.1. *The pair* $(\nu_\cdot^N, \exp(A_\cdot^N))$*, obtained by the transformation*

$$(\mu_t^N, A_t^N) \to (\exp(A_t^N)\mu_t^N, \exp(A_t^N)), \quad t \geq 0$$

is C *- tight and has hydrodynamic limit, in the sense of Definition 3.1.2, componentwise, the solution to (3.2.4)-(3.2.5), respectively its total mass* $n_t = \langle \nu_t, 1 \rangle$.

Proof. We write Ito's formula for semi-martingales [22]. Tightness follows from the tightness of the pair $(\mu_\cdot^N, A_\cdot^N)$ (Theorem 6.2.1) and the fact that all possible integrands in (5.2.6), including in the quadratic variations of the martingales, are dominated by $const \exp A_T^N$ or $const A_T^N \exp A_T^N$, both bounded above by $\exp 2A_T^N$ which we have from Proposition 8.2.1.

The same bounds will show, in addition, that for any ϕ satisfying the boundary condition (3.2.5) and (3.2.6). Assuming that, the same bounds on the integrands, together with Doob's maximal inequality applied to the martingale part will show that, for the functional in (8.3.15)

$$\lim_{N \to \infty} E[\Phi(\nu_\cdot^N)] = 0 \tag{8.4.16}$$

Let $(\nu., n.)$ be limit points of the tight pair of transformed processes. Since $(\nu_\cdot^N, n_\cdot^N) \Rightarrow (\nu., n.)$ and Φ is continuous and bounded, we obtained that

$$E[\Phi(\nu.)] = 0 \quad \text{and then} \quad \Phi(\nu.) = 0 \quad a.s. \tag{8.4.17}$$

It is sufficient to remark that, being C - tight, the limit is continuous in time. It follows that we can pick a set of measure zero, common to all $t \in [0,T]$, and as a consequence, common for all $t \in [0,\infty)$ by choosing $T = r$, $r \in \mathbb{N}$, so that $\Phi(\nu_\cdot) = 0$ on its complement. We proved that $\nu_\cdot$ solves (3.2.4),(3.2.5) and (3.2.6). By uniqueness, we are done with the claim on ν_t^N. When D is bounded, it is sufficient to integrate against the constant 1 (a variation of the argument with approximations of indicator functions of a sequence of nested compacts will prove the same if D is unbounded) to see that if

$$n_t^N = \langle \nu_t^N, 1\rangle = \exp(A_t^N)\,,$$

then $n_\cdot^N \Rightarrow n_\cdot$. Finally, since the convergence is uniform in t over $[0,T]$, and the limit is a delta function (i.e. delta concentrated at the unique deterministic solution), we have that convergence in distribution implies convergence in probability. $\square$

8.5 Proof of Theorem 3.1.3

Proof. The preceding chapters and the results in the current chapter essentially prove the main theorem 3.1.3. The first part of the theorem (tightness) is proven in Chapter 10.2.7. The second part, where μ_t is determined based on ν_t, the expected value of empirical measure of the auxiliary process, is done the previous sections of this chapter. Finally, the formula for A_t is determined in Proposition 8.4.1. We notice that the uniqueness of the pde (Definition 3.2.1) is proven in Proposition 7.3.1. $\square$

Chapter 9

Quasi-Stationarity

This chapter looks at *quasi-stationary distributions* associated to our hydrodynamic limit. Quasi-stationarity is natural whenever a dynamical system is either dissipative or accretive, in other words, when mass is not conserved, and by normalization we can obtain a probability measure that is approached in the Yaglom limit (9.1.7) sense as $t \to \infty$.

9.1 General Setup for QSD

We investigate quasi-stationarity of the auxiliary process at equilibrium In the following, S_t will be a strongly Feller semigroup, i.e. for any $t \geq 0$

$$
\begin{aligned}
&(i) \quad \forall \phi \in C_b(D) \qquad S_t\phi \in C_b(D) \qquad\qquad (9.1.1)\\
&(ii) \quad \forall t, t' \geq 0\,, \quad \forall \phi \in C_b(D) \qquad S_{t+t'}\phi = S_t S_{t'}\phi\\
&(iii) \quad \forall t \geq 0 \qquad t \to S_t\phi \quad \text{is continuous in the supremum norm}\,.
\end{aligned}
$$

Most results hold true if condition (i) is replaced with a weaker condition:

$$(i') \quad S_t 1 \in C_b(D).$$

We shall assume that there exists $\alpha_1 > -\infty$ such that

$$\forall \alpha > \alpha_1 \qquad \sup_{x \in D} \int_0^\infty e^{-\alpha t} S_t 1(x) ds < +\infty \,. \tag{9.1.2}$$

A stronger condition is that there exists $\alpha' > -\infty$ such that $e^{-\alpha' t} S_t$ is a contraction semigroup.

Definition 9.1.1. A probability measure $\nu(dx)$ on D is said a *quasi-stationary distribution* (qsd) for the semigroup S_t if

$$\langle \nu S_t, \phi \rangle = \langle \phi, \nu \rangle \langle \nu, S_t 1 \rangle \,, \qquad \forall \quad t \geq 0 \,. \tag{9.1.3}$$

In the context of the process (ζ_t), we define its (marginal) semigroup applied to test functions $F \in C_b(M_F(D))$ of the special form $F(\mu) = \langle \mu, \phi \rangle$, where $\phi \in C_b(D)$

$$S_t \phi(x) = E_x[\langle \zeta_t, \phi \rangle] \,, \tag{9.1.4}$$

$$\text{with the notation} \quad E_x[F(\zeta_t)] = E[F(\zeta_t) | \zeta_0 = \delta_x] \,.$$

Then (9.1.3) reads explicitly as

$$E_\nu[\sum_{i=1}^{N_t} \phi(X_t^i)] = E_\nu[N_t] \cdot \langle \phi, \nu \rangle \,, \qquad \forall \quad t \geq 0 \,. \tag{9.1.5}$$

Equivalently, we can define a qsd by the property that for any two test functions $\phi, \psi \in C_b(D)$

$$\frac{\langle \nu, S_t \phi \rangle}{\langle \nu, S_t \psi \rangle} = \frac{\langle \nu, \phi \rangle}{\langle \nu, \psi \rangle} = \text{constant in } t \geq 0 \,. \tag{9.1.6}$$

A notion that is closely related to quasi-stationary distribution is *Yaglom limit*.

A probability measure $\nu(dx)$ on D is said a *Yaglom limit* for the process (Z_t) if there exists a probability measure ν' such that, for all $\phi \in C_b(D)$

$$\lim_{t\to\infty} \frac{\langle \nu', S_t\phi\rangle}{\langle \nu', S_t 1\rangle} = \langle \nu, \phi\rangle \,. \tag{9.1.7}$$

In that case we say ν' is in the *domain of attraction* of ν. If a Yaglom limit has domain of attraction all delta functions, or equivalently, any probability measure ν' on D, it is said a *strong Yaglom limit*.

Theorem 9.1.2. *The expected value of the total number of particles for the auxiliary process, $\langle \nu, S_t 1\rangle$, is exponential. A Yaglom limit is a qsd. A qsd is in its own domain of attraction. A strong Yaglom limit, if it exists, is unique.*

Proof. Using $\phi = S_s\psi$ and $\psi = 1$ we obtain that $t \to \nu S_t 1 = n_t^\nu$ is exponential. In case the semigroup is dissipative, the time to extinction is exponential. Now let t and t' be positive. Then, applying the definition (9.1.7) with $\phi \to S_t\phi$

$$\lim_{t\to\infty} \frac{\langle \nu', S_t S_{t'}\phi\rangle}{\langle \nu', S_t 1\rangle} = \langle \nu, S_{t'}\phi\rangle \,. \tag{9.1.8}$$

$$\frac{\langle \nu', S_t S_{t'}\phi\rangle}{\langle \nu', S_t 1\rangle} = \frac{\langle \nu', S_{t+t'}\phi\rangle}{\langle \nu', S_{t+t'} 1\rangle} \cdot \frac{\langle \nu', S_t S_{t'} 1\rangle}{\langle \nu', S_t 1\rangle} \,. \tag{9.1.9}$$

Let $t \to \infty$. The first factor converges to $\langle \nu, \phi\rangle$ as $t + t' \to \infty$ and the second factor uses (9.1.7) with $\phi \to S_t 1$ to converge to $\langle \nu, S_{t'}\phi\rangle$. The two limits being equal shows that ν is a qsd. □

A quasi-invariant distribution $\nu(dx)$ of the semigroup defined by the operator in (3.3.8) with the boundary condition $2\phi(c) = \phi(0)$ and $\phi'(1) = 0$ defined in (7.2.6, which is is the semigroup giving the time evolution of the expected value of the empirical process (ζ_t) from Chapter 2.3, is thus a left-side eigenfunction of the semigroup. This is shown in Proposition 9.1.3 but we proceed to discuss heuristically, at first, how the preceding discussion on Yaglom limits relates to this property.

Suppose, formally we solved (3.3.8) and $\rho(t,x)$ is the weak solution (we can prove it is a function).Take the Yaglom limit

$$\lim_{t\to\infty} \frac{E[\sum_{i=1}^{N_t} \phi(X_t^i)]}{E[N_t]} = \langle \phi, \nu \rangle \,, \tag{9.1.10}$$

where $\nu(dx)$ is the quasi-invariant measure in Yaglom limit sense.

Heuristically, denote $a_t = E[N_t]$ and $\nu_t = \mu_t / a_t$. Then

$$\begin{aligned} \partial_t \mu_t &= L^* \mu_t \,, \\ \partial_t \nu_t &= (L^* \mu_t a_t - \mu_t a_t') a_t^{-2} \\ \partial_t \nu_t &= L^* \nu_t - (\ln a_t)' \nu_t \,. \end{aligned} \tag{9.1.11}$$

As $t \to \infty$, the left hand side does not depend on t, as well as the derivative of $\ln a_t$. In principle

$$a_t \sim e^{\lambda_* t} \,, \qquad 0 = L^* \nu - \lambda_* \nu \,. \tag{9.1.12}$$

The following proposition rigorously sums up the heuristic above and thus proves Theorem 3.3.1.

Proposition 9.1.3. *Any qsd $\nu(dx)$ is a left eigenvalue of L with the boundary conditions (3.2.5) and (3.2.6).*

Remark. The first limit in (9.1.15) will be shown to exist when we prove the sufficient conditions on $\nu(dx)$ as well as the Yaglom limit (9.1.10).

Proof. Let ζ_t be the empirical measure (2.3.10). For any test function ϕ satisfying the b.c. we write the martingale

$$M_t^{\phi,\zeta} = \langle \phi, \zeta_t \rangle - \langle \phi, \zeta_0 \rangle - \int_0^t \langle L\phi, \zeta_s \rangle ds \,. \tag{9.1.13}$$

Notice that the mapping defined for $\psi \in C_b(D)$

$$t \to E_x[\langle \psi, \zeta_t \rangle] = S_t\phi(x) \tag{9.1.14}$$

is continuous. Pick $\phi \in C^2(\bar{D})$ (bounded with two bounded derivatives up to the boundary) satisfying the boundary conditions (3.2.5). We note that this is a *determining class* for the finite measures on D. It follows that $\psi = L\phi \in C_b(D)$. The expected value of $M_t^{\phi,\zeta}$, when the initial value x has distribution $\nu(dx)$, the qsd satisfying (3.3.8), divided by $t > 0$, will have a limit as $t \downarrow 0$, due to the fact that ϕ is bounded and the integrand is continuous in time. We obtained

$$\begin{aligned} \langle \phi, \nu \rangle \lim_{t\to\infty} \frac{1}{t}\Big(E_\nu[N_t] - E_\nu[N_0] \Big) &= \lim_{t\to\infty} \frac{1}{t}\Big(\int_0^t E_\nu[\langle L\phi, \zeta_s \rangle ds \Big) \\ &= E_\nu[\langle L\phi, \zeta_0 \rangle] = \langle L\phi, \nu \rangle \,, \end{aligned} \tag{9.1.15}$$

where we know that the second limit exists. It follows that it is necessary that the limit on the left hand side exists as well. Let it be $\lambda_* \geq 0$ (it must be nonnegative). Then

$$\lambda_* \langle \phi, \nu \rangle = \langle L\phi, \nu \rangle \,, \tag{9.1.16}$$

which implies the conclusion in view of Proposition 8.4.1 and equation (9.1.12). $\square$

9.2 QSD - Reflected Brownian Motion

We start with the case when the driving motion is on $D = (0, 1)$, drift $\mu = 0$ with reflection at $x = 1$, the original setup of the problem.

To investigate the left side eigenfunctions with eigenvalue $\lambda \in \mathbb{R}$, we use integration against a test function $\phi \in C^2([0, 1])$ satisfying the boundary conditions (3.2.5)-(3.2.6), which impose conditions (3.2.7) on the the eigenfunction g, verifying

$$2\lambda \int_0^1 \phi(x)g(x)dx = \int_0^1 \phi''(x)g(x)dx = \int_0^c \phi''(x)g(x)\,dx + \int_c^1 \phi''(x)g(x)dx\,.$$

Implies that any such eigenfunction g and test function ϕ satisfies

$$[\phi'(c)g(c-) - \phi'(0)g(0)] - [\phi(c)g'(c-) - \phi(0)g'(0)]$$

$$+[\phi'(1)g(1) - \phi'(c)g(c+)] - [\phi(1)g'(1) - \phi(c)g'(c+)]$$
$$= -\int_0^1 (2\lambda g(x) - g''(x))\phi(x)dx\,.$$

It is important to not consider g smooth at c, as we see from the one-sided limits. Inside the intervals, we obtain $g'' = 2\lambda g$. We are interested in $\lambda > 0$. Take $g(x) = c_1 e^{\beta x} + c_2 e^{-\beta x}$. The boundary conditions derived from the equations above are given earlier in equation (3.2.7)

$$g(c+) = g(c-) \qquad g(0) = 0 \qquad g'(1) = 0$$
$$(g'(c+) - g'(c-)) + 2g'(0) = 0\,. \tag{9.2.17}$$

For λ to exist, $g(x)$ has the form $g(x) = c_1 e^{\beta x} + c_2 e^{-\beta x}$ with possibly different constants on $[0,c]$ and $[c,1]$. On the upper, respectively lower part we have

$$g(x) = e^{\beta x} + e^{2\beta} e^{-\beta x}\,, \qquad c \le x \le 1 \tag{9.2.18}$$
$$g(x) = q(e^{\beta x} - e^{-\beta x})\,, \qquad 0 \le x \le c\,, \qquad q = \frac{e^{\beta c} + e^{2\beta} e^{-\beta c}}{e^{\beta c} - e^{-\beta c}}$$

where q is obtained from the continuity at c. The last boundary condition is

$$[(e^{\beta c} - e^{2\beta} e^{-\beta c}) - q(e^{\beta c} + e^{-\beta c})] + 2Kq = 0\,, \qquad K = 2$$

where we allowed a general K representing the number of individuals in the next generation, in case the branching has a different rate. This may influence the nature (existence, multiplicity) of the quasi-stationary measure g.

Denote $e^{\beta c} = z$. After simplification, we obtain

$$e^{2\beta - 1} = z^{\omega} = \frac{Kz - 1}{z - K}\,, \qquad \omega = \frac{2}{c} - 1\,. \tag{9.2.19}$$

For a quick look at the nature of the solutions, take c rational, and more precisely $c = \frac{1}{2}$. We obtain $P(z) = 0$, for $P(z) = z^4 - Kz^3 - Kz + 1$. Then $P(0) = 1 > 0$; $P(1) = 2 - 2K < 0$ if $K > 1$ and $P(K) < 0$. It is easy to see that $P(K+1) > 0$. Looking at the derivatives, we check that there exist only two real roots, one in $(0,1)$ and one, denoted by z_*, in $(K, K+1)$.

More generally, for any $K > 1$ and $\omega \in (1, \infty)$ we verify $P(0) = 1 > 0$ and $P(K) < 0$ while $P(2K) > 0$. This guarantees the existence of a root greater than one. In fact, for $\omega \downarrow 1$ the roots approach the roots of the limiting case $\omega = 1$ equal to $K \pm \sqrt{K^2 - 1} \le 2K$. For $c < \frac{2}{3}$ we can show that $P(K+1) > 0$, improving the upper bound.

An interesting feature is that as $\omega \to \infty$, i.e. $c \downarrow 0$, one could expect $\beta \to \infty$, which is the same as the root greater than one be very large. However, this does not happen, as it is bounded above by $K+1$. The interpretation is that the quasi-stationary profile, as well as the exponential rate of growth of the total mass are regulated dominantly by the branching parameter K and not by the branching rate, here determined by how close the source c is from the boundary $x=0$.

Proposition 9.2.1. *The root $z_* > 1$ gives a $\beta_* > 0$, which in turn gives the explicit quasi-invariant measure $\nu(dx) = g_*(x)dx$ for β_* in equation (9.2.18).*

Proof. The proof is complete, due to the uniqueness of the solution to equation (9.2.19) obtained in the preceding paragraph and Proposition 9.1.3. □

9.3 Generalization and Resolvent Formula

Theorem 9.3.1 and the following calculations are proven in our upcoming paper [18].

In the following, $\bar{K}$ is the expected value of the *random number K of individuals born at branching time.* Throughout the main part of the present work, $D=(0,1)$, the boundary can be considered to fix ideas $\{0\}$, $K \equiv 2$ (nonrandom), $\bar{K}=2$, and $\gamma(dx) = \delta_c(dx)$, the redistribution measure of location where the K new particles are born. However, the derivation is similar and without loss of generality, we obtain relation (9.3.20) following the reasoning used in Chapter 7.

The existence of Yaglom limit is done through analyzing the resolvents of $(L, \mathcal{D}(L))$ defined in (9.1.14). Let R_α^D and R_α be the resolvents of $(L, \mathcal{D}(L))$ and the semigroup S_t defined in (9.1.14).

$$S_t\phi(x) = S_t^D\phi(x) + \bar{K}\int_0^t \int_D S_{t-s}\phi(x')\gamma(dx')f_D(s,x)\,ds\,. \tag{9.3.20}$$

Here $f_D(s,x)$ is the density function of the hitting time of the boundary τ^D,

$$P_x(\tau^D \in dt) = f(t,x)dt \quad t > 0\,, \quad x \in D\,,$$

with the well known relation

$$P_x(\tau^D > t) = S_t^D \mathbf{1}\,.$$

The resolvents satisfy

$$R_\alpha \phi(x) = R_\alpha^D \phi(x) + \bar{K}(\gamma R_\alpha \phi)\hat{f}_D(\alpha, x) \tag{9.3.21}$$

and again applying γ

$$\gamma R_\alpha \phi = \gamma R_\alpha^D \phi + \bar{K}(\gamma R_\alpha \phi)\hat{f}_D(\alpha, \gamma)\,, \tag{9.3.22}$$

solving,

$$\gamma R_\alpha \phi = \frac{\gamma R_\alpha^D \phi}{1 - \bar{K}\hat{f}_D(\alpha, \gamma)}\,, \tag{9.3.23}$$

and plugging back in (9.3.21) we obtain

$$R_\alpha \phi(x) = R_\alpha^D \phi(x) + \frac{\bar{K}\hat{f}_D(\alpha, x)}{1 - \bar{K}\hat{f}_D(\alpha, \gamma)} \gamma R_\alpha^D \phi\,. \tag{9.3.24}$$

Let $\alpha = \lambda_*$ be the solution of

$$1 - \bar{K}\hat{f}_D(\alpha, \gamma) = 0\,. \tag{9.3.25}$$

Theorem 9.3.1. *When $\bar{K} > 0$, the Yaglom limit exists and is equal to $\gamma R_{\lambda^*}^D$*

modulo a normalization constant. More precisely

$$\nu(dx) = C(\lambda^*) \int_D \gamma(dx') R^D_{\lambda^*}(x', dx)\,, \qquad C(\lambda^*) = \frac{\lambda^*}{1 - \frac{1}{K}}\,. \tag{9.3.26}$$

9.3.1 Proof of Theorem 3.3.1

By monotonicity of the Laplace transform, we can check that $\lambda_* > 0$. Notice that we already know from (7.1.2) that this is the critical number for the growt rate of $N_t \sim e^{\lambda_* t}$.

Theorem 9.3.1 proves the Yaglom limit and identifies λ_*. Proposition 9.2.1 identifies the corresponding left eigenfunction. □

9.3.2 Special Cases and Numerical Results

When $\gamma(dx) = \delta_c(dx)$ and $K = \bar{K} = 2$, the resolvent $R^D_{\lambda_*}$ in (9.3.26) above has a kernel (density function) which can be explicitly calculated as

$$R^D_{\lambda_*}(c, x) = \begin{cases} \frac{2\sqrt{2\lambda}\sinh\sqrt{2\lambda}c}{\cosh\sqrt{2\lambda}} \cosh\sqrt{2\lambda}(1-x) & \text{if } c \le x \le 1 \\ \frac{2\sqrt{2\lambda}\cosh\sqrt{2\lambda}(1-c)}{\cosh\sqrt{2\lambda}} \sinh\sqrt{2\lambda}x & \text{if } 0 \le x < c \end{cases} \tag{9.3.27}$$

where the normalizing factor $C(\lambda) = 2\sqrt{2\lambda}$ and λ, c are connected by the formula

$$\frac{1}{K} = \frac{1}{2} = \frac{\cosh\sqrt{2\lambda}(1-c)}{\cosh\sqrt{2\lambda}} \tag{9.3.28}$$

and, to simplify notation, we omitted the star subscript $\lambda = \lambda_*$ from Theorem 9.3.1, but the critical value is the same.

When c is small, that is, very close to 0, the frequency with which the particle system updates itself will be relatively greater. Therefore we expect to see the mass concentrate around a small vicinity of c, rendering $R^D_{\lambda_*}(c, x)$ bear the shape

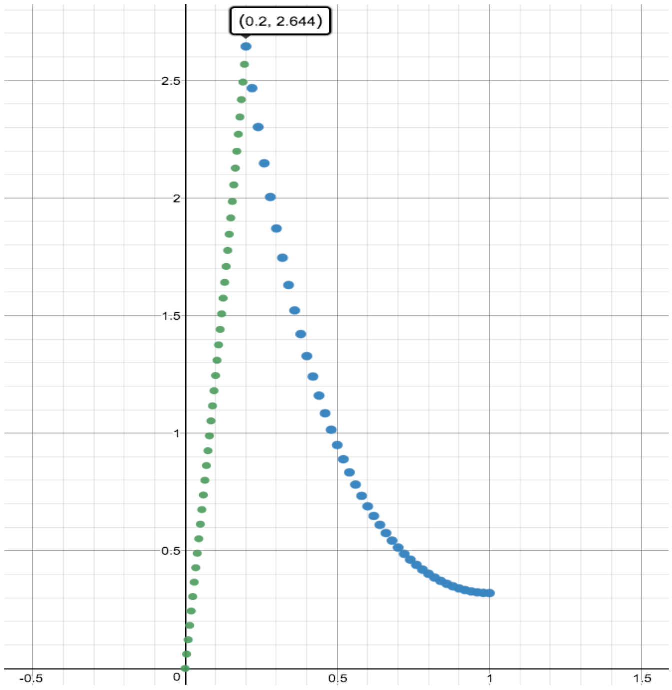

Figure 9.1: $R^D_{\lambda_*}(0.2, x)$ with $K = 2$

of a Dirac delta function $\delta_c(dx)$. When $c >> 0$ is relatively large, the whole particle system will "slow down", making $R^D_{\lambda_*}(c, x)$ more flat. In the numerical representations below, we illustrate this point by observing the contrast between Figure 9.1, Figure 9.2 and Figure 9.3. As c gets smaller, the graph of $R^D_{\lambda_*}(c, x)$ forms a sharper "angle" at c.

Another interesting feature that is inherent in (9.3.28) is that $\sqrt{2\lambda}c$ stabilizes to $\ln 2 \approx 0.69$, i.e. the solution of the equation is such that

$$\lambda_* \sim \frac{(\ln 2)^2}{2c^2} \qquad c \downarrow 0 \,.$$

Since λ_* is the analogue of the self-organizing criticality from (1.1.1), we obtain a full range of possible critical values, extending from the solution to (9.3.28) at $c \uparrow 1$ to $+\infty$ when $c \downarrow 0$.

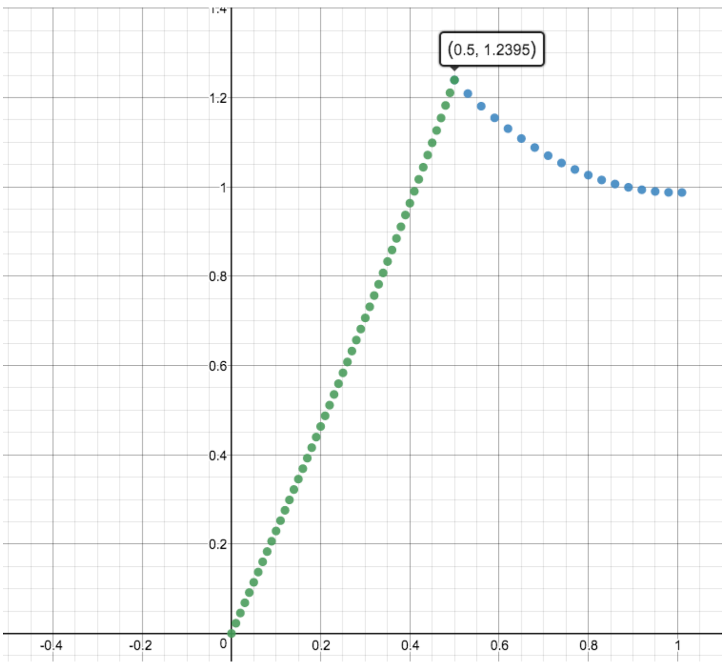

Figure 9.2: $R^D_{\lambda_*}(0.5, x)$ with $K = 2$

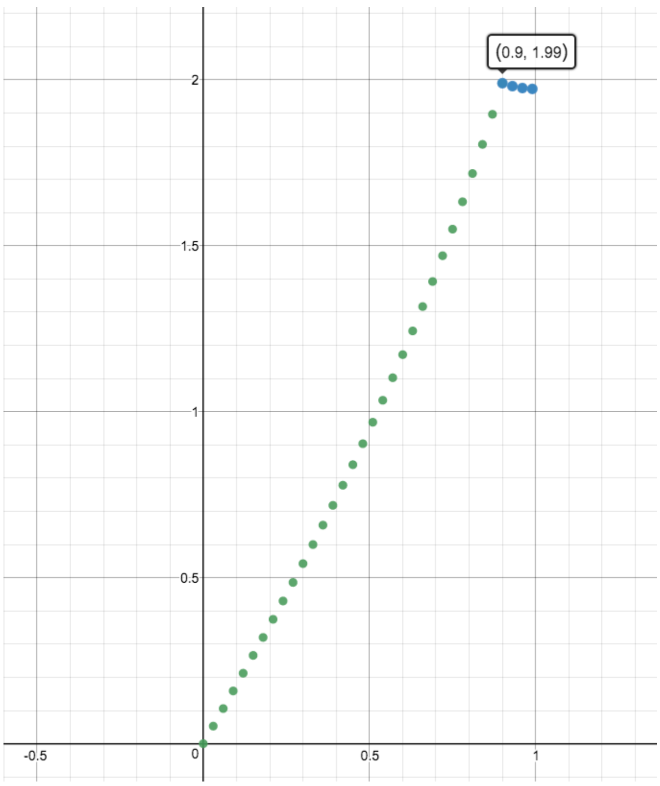

Figure 9.3: $R^D_{\lambda_*}(0.9, x)$ with $K = 2$

9.4 Generalizations, Alternative Models, and Future Directions

Higher dimensional case (cite) with general diffusion instead of standard BM, soft potential V, simply a Brownian motion with negative drift on $[0, \infty)$, Ornstein-Uhlenbeck, Galton-Watson (birth-death)[2][3] and many generalizations of the resampling distribution. Some of these cases, including the one dimensional ones, are interesting in regard to quasi-stationarity since it is well known that BM on the half line does not have a unique qsd.

Chapter 10

Preliminary Theory of Stochastic Processes

10.1 Important Probabilistic Inequalities

We introduce below some inequalities which are useful for estimating martingale involved processes. To start with, we need to define continuous time martingales.

Definition 10.1.1. [Definition 3.1 from [23]] Consider a real-valued process $X = \{X_t; 0 \leq t < \infty\}$ on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a given filtration $\{\mathcal{F}_t\}$ and such that $E|X_t| < \infty$ holds for every $t \geq 0$. The process $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is said to be a *submartingale* (respectively, a *supermartingale*) if, for every $0 \leq s < t < \infty$, we have, a.s. $E[X_t|\mathcal{F}_s] \geq X_s$ (respectively, $E[X_t|\mathcal{F}_s] \leq X_s$).
We shall say that $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a *martingale* if it is both a submartingale and a supermartingale.

The two inequalities in next theorem prove to be very useful in the proof of tightness of A_t^N in chapter 6. We use these two inequalities to show that both jump and continuous martingales $\mathcal{M}_t^{N,J,\phi}$, $\mathcal{M}_t^{N,c,\phi}$ are of order N^{-1}.

Theorem 10.1.2. *[Theorem 3.8 from [23]] Let $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a submartingale whose every path is right continuous. Given a subinterval $[a, b]$*

of $[0, \infty)$ *and let* $\lambda > 0$ *be real number. We have the following:*

(i) Submartingale inequality:

$$P\big(\sup_{a \le t \le b} X_t \ge \lambda\big) \le \frac{E[X_t^p]}{\lambda^p}\,, \qquad p \ge 1 \tag{10.1.1}$$

(ii) Doob's Maximal Inequality:

$$E\Big[\big(\sup_{a \le t \le b} X_t\big)^p\Big] \le \Big(\frac{p}{p-1}\Big)^p E\big[x_b^p\big]\,, \qquad p > 1 \tag{10.1.2}$$

Remark 10.1.3. Notice that Doob's Maximal Inequality (10.1.2) is analogous to Chebyshev's inequality for simple random variables.

10.2 Tightness and C-tightness in Skorokhod Spaces

This presentation follows very closely the material presented in the classic reference text on the subject [8].

10.2.1 The J_1 Topology on Skorokhod Space $D[0, 1]$

Unlike the space that houses the sample paths of Brownian motion, which are continuous, in our model, however, the sample paths of each particle are discontinuous with countable jumps, triggered by boundary hits at 0. The standard function space that these sample paths reside in is the space $D[0, \infty)$, which is the space of all real-valued functions defined over $[0, \infty)$ that are right continuous with left limits. The foundation of the tightness is a topology naturally imposed on this space, which we present below.

We start with the space $D = D[0,1]$, from which we obtain corresponding structures of $D[0,\infty)$ by extending the interval.

Definition 10.2.1. [8] Let $D = D[0,1]$ be the space of real functions x on $[0,1]$ that are right-continuous and have left-hand limits:

(i) For $0 \leq t < 1$, $x(t+) = \lim_{s \downarrow t} x(s)$ exists and $x(t+) = x(t)$.

(ii) For $0 < t \leq 1$, $x(t-) = \lim_{s \uparrow t} x(s)$ exists.

In particular, functions with these two properties are called *cadlag*.

For $x \in D$ and $T \subseteq [0,1]$, put

$$w_x(T) = w(x,T) = \sup_{s,t \in T} |x(s) - x(t)|$$

The modulus of continuity of x can be written as

$$w_x(\delta) = w(x,\delta) = \sup_{0 \leq t \leq 1-\delta} w_x[t, t+\delta] \tag{10.2.3}$$

In addition to the modulus above, another form of modulus turns out to be very useful. Given a partition, $\{t_i\}$, of $[0,1]$ where $0 = t_0 < t_1 < \ldots < t_v = 1$, we call $\{t_i\}$ δ-sparse if it satisfies $\min_{1 \leq i \leq v}(t_i - t_{i-1}) > \delta$. Now define, for $0 < \delta < 1$,

$$w'_x(\delta) = w'(x,\delta) = \inf_{\{t_i\}} \max_{1 \leq i \leq v} w_x[t_{i-1}, t_i), \tag{10.2.4}$$

where the infimum extends over all δ-sparse sets t_i.

The two moduli (10.2.3) and (10.2.4) can be linked by two equations. We have the following.

$$w'_x(\delta) \leq w_x(2\delta), \qquad \text{if} \quad \delta < 1/2$$

$$w_x(\delta) \leq 2w'_x(\delta), \qquad \text{if} \quad x \in C := C[0,1]$$

Let Λ denote the class of strictly increasing, continuous mappings of $[0,1]$ onto itself. If $\lambda \in \Lambda$, then $\lambda(0) = 0$ and $\lambda(1) = 1$. For any x and y in D, define $d(x,y)$ to be the infimum of those positive ϵ for which there exists a λ in Λ satisfying

$$\sup_t |\lambda(t) - t| = \sup_t |t - \lambda^{-1}(t)| < \epsilon \tag{10.2.5}$$

and

$$\sup_t |x(t) - y(\lambda(t))| = \sup_t |x(\lambda^{-1}(t) - y(t)| < \epsilon \tag{10.2.6}$$

In particular, for $y\lambda$ denoting the composition $y \circ \lambda$

$$d(x,y) := \inf_{\lambda \in \Lambda} \{||\lambda - I|| \vee ||x - y\lambda||\}.$$

where I is the identity function on $[0,1]$ and $||x(t)|| := \sup_t |x(t)|$ for $x(t) \in D$. In the same monograph (c.f. [8]) it is shown that d is a metric. With this construction, the topology on D induced by d, is known as the *Skorokhod topology* or J_1 *topology.* The uniform distance $||x-y||$ is defined as the infimum of those positive ϵ for which $\sup_t |x(t) - y(t)| < \epsilon$. Notice that the λ in (10.2.5) and (10.2.6) represents the uniformly small "deformation" of the time scale done in this section.

One issue with the metric d is that the induced topological space D is separable but not complete. However, completeness is essential in our application of Aldous Criterion to prove A_t^N is tight. In order to get completeness, we need to introduce a time rescaling factor λ into d. The resulting metric d° is equivalent to d and it will give us completeness.

If λ is a nondecreasing function on $[0,1]$ satisfying $\lambda(0)=0$ and $\lambda(1)=1$, put

$$||\lambda||^\circ := \sup_{s<t}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right|$$

Let $d^\circ(x,y)$ be the infimum of those positive ϵ for which Λ contains some λ such that $||\lambda||^\circ<\epsilon$ and (10.2.6) holds. In other words, let

$$d^\circ(x,y)=\inf_{\lambda\in\Lambda}\{||\lambda||^\circ\vee||x-y\lambda||\}$$

It is shown in the monograph [8] that the topological space D induced by metric d° is complete and separable. Now we're ready to develop compactness and tightness in space D.

10.2.2 Characterization of Compactness in $D[0,1]$

Using the modulus (10.2.4), we can prove an analogue of the Arzela-Ascoli Theorem.

Theorem 10.2.2. *[Theorem 12.3 from [8]] A necessary and sufficient condition for a set A to be relatively compact in the Skorokhod topology is that*

$$\sup_{x\in A}||x||<\infty\,. \tag{10.2.7}$$

and

$$\lim_{\delta \to 0} \sup_{x \in A} w'_x(\delta) = 0\,. \tag{10.2.8}$$

Notice that the difference between this theorem and the Arzela-Ascoli theorem is that for no single t do $\sup_{x\in A} |x(t)| < \infty$ and (10.2.8) together imply (10.2.7). The more useful part of the theorem is the sufficiency, which we use to prove tightness.

10.2.3 Weak Convergence and Tightness in $D[0, 1]$

A typical way to prove weak convergence in function space is to prove weak convergence of the finite-dimensional distributions and then prove tightness. Since D is separable and complete under the metric d°, a family of probability measures on $(D, \mathcal{D})$, where $\mathcal{D}$ is the corresponding σ-algebra, is relatively compact if and only if it is tight.

Theorem 10.2.3. *[Theorem 13.2 from [8]] Let $\{P_n\}$ be a sequence of probability measures on $(D, \mathcal{D})$. The sequence $\{P_n\}$ is tight if and only if these two conditions hold:*

(i) We have

$$\lim_{a\to\infty} \limsup_{n} P_n(x : ||x|| \geq a) = 0\,.$$

(ii) For each ϵ,

$$\lim_{\delta} \limsup_{n} P_n(x : w'_x(\delta) \geq \epsilon) = 0\,.$$

The norm $|| \cdot ||$ involved in condition (i) can be replace with $| \cdot |$.

Corollary 10.2.4. *The following condition can be substituted for (i) in Theorem 10.2.3: (i') For each t in a set T that is dense in $[0,1]$ and contains 1,*

$$\lim_{a\to\infty}\limsup_{n} P_n(x : |x(t)| \geq a) = 0$$

10.2.4 Extension From $D[0,1]$ to $D[0,\infty)$

Denote, for $t > 0$, $D_t = D[0,t]$, the space of cadlag functions on $[0,t]$. All the definitions for D_1 extend naturally to D_t. So do all the theorems in D_1. If x is an element of $D_\infty = D[0,\infty)$, or if x is an element of D_u and $t < u$, then x can also be regarded as an element of D_t by restricting its domain of definition.

One natural way to define Skorokhod convergence in D_∞ is by $d^\circ(x_n, x) \to_n 0$ for each finite, positive t. But counterexample exists due to discontinuities when restricting domains. We need the following lemma:

Lemma 10.2.5. [Lemma 1 of Section 16 from [8]] Let x_n and x be elements of D_u. If $d_u^\circ(x_n, x) \to_n 0$ and $m < u$, and if x is continuous at m, then $d_m^\circ(x_n, x) \to_n 0$.

With the lemma established, we can now define the metric $d_\infty^\circ(x, y)$ on D_∞.

$$d_\infty^\circ(x,y) = \sum_{m=1}^{\infty} 2^{-m}(1 \wedge d_m^\circ(x^m, y^m))$$

Based on the above definition, we have

Theorem 10.2.6. *[Theorem 16.2 from [8]] There is convergence $d_\infty^\circ(x_n, x) \to 0$ in D_∞ if and only if $d_t^\circ(x_n, x) \to 0$ for each continuous point t of x.*

With this characterization of convergence, we can prove the following tightness theorem.

Theorem 10.2.7. *[Theorem 16.5 from [8]] The sequence $\{P_n\}$ is tight if and only if the following two conditions hold:*

(i) For each m,

$$\lim_{a\to\infty} \limsup_{n} P_n(x : ||x||_m \geq a) = 0.$$

(ii) For each m and ϵ,

$$\lim_{\delta} \limsup_{n} P_n(x : w'_m(x, \delta) \geq \epsilon) = 0.$$

Finally this theorem will lead us to the Aldous Criterion stated at the beginning of Chapter 6. However, to prove that the limit point is continuous, we need the following theorem from [32], from which we adapted Proposition 6.0.1.

C-tightness is a stronger criterion that allows to state that not only the sequence of probability laws under consideration is *tight*, that is, precompact, but its limit points are concentrated on the subsapece of continuous functions (paths) of the Skorokhod space. Since in our work the original process presents jumps, but the limiting process is continuous, this test is the most appropriate for our goals.

Theorem 10.2.8. *[Criterion for C-tightness, from [32]] Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of real valued processes on the Skorokhod space (with rcll paths). Then the sequence of probability measures $\{P_n\}_{n=1}^{\infty}$ induced on $\mathbb{D}[0,\infty)$ by $\{X_n\}_{n=1}^{\infty}$ is tight and any weak limit point of this sequence is concentrated on $\mathbb{C}[0,\infty)$ if and only if the following two conditions hold for each $T > 0$ and $\epsilon > 0$:*

(i) $\lim_{K\to\infty} \limsup_{n\to\infty} P(||X_n||_T \geq K) = 0\,,$

(ii) $\lim_{\delta\to 0}\limsup_{n\to\infty} P(w(X_n,\delta,T)\geq\epsilon)=0$ *, where for* $x\in\mathbb{D}[0,\infty)$*,*

$$w(x,\delta,T)=\sup_{t\geq 0}\left\{\sup_{u,v\in[t,t+\delta]}|x(u)-x(v)| : 0\leq t<t+\delta\leq T\right\}. \tag{10.2.9}$$

Remark. Throughout this work, the actual index n is N, the scaling constant $N\to\infty$.

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