2017-07-20

On Statistical Solutions of Evolution Equations

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UNIVERSITY OF MIAMI

ON STATISTICAL SOLUTIONS OF EVOLUTION EQUATIONS

By

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A DISSERTATION

Submitted to the Faculty
of the University of Miami
in partial fulfillment of the requirements for
the degree of Doctor of Philosophy

Coral Gables, Florida

August 2017

UNIVERSITY OF MIAMI

A dissertation submitted in partial fulfillment of
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ON STATISTICAL SOLUTIONS OF EVOLUTION EQUATIONS

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CARDONA, JORGE EDUARDO (Ph.D. Mathematics)
On Statistical Solutions of Evolution Equations (August 2017)

Abstract of a dissertation at the University of Miami.

Dissertation supervised by Professor Lev Kapitanski.
No. of pages of text. (94)

I study different types of statistical solutions (Hopf, Foiaș, Vishik-Fursikov) for nonlinear evolution equations. As a test equation, I use the nonlinear Schrödinger equation with power-like nonlinearity in the case where the proofs of uniqueness are not available. When there is no uniqueness in the original equation, statistical solutions are not unique. For autonomous differential equations, there is a formal semigroup property. I propose to look for statistical solutions with an analogous property. For statistical solutions, this should be the homogeneous Markov property. I call such solutions Markov statistical solutions.

The proofs of the existence of the Markov statistical solutions rely on the Markov selection theorem. N.V. Krylov was the first to realize the importance of the Markov selection in the context of Markov processes. D.W. Stroock, and S.R.S. Varadhan re-framed Krylov's selection in the context of solutions of the martingale problem. Recently, their results have been used by F. Flandoli and M. Romito, and Goldys et al., for the analysis of the Navier-Stokes equation with additive noise. I use the Markov selection theorem to prove the existence of Markov statistical solutions. I give a new proof of the Markov selection theorem. This proof has prompted me to look back at the set-valued solutions of deterministic equations, where the analog of the homogeneous Markov property should be the semigroup property. It turned out that no theorems of existence of selections satisfying the semigroup property were known. I state and prove a selection theorem for measurable selections with the semigroup property. Such result is important in its own right. I use it here to give a second proof of the existence of Vishik-Fursikov measures.

A mi madre, de quién sigo aprendiendo todos los días,

y a mi padre, cuya memoria lo permea todo.

Acknowledgements

This dissertation represents the first milestone in a long journey of discovering the world of mathematics. It is, in essence, a personal journey, possible only thanks to the help of many people around me.

First, I want to thank my advisor, Dr. Lev Kapitanski. During the last 3 years, his guidance and support have been of great help. I enjoy all of our research meetings, and I feel extremely fortunate to have worked with him. I have learned with him not only some really deep ideas in mathematics, but also, I have obtained a better understanding of what makes a mathematician.

During the years as a graduate student, I had the fortune of having great friends in the program whom helped me to have an easier time in the moments of doubt and stress. I will be in great debt with them. As well, the stuff members in the department were always helpful in any moment and make us feel at home with their encouraging comments and support.

During the last 3 years of my Ph.D. I had the suuport of the Colombian Administrative Department of Sciece, Technology and Innovation, Colciencias. I am very thankful for their support.

Last but not least, I want to thank my family. Specially, my mother, Luz Marina, who has been supportive on all my personal projects, I own her more than I can describe here. Also, my father, whose memories will remain forever. My current passion for science and mathematics is only a result of his supportive and lovely encoragement while I was growing up. Finally, my brother and sister, Alejandro and Ana Mercedes, for their support during all these years far from them.

Contents

Introduction

The motion of a deterministic mechanical system is governed by a differential equation which has the symbolic form

$$\frac{du}{dt} = F(u)\,, \tag{1}$$

where $u(t)$ is a set of parameters that completely characterizes the state of the system at time t. If the number of parameters is very large or if they cannot be measured accurately, it is useful to study probability distributions on the space of states of the system and their evolution in time resulting from the individual motions. Over the last seventy years, several different forms of equations for evolution of probability measures have been proposed, most notable, by E. Hopf, C. Foiaș, and M. I. Vishik and A. Fursikov. The solutions of those equations are called statistical solutions of the original equation (1).

If solutions of (1) are unique and $S_t : a \mapsto S_t(a)$ is the corresponding evolution of the initial state a, then, the evolution of any given probability distribution on initial states is unique: $S_t^* : \mu_0 \mapsto \mu_t = \mu_0(S_t^{-1}(\cdot))$. If there is no uniqueness for (1), one can study the corresponding equations (Hopf, Foiaș) for probability measures from the point of view of existence, uniqueness, and the properties of solutions.

I study different types of statistical solutions for nonlinear evolution equations. As a test equation (1) I use the nonlinear Schrödinger equation (NLS) with power-like nonlinearity in the case where the proofs of uniqueness are not available. When there is no uniqueness in the original equation, statistical solutions are not unique. For autonomous differential equations like (1), there is a formal semigroup property. I propose to look for statistical solutions with an analogous property. For statistical solutions, this should be the homogeneous Markov property. I call such solutions Markov statistical solutions.

The proofs of the existence of the Markov statistical solutions rely on the Markov selection theorem. N.V. Krylov [21] was the first to realize the importance of the Markov selection in the context of Markov processes. D.W. Stroock, and S.R.S. Varadhan [29] re-framed Krylov's selection in the context of solutions of the martingale problem. Recently, their results have been used by F. Flandoli and M. Romito [10], and Goldys et al. [15], for the analysis of the Navier-Stokes equation with additive noise. I use the Markov selection theorem to proof the existence of Markov statistical solutions. I give a new proof of the Markov selection theorem. This proof has prompted me to look back set-valued solutions of deterministic equations, where the analog of the homogeneous Markov property should be the semigroup property. It turned that no theorems of existence of selections satisfying the semigroup property were known. I state and prove a selection theorem for measurable selections with the semigroup property. Such result is important in its own right. I use it here to give a second proof of the existence of Vishik-Fursikov measures.

Outline of the Thesis

In Chapter 1 I describe the Cauchy problem associated with the nonlinear Schrödinger equation (1.10). With the help of some a priori estimates I define the concept of weak solutions to (1.10) and find a suitable function space for solutions.

In Chapter 2 a measurable semigroup selection theorem is proved. This is one of the main results of my thesis.

Chapter 3 is devoted to the Hopf equation, and the associated Hopf statistical solutions. Existence, but not uniqueness, of such solutions is presented. An explicit example of the Hopf equation for the Burgers equation is given as well. The Foiaș equation is also presented. The Foiaș statistical solution of an ODE with non-uniqueness is presented.

In Chapter 4 I introduce Vishik-Fursikov measures, as described by C. Foiaș et al. [13]. The existence of such solutions to NLS is proved.

In Chapter 5, I introduce a new notion of the Markov statistical solutions, and I use the Markov selection theorem as presented by Goldies et al. to obtain a Markov statistical solution to NLS. Then, I give a new proof of the Markov selection theorem of N.V. Krylov [21], and D. Stroock and S.R.S. Varadhan [29], which is close in spirit to the abstract selection theorem of Chapter 2.

Chapter 1

Nonlinear Schrödinger Equation

1.1 Function Spaces over Riemannian Manifolds

In this thesis, the main results are illustrated with the semilinear Schrödinger equation on a closed Riemmanian manifold (M, g) of dimension d. The Lebesgue spaces $L^p(M)$ and the L^2-Sobolev spaces $H^s(M)$ are defined in a more or less standard way. The L^p-norm of a function ξ is denoted $\|\xi\|_p$, where

$$\|\xi\|_p^p = \int_M |\xi(x)|^p dx\,, \tag{1.1}$$

where dx is the volume element on M. In $L^2(M)$ the inner product is denoted by (ξ, η),

$$(\xi, \eta) = \int_M \xi(x)\,\overline{\eta(x)}\,dx\,.$$

Denote by Δ the Laplace-Beltrami operator on M (in analysis, Δ is negative definite.)

Theorem 1.1.1. *For a closed d-manifold M, there exists a complete orthonormal basis $W = \{w_0, w_1, \ldots\}$ of L^2 formed by eigenfunctions of the Laplace-Beltrami op-*

erator, $-\Delta w_i = \lambda_i w_i$, *for* $i = 0, 1, \dots,$ *with the eigenvalues*

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \,.$$

Every function $u \in L^2(M)$ can be represented by its Fourier series

$$u = \sum_{k \geq 0} \hat{u}_k w_k \,,$$

where $\hat{u}_k = (u, w_k)$. In terms of Fourier coefficients

$$(u, v) = \sum_{k=0}^{\infty} \hat{u}_k \overline{\hat{v}_k} \,.$$

and

$$\|u\|_2^2 = \sum_{k=0}^{\infty} |\hat{u}_k|^2 \,.$$

The Sobolev space $H^s(M)$, for $s \in \mathbb{R}$, is made of (as a distribution, if $s < 0$) Fourier series

$$u = \sum_{k \geq 0} \hat{u}_k w_k \,,$$

such that

$$\|u\|_{H^s}^2 = \sum_{k=0}^{\infty} \langle \lambda_k \rangle^{2s} |\hat{u}_k|^2 < \infty \,,$$

where $\langle \lambda \rangle = (1 + \lambda^2)^{\frac{1}{2}}$. The dual of $H^s(M)$ with respect to $L^2(M)$ is the space $H^{-s}(M)$. The duality pairing between $H^s(M)$ and $H^{-s}(M)$ is

$$\langle u, v \rangle = \sum_{k \geq 0} \hat{u}_k \, \overline{\hat{v}_k} \,.$$

As it is usual for Hilbert spaces, we will define the finite dimensional projection P^m as follows.

Definition 1.1.1. For any integer $m > 0$, define the projection operator P^m, acting in the whole scale of Sobolev spaces H^s, $s \in \mathbb{R}$, as

$$P^m[\phi] = \sum_{k=0}^{m} \hat{\phi}_k w_k \tag{1.2}$$

for any $\phi = \sum_{k\geq 0} \hat{\phi}_k w_k \in H^s$. If the context allows we will simply denote ϕ^m instead of $P^m[\phi]$.

Lemma 1.1.2. *If $s \in \mathbb{R}$, and $\phi \in H^s$, then, $\phi^m \to \phi$ as $n \to \infty$, strongly in H^s.*

In what follows we write $M_T = [0,T] \times M$, and identify $L^2([0,T] \to L^2(M))$ with the space $L^2(M_T)$. Similarly, $M_\infty = [0,\infty) \times M$.

The following Lemma will be useful, see [23].

Lemma 1.1.3 (Friedrichs)**.** *Let (u_k) be a sequence in $L^2([0,T] \to H^1(M))$ such that $u_k \rightharpoonup u$ as $k \to \infty$, then $u_k \to u$ in $L^2(M_T)$ as $k \to \infty$.*

Proof. Since $u_m \rightharpoonup u$ in $L^2([0,T) \to H^1)$ we have

$$u_m \rightharpoonup u \in L^2(M_T)\,, \tag{1.3}$$

and

$$\nabla u_m \rightharpoonup \nabla u \in L^2(M_T)\,. \tag{1.4}$$

For any $\phi \in H^1$, the finite-dimensional projections ϕ^m converge $\phi^m \to \phi$ strongly in H^1 to ϕ. Also,

$$\|\nabla \phi^m\|_2^2 = \sum_{n=0}^{m} \lambda_n |(\phi, w_n)|^2 \leq \sum_{n\geq 0} \lambda_n |(\phi, w_n)|^2 = \|\nabla \phi\|_2^2\,. \tag{1.5}$$

Since $\lambda_n \to \infty$, for any $\epsilon > 0$ there exists an M_ϵ such that $\frac{1}{\lambda_n} < \epsilon$ for any $n > M_\epsilon$. Then, for $m > M_\epsilon$, we have

$$\|\phi - \phi^m\|_2^2 = \sum_{n>m} |(\phi, w_n)|^2 \leq \epsilon \sum_{n>m} \lambda_n |(\phi, w_n)|^2 \leq \epsilon \|\nabla \phi\|_2^2 . \tag{1.6}$$

For any $k > 0$ consider the following

$$\begin{aligned} \int_0^T \|u(t) - u_k(t)\|_2^2 dt &= \int_0^T \sum_{n\leq m} |(u(t) - u_k(t), w_n)|^2 dt + \int_0^T \sum_{n>m} |(u(t) - u_k(t), w_n)|^2 dt \\ &\leq \int_0^T \sum_{n\leq m} |(u(t) - u_k(t), w_n)|^2 dt + \epsilon \|\nabla(u - u_k)\|_{L^2(M_T)}^2 . \end{aligned} \tag{1.7}$$

Since $\nabla u_k \rightharpoonup \nabla u$ in $L^2(M_T)$, then the $L^2(M_T)$ norm of $\nabla(u - u_k)$ is uniformly bounded, hence the second term can be made arbitrarily small by picking a sufficiently small ϵ. For a fixed $m > M_\epsilon$ the first term goes to 0 as $k \to \infty$, due (1.3). Thus,

$$\int_0^T \left\|u(t) - u_k(t)\right\|_2^2 dt \to 0$$

as $k \to \infty$, i.e. $u_k \to u$ in $L^2(M_T)$. □

In the case of the L^p spaces, for $p > 2$, the family of eigenfunctions $\{w_0, w_1, \cdots\}$ forms a Schauder basis, see [35, Part II.B.5]. The following is a conclusion of [35, Proposition II.B.6],

Lemma 1.1.4. *If $p > 2$, and $\phi \in L^p$, then $\phi^m \to \phi$, strongly in L^p as $m \to \infty$.*

Moreover, there is a constant C_p, such that, for any $\phi \in L^p$

$$\|\phi^m\|_p \leq C_p \|\phi\|_p . \tag{1.8}$$

The following two results are of great importance in the treatment of evolution equations and can be found in [30, Chapter III, Lemma 1.1. and 1.2].

Lemma 1.1.5. *Let X be a given Banach space with dual X^* and let u and g be two functions belonging to $L^1([a,b] \to X)$. Then, the following three conditions are equivalent*

1. *u is a.e. equal to a primitive function of g,*

$$u(t) = \xi + \int_0^t g(s)\, ds\,, \quad \xi \in X, t \in [a,b]\,.$$

2. *For each test function $\phi \in \mathscr{D}((a,b))$,*

$$\int_a^b u(s)\, \phi'(s)\, ds = -\int_a^b g(s)\, \phi(s)\, ds\,.$$

3. *For each $\eta \in X^*$,*

$$\frac{d}{dt}\langle u(t), \eta\rangle = \langle g(t), \eta\rangle\,,$$

   in the scalar distribution sense on (a,b).

If (1-3) are satisfied u, in particular, is a.e. equal to a continuous function from $[a,b]$ into X.

Lemma 1.1.6. *Let $V \subset H \subset V^*$ be a Gel'fand triple and let $\langle f, g\rangle$ be the pairing between V and V^* that agrees with the inner product in H. If $u \in L^2([a,b] \to V)$ and $u_t \in L^2([a,b] \to V^*)$, then $u \in C([a,b] \to H)$ and*

$$\frac{d}{dt}\|u(t)\|_H^2 = 2\,\mathrm{Re}\langle u_t(t), u(t)\rangle \quad a.e.\ t\,. \tag{1.9}$$

1.2 Cauchy Problem of the Nonlinear Schrödinger Equation

In this chapter we consider the Cauchy problem for the nonlinear Schrödinger equation (NLS)

$$iu_t(t,x) + \Delta u(t,x) - |u(t,x)|^{(p-2)}u(t,x) = 0 \tag{1.10}$$

on a closed Riemannian d-manifold M, $d \geq 1$. The exponent p it is assumed to satisfy $p \geq 2$.

Let $u(0,x) = \phi(x)$ be the initial condition for (1.10). First, I will derive basic a priori estimates. Assume there is a solution $u(t,x)$ of (1.10) sufficiently regular to perform the following operations. Multiply (1.10) by $\overline{u(t,x)}$ and integrate over M to obtain

$$\int_M iu_t(t,x)\overline{u(t,x)} - \nabla u(t,x)\overline{\nabla u(t,x)} - |u(t,x)|^p dx = 0\,. \tag{1.11}$$

Take the conjugate of (1.10), and multiply by $u(t,x)$ to obtain

$$\int_M -i\overline{u_t(t,x)}u(t,x) - \overline{\nabla u(t,x)}\nabla u(t,x) - |u(t,x)|^p dx = 0\,. \tag{1.12}$$

Finally, take the difference of the last two equations to obtain

$$\frac{d}{dt}\int_M |u(t,x)|^2 dx = 0\,, \tag{1.13}$$

which represents the **conservation of mass** for NLS:

$$\|u(t)\|_2 = \|u(0)\|_2\,, \qquad \forall t \geq 0\,. \tag{1.14}$$

Next, multiply (1.10) by $\overline{u_t(t,x)}$ and integrate over M to obtain

$$\int_M i|u_t(t,x)|^2 - \nabla u(t,x)\overline{u_{t,x}(t,x)} - |u(t,x)|^{p-2}u(t,x)\overline{u_t(t,x)}dx = 0\,. \tag{1.15}$$

Take the conjugate of (1.10) and multiply by $u_t(t,x)$ to obtain

$$\int_M -i|u_t(t,x)|^2 - \overline{\nabla u(t,x)}u_{t,x}(t,x) - |u(t,x)|^{p-2}\overline{u(t,x)}u_t(t,x)dx = 0\,. \tag{1.16}$$

Sum the last two equations, and divide by two, to obtain

$$\frac{d}{dt}\int_M \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{p}|u(t,x)|^p dx = 0\,, \tag{1.17}$$

which represents the **conservation of energy** for NLS.

Definition 1.2.1 (Energy space and energy functional)**.** The space $V = H^1 \cap L^p$ will be called the **energy space**. The norm in V is

$$\|\xi\|_V = \max\{\|\nabla\xi\|_2, \|\xi\|_p\}\,. \tag{1.18}$$

The dual space of V is $V^* = H^{-1} + L^q$, with $\frac{1}{p} + \frac{1}{q} = 1$, and norm

$$\|\xi\|_{V^*} = \inf\left\{ \|f\|_{H^{-1}} + \|g\|_q \,\middle|\, f + g = \xi\,, f \in \dot{H}^{-1}\,, g \in L^{\frac{p}{p-1}} \right\}\,. \tag{1.19}$$

The **energy functional** $E : V \to \mathbb{R}$ is defined as

$$E[\xi] = \frac{1}{2}\|\nabla\xi\|_2^2 + \frac{1}{p}\|\xi\|_p^p\,.$$

It follows from (1.17) that for sufficiently smooth solution u of (1.10) we have,

$$E[u(t)] = E[u(0)]\,, \qquad \forall t \geq 0\,. \tag{1.20}$$

Weak solutions we define next will satisfy the conservation of mass property (1.14), but instead of the conservation of energy (1.20), we require the energy inequality,

$$E[u(t)] \leq E[u(0)]\,, \qquad \forall t \geq 0\,. \tag{1.21}$$

This would imply that $u \in L^\infty([0,T] \to V)$. Since the Laplacian maps $\dot{H}^1$ to $\dot{H}^{-1}$, we see from (1.10) that $u_t \in L^\infty([0,\infty] \to V^*)$. Clearly, $u \in L^2([0,T] \to V)$. In view of Lemma 1.1.6 we conclude that $u \in C([0,T] \to L^2)$.

Pick a regular enough test function $\eta(x)$. Multiply (1.10) by $\overline{\eta(x)}$, and integrate over M using integration by parts when possible to obtain

$$\int_M iu_t(t,x)\,\overline{\eta(x)} - \nabla u(t,x)\,\nabla\overline{\eta(x)} - |u(t,x)|^{p-2}u(t,x)\,\overline{\eta(x)}dx = 0\,, \tag{1.22}$$

and rewrite (1.22) as

$$(u_t(t),\eta) = -i(\nabla u(t),\nabla\eta) - i\Big\langle |u(t)|^{p-2}u(t),\eta\Big\rangle\,, \tag{1.23}$$

where the pairing $\langle\cdot,\cdot\rangle$ is between the spaces L^q and L^p, with $\frac{1}{p}+\frac{1}{q}=1$. Given $t \in [0,T]$, integrate from 0 to t to obtain

$$(u(t),\eta) = (u(0),\eta) - i\int_0^t (\nabla u(s),\nabla\eta) + \Big\langle |u(s)|^{p-2}u(s),\eta\Big\rangle ds\,. \tag{1.24}$$

For this equation to make sense, it is sufficient to have $\eta \in V$. We get to our first definition of a weak solution on the interval $[0,T]$.

Definition 1.2.2. Given $\phi \in V$ and $T > 0$, a function $u : [0,T] \to V$ will be said to be a weak solution of (1.10) with intial condition ϕ, if it satisfies the following:

1. The function u belongs to $C([0,T] \to L^2) \cap L^\infty([0,T] \to V)$.

2. For any $\eta \in V$,

$$(u(t), \eta) = (\phi, \eta) - i \int_0^t (\nabla u(s), \nabla \eta) + \left\langle |u(s)|^{p-2} u(s), \eta \right\rangle ds\,. \tag{1.25}$$

3. The mass is conserved, i.e., for any $t \in [0, T]$, $\|u(t)\|_2 = \|\phi\|_2$.

4. The energy inequality

$$E[u(t)] \leq E[\phi]\,, \qquad \forall t \geq 0\,,$$

   is satisfied.

1.2.1 Construction of Weak Solutions

We use Faedo-Galerkin method to construct weak solutions to (1.10). For any $m > 0$ we will define a new finite dimensional Cauchy problem associated with (1.10). The solutions of this new problem will approximate the weak solution, and converge to it in the appropriate limit.

The Faedo-Galerkin approximations are the functions of the form

$$u^m(t, x) = \sum_{n=0}^{m} u_n^m(t)\, w_n(x)\,, \tag{1.26}$$

satisfying

$$\frac{d}{dt}(u^m(t), w_n) = -i\,(\nabla u^m(t), \nabla w_n) - i \left\langle |u^m(t)|^{p-2} u^m(t), w_n \right\rangle, \tag{1.27}$$

for all $n = 0, \ldots, m$, with the initial conditions

$$(u^m(0), w_n) = (\phi, w_n)\,. \tag{1.28}$$

Consider the map $F : V \to V^*$ defined by

$$F(\xi) = i\Delta\xi - |\xi|^{p-2}\,\xi\,,$$

and define $F^m(\xi) = P^m(F(P^m(\xi)))$. Then, (1.27) and (1.28) can be written as the finite-dimensional Cauchy problem

$$\frac{d}{dt}u^m(t) = F^m(u^m(t))\,, \quad u^m(0) = \phi^m\,. \tag{1.29}$$

Equation (1.29) is a system of $m+1$ first-order ODEs. To see this more clearly notice that

$$\begin{aligned}&\Big\langle |u^m(t,x)|^{p-2}u^m(t,x), w_n\Big\rangle = \\ &\qquad \sum_{j_1=0}^{m}\sum_{j_2=0}^{m}\cdots\sum_{j_{p-1}=0}^{m} u^m_{j_1}(t)\,\overline{u^m_{j_2}(t)}\,\cdots\,\overline{u^m_{j_{p-2}}(t)}\,u^m_{j_{p-1}}(t)\Gamma^n_{j_1,\bar{j}_2,\cdots,\bar{j}_{p-2},j_{p-1}}\,,\end{aligned} \tag{1.30}$$

where

$$\Gamma^n_{j_1,\bar{j}_2,\cdots,\bar{j}_{p-2},j_{p-1}} = \Big(w_{j_1}\overline{w_{j_2}}\cdots\overline{w_{j_{p-2}}}w_{j_{p-1}}, w_n\Big)\,,$$

a quantity that depends solely on the manifold M and the family W. Equation (1.30) is just a polynomial in $m+1$ variables, hence continuous, and locally Lipschitz. For $n = 0, 1, \ldots, m$, define the map $F^m_n : \mathbb{C}^{m+1} \to \mathbb{C}$, as

$$F^m_n(z_0, z_1, \ldots, z_m) = \left(F^m\left(\sum_{k=0}^{m} z_k\, w_k\right), w_n\right).$$

The system (1.29) is,

$$\frac{d}{dt}u^m_n(t) = F^m_n(u^m_0(t), u^m_1(t), \ldots, u^m_m(t))\,, \quad n = 0, 1, \ldots, m\,, \tag{1.31}$$

where, explicitly,

$$
\begin{aligned}
F_n^m(u_1^m(t), u_1^m(t), \dots, u_m^m(t)) &= i\lambda_n u_n^m(t) \\
&- i \sum_{j_1=1}^m \sum_{j_2=1}^m \cdots \sum_{j_{p-1}=1}^m u_{j_1}^m(t)\, \overline{u_{j_2}^m(t)} \cdots \overline{u_{j_{p-2}}^m(t)}\, u_{j_{p-1}}^m(t) \Gamma^n_{j_1,\bar{j}_2,\cdots,\bar{j}_{p-2},j_{p-1}}\,,
\end{aligned}
\tag{1.32}
$$

for $n = 0, 1, \dots, m$. By Picard–Lindelöf theorem, there is a unique local in time u^m satisfying the system (1.29). In order to extend this solution for all $t \geq 0$, we show that

$$
\sum_{n=0}^m |\hat{u}_n^m(t)|^2 = \|u^m(t)\|_2^2
$$

is a priori bounded.

Rewrite (1.29) in the form

$$
i\frac{d}{dt}(u^m(t), w_n) = (\nabla u^m(t), \nabla w_n) + \left\langle |u^m(t)|^{p-2} u^m(t), w_n \right\rangle, \tag{1.33}
$$

and

$$
-i\frac{d}{dt}(w_n, u^m(t)) = (\nabla w_n, \nabla u^m(t)) + \left\langle w_n, |u^m(t)|^{p-2} u^m(t) \right\rangle, \tag{1.34}
$$

take the difference, multiply by $\langle u^m(t), w_n \rangle$ and sum over $n = 0, 1, \dots, m$ to obtain

$$
\frac{d}{dt}\|u^m(t)\|_2^2 = 0\,. \tag{1.35}
$$

Hence, the Galerkin approximations satisfy the conservation of mass, i.e.,

$$
\|u^m(t)\|_2 = \|\phi^m\|_2\,, \qquad \forall t \geq 0\,, \tag{1.36}
$$

and

$$
\|u^m(t)\|_2 \leq \|\phi\|_2\,, \qquad \forall t \geq 0\,. \tag{1.37}
$$

This proves that the solution to (1.29) can be extended to a global in time solution. Since u^m is of the form

$$u^m(t,x) = \sum_{n=0}^{m} \hat{u}_n^m(t) w_n(x)\,,$$

with $\hat{u}_n^m(t) = (u^m(t), w_n)$, then $u^m(t,\cdot)$ is a $C^\infty(M)$ function. Thus, the functions $\nabla u^m(t)$ and $\Delta u^m(t)$ exist in the classical sense. For $n = 0, 1, \ldots, m$, the following integral equation is satisfied by the Galerkin approximations,

$$(u^m(t), w_n) = (\phi^m, w_n) - i\int_0^t (\nabla u^m(s), \nabla w_n) + \Big\langle |u^m(s)|^{p-2} u^m(s), w_n \Big\rangle ds\,. \tag{1.38}$$

Remark 1.2.1. *For future reference, for $m > 0$, denote by S_t^m the map $V \to V$ such that the path $t \mapsto S_t^m(a)$ is the unique solution of (1.29) starting at $P^m(a)$.*

In addition, sum (1.33) and (1.34), multiply by $\frac{d}{dt}\langle u^m(t), w_n\rangle$ and sum over $n = 0, 1, \ldots, m$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^m(t)\|_2^2 + \frac{1}{p}\frac{d}{dt}\|u^m(t)\|_p^p = 0\,. \tag{1.39}$$

Hence, the Galerkin approximations satisfy the conservation of energy, i.e.,

$$E[u^m(t)] = E[\phi^m]\,, \qquad \forall t \geq 0\,. \tag{1.40}$$

Let us summarize the bounds on the approximate solutions u^m. For each $m > 0$ the initial condition of the system (1.27) satisfies

$$\|\phi^m\|_2 \leq \|\phi\|_2\,, \quad \|\nabla\phi^m\|_2 \leq \|\nabla\phi\|_2 \quad \text{and} \quad \|\phi^m\|_{L^p} \leq C_p\|\phi\|_{L^p}\,, \tag{1.41}$$

where C_p is the constant defined in Lemma 1.1.4. In turn, this means that the energy of ϕ^m can be estimated as

$$E[\phi^m] \leq C_p^p E[\phi]\,. \tag{1.42}$$

As a result of the conservation of mass and energy, together with (1.41) and (1.42), we have the following:

- The family $(u^m)_{m\geq 0}$ is uniformly bounded in L^2, i.e., for any $t > 0$,

$$\|u^m(t)\|_2 = \|\phi^m\| \leq \|\phi\|_2 . \tag{1.43}$$

- The family $(\nabla u^m)_{m\geq 0}$ is uniformly bounded in L^2, i.e., for any $t > 0$,

$$\|\nabla u^m(t)\|_2 \leq \sqrt{2E[u^m(t)]} = \sqrt{2E[\phi^m]} \lesssim_p E[\phi]^{\frac{1}{2}} , \tag{1.44}$$

  with constant $2^{\frac{1}{2}} C_p^{\frac{p}{2}}$.

- The family $(u^m)_{m\geq 0}$ is uniformly bounded in L^p, i.e., for any $t > 0$,

$$\|u^m(t)\|_p \leq \sqrt[p]{pE[u^m(t)]} = \sqrt[p]{pE[\phi]} \lesssim_p E[\phi]^{\frac{1}{p}} , \tag{1.45}$$

  with constant $p^{\frac{1}{p}} C_p$.

The last two can be combined to show that the family $(u_m)_{m\geq 0}$ is uniformly bounded in V, i.e., for any $t > 0$,

$$\|u^m(t)\|_V \leq C_p^{\frac{p}{2}} \exp(E[\phi]/e) . \tag{1.46}$$

The bounds described in (1.43), (1.44) and (1.45) give us a hint of what function spaces to consider, in particular,

- the family of functions $(u^m)_{m\geq 0}$ belongs to the space $L^\infty([0,\infty) \to L^2)$, i.e.,

$$\|u^m\|_{L^\infty([0,\infty)\to L^2)} = \operatorname*{ess\,sup}_{t\geq 0} \|u^m(t)\|_2 \leq \|\phi\|_{L^2} . \tag{1.47}$$

- the family of functions $(u^m)_{m\geq 0}$ belongs to the space $L^\infty([0,\infty)\to V)$, i.e.,

$$\|u^m\|_{L^\infty([0,\infty)\to V)} = \operatorname*{ess\,sup}_{t\geq 0} \|u^m(t)\|_V \leq C_P^{\frac{p}{2}} \exp(E[\phi]/e)\,. \tag{1.48}$$

Limiting Argument For the limiting argument we use the space $L^1([0,\infty)\to V^*)$ and its dual, the space $L^\infty([0,\infty)\to V)$. This setting allow us to use the Banach-Alaoglu theorem, which helps to tame the nonlinearity.

Lemma 1.2.2. *Given a bounded sequence u^m in $L^\infty([0,\infty)\to V)$, we can extract a subsequence u^{m_k} and find $u\in L^\infty([0,\infty)\to V)$ such that*

(a) $u^{m_k} \overset{\star}{\rightharpoonup} u \in L^\infty([0,\infty)\to V)$.

(b) $u^{m_k} \to u \in L^2(M_T)$, *for any* $T>0$, *i.e., in* $L^2_{loc}(M_\infty)$.

(c) $u^{m_k}(t,x)\to u(t,x)$ *for a.e.* $(t,x)\in M_\infty$.

Proof. By means of the Banach-Alaoglu Theorem (see [6]) we can extract a subsequence u^{m_k} and find $u\in L^\infty([0,\infty)\to V)$ such that

$$u^{m_k} \overset{\star}{\rightharpoonup} u$$

in $L^\infty([0,\infty)\to V)$. Since $V=\dot{H}^1\cap L^p$, the weak-$\star$ convergence of u^{m_k} means that

$$u^{m_k} \rightharpoonup u$$

in $L^2([0,T]\to H^1)$, for any $T>0$. Due to Lemma 1.1.3 we conclude that $u^{m_k}\to u$ strongly in the space $L^2(M_T)$, for any $T>0$, i.e., the sequence $(u^{m_k})_{k>0}$ satisfies **(a)** and **(b)**.

Let $T=1$, from the strong convergence of $u^{m_k}\to u$ in the space $L^2(M_T)$ we can extract a subsequence that converges almost everywhere in M_T. Extract further subsequences with $T=2,3,\ldots$ and so on.

Consider the subsequence in the diagonal, and recall it u^{m_k}. For any M_T with $T = 1, 2, \ldots$, the sequence $(u^{m_k})_{k>0}$ converges almost everywhere in M_T, i.e., for each T there exists a null set $m_T \subset M_T$, such that $u^{m_k}(t, x) \to u(t, x)$ for $(t, x) \in M_T \setminus m_T$. The countable union of null-sets is also a null set, hence, the sequence $(u^{m_k})_{k>0}$ satisfies **(a)**, **(b)** and **(c)**.

□

The limiting argument is summarized in the following.

Lemma 1.2.3. *For any $\phi \in V$ there is a path $u : [0, \infty) \to V$, such that*

$$u \in L^\infty([0, \infty) \to V) \cap C([0, \infty) \to L^2)\,,$$

that satisfies

$$(u(t), v) = (\phi, v) - i \int_0^t (\nabla u(s), \nabla v) + \left\langle |u(s)|^{p-2} u(s), v \right\rangle ds\,, \quad \forall v \in V\,, \forall t \geq 0\,. \tag{1.49}$$

Proof. Thanks to the bound described in (1.48) there is a subsequence u^{m_k} and $u \in L^\infty([0, \infty) \to V)$ satisfying **(a)**, **(b)** and **(c)** of Lemma 1.2.2.

For $T > 0$, take any $\eta \in L^\infty([0, T] \to \mathbb{R})$, and $w_n \in W$. Define

$$w(s, x) = \eta(s)\, w_n(x)\,,$$

a function in $L^1([0, \infty) \to V)$. Since $u^{m_k} \to u$ a.e. in M_T, we apply Egorov's theorem. For any $\varepsilon > 0$, there is a set $m_T \subset M_T$, such that, $|m_T| < \varepsilon$ and $u^{m_k} \to u$ uniformly on $M_T \setminus m_T$. We have

$$\begin{aligned} \int_{m_T} |u^{m_k}(s, x)|^{p-1} |w(s, x)|\, dx\, ds &\leq \left(\int_{m_T} |u^{m_k}(s, x)|^p\, dx\, ds \right)^{\frac{p-1}{p}} \left(\int_{m_T} |\eta(s) w_n(x)|^p\, dx\, ds \right)^{\frac{1}{p}} \\ &\leq T^{\frac{p-1}{p}} \|u^{m_k}\|^{p-1}_{L^\infty([0,\infty) \to V)} \left(\int_{m_T} |\eta(s) w_n(x)|^p\, dx\, ds \right)^{\frac{1}{p}}\,. \end{aligned}$$

Since $\eta \in L^\infty([0,T] \to \mathbb{R})$ and $w_n \in C^\infty(M)$,

$$\int_{m_T} |\eta(s) w_n(x)|^p \, dx \, ds \to 0 \,, \qquad \text{as} \qquad |m_T| \to 0 \,.$$

At the same time, $\|u^{m_k}\|_{L^\infty([0,\infty)\to V)}$ is uniformly bounded, then,

$$\int_{m_T} |u^{m_k}(s,x)|^{p-1} |w(s,x)| \, dx \, ds \to 0 \,, \qquad \text{as} \qquad |m_T| \to 0 \,.$$

The function u enjoys the same limit. By the uniform convergence on $M_T \setminus m_T$, we have, as $k \to \infty$,

$$\int_0^\infty \int_M |u^{m_k}(s,x)|^{p-2} u^{m_k}(s,x) w(s,x) ds \, dx \to \int_0^\infty \int_M |u(s,x)|^{p-2} u(s,x) w(s,x) ds \, dx \,. \tag{1.50}$$

Consider now,

$$\int_0^\infty \int_M \nabla u^{m_k}(t,x) \nabla w(t,x) dt dx \,, \tag{1.51}$$

since $u^{m_k} \overset{\star}{\rightharpoonup} u \in L^\infty([0,\infty) \to V)$ we have $\nabla u^{m_k} \overset{\star}{\rightharpoonup} \nabla u \in L^\infty([0,\infty) \to L^2)$, hence

$$\int_0^\infty \int_M \nabla u^{m_k}(t,x) \nabla w(t,x) dt dx \to \int_0^\infty \int_M \nabla u(t,x) \nabla w(t,x) dt dx \,. \tag{1.52}$$

For $t > 0$, let $\eta(s) = 1_{[0,t]}(s)$, the indicator function of the interval $[0,t]$. Fix $n > 0$, with $w(s,x) = 1_{[0,t]}(s) w_n(x)$ we have

$$\int_0^t \Big\langle |u^{m_k}(s)|^{p-2} u^{m_k}(s), w_n \Big\rangle \, ds \to \int_0^t \Big\langle |u(s)|^{p-2} u(s), w_n \Big\rangle \, ds \,,$$

and

$$\int_0^t (\nabla u^{m_k}(s), \nabla w_n) \, ds \to \int_0^t (\nabla u(s), \nabla w_n) \, ds \,.$$

For all $m_k > n$, the function u^{m_k} satisfies (1.38), i.e.

$$(u^{m_k}(t), w_n) = (\phi^{m_k}, w_n) - i\int_0^t (\nabla u^{m_k}(s), \nabla w_n) + \Big\langle |u^{m_k}(s)|^{p-2} u^{m_k}(s), w_n \Big\rangle ds\,. \quad (1.53)$$

In the limit we obtain, for a.e. $t \in [0, T]$

$$(u(t), w_n) = (\phi, w_n) - i\int_0^t (\nabla u(s), \nabla w_n) + \Big\langle |u(s)|^{p-2} u(s), w_n \Big\rangle ds\,, \quad (1.54)$$

where the right hand side is an absolutely continuous function, and the left hand side is only defined almost everywhere in time because we use the convergence obtained in **(c)** of Lemma 1.2.2.

We can improve (1.54), for any $v \in V$, let $v^m = P^m v$, i.e.,

$$v^m = \sum_{n=0}^{m} (v, w_n) w_n\,.$$

For any n, multiply (1.54) by (v, w_n) and sum over $n = 0, \ldots, m$, to obtain, for almost every $t \geq 0$,

$$(u(t), v^m) = (\phi, v^m) - i\int_0^t (\nabla u(s), \nabla v^m) + \Big\langle |u(s)|^{p-2} u(s), v^m \Big\rangle ds\,. \quad (1.55)$$

Recall that $v^m \to v$, strongly in V, that includes $v^m \to v$ in L^2, L^p and $\nabla v^m \to \nabla v$ in L^2, hence we can pass to the limit in (1.55) to obtain, for any $v \in V$,

$$(u(t), v) = (\phi, v) - i\int_0^t (\nabla u(s), \nabla v) + \Big\langle |u(s)|^{p-2} u(s), v \Big\rangle ds\,. \quad (1.56)$$

In view of Lemma 1.1.5 the map $t \mapsto u(t)$ has a weak derivative with respect to t, taking values in V^*. Hence, we can apply Lemma 1.1.6 to conclude that $u \in C([0, \infty) \to L^2)$. Again, this just means that, for almost every $t \in [0, T]$, u is equal to a continuous function $[0,T] \to L^2$, nevertheless, we can always construct the representative element that is continuous, and we will assume u is that one.

□

Conservation Laws We turn now to the study of the conservation laws satisfied by the path constructed in Lemma 1.2.3. In particular, we are interested in the conservation laws enjoyed by the Galerkin approximations, i.e., the conservation of mass (1.36) and energy (1.40).

For any $k > 0$, the function u^{m_k} satisfies the conservation of mass (1.36) and energy (1.40). From the convergence obtained in **(c)** of Lemma 1.2.2 we have that, for almost every $t \geq 0$,

$$\|u^{m_k}(t) - u(t)\|_2 \to 0 \quad \text{as} \quad k \to \infty\,.$$

Since $\phi^{m_k} \to \phi$ in L^2, for almost every $t \geq 0$, we have

$$\|u^{m_k}(t)\| = \|\phi^{m_k}\| \to \|u(t)\|_2 = \|\phi\| \quad \text{as} \quad k \to \infty\,.$$

Finally, using the fact that $u \in C([0,\infty) \to L^2)$ we have, for every $t \geq 0$,

$$\|u(t)\|_2 = \|\phi\|_2\,. \tag{1.57}$$

Regarding the conservation of energy, we have that for n, thanks to (1.54), the map $t \mapsto (u^{m_k}(t), w_n)$ is absolutely continuous, for all $m_k > n$. Moreover, because of the convergence in **(c)** of Lemma 1.2.2 we have $(u^{m_k}(t), w_n) \to (u(t), w_n)$ for almost every $t \geq 0$. Notice that for any $m_k > n$,

$$
\begin{aligned}
|(u^{m_k}(t+\Delta t) - u^{m_k}(t), w_n)| &\leq \int_t^{t+\Delta t} \Big| (\nabla u^{m_k}(s), \nabla w_n) + \langle |u^{m_k}(s)|^{p-2} u^{m_k}(s), w_n \rangle \Big| ds \\
&\leq \|w_n\|_V \int_t^{t+\Delta t} \|\nabla u^{m_k}(s)\|_2 + \|u^{m_k}(s)\|_p^{p-1} ds \\
&\leq \|w_n\|_V \left((2E[\phi])^{\frac{1}{2}} + (pE[\phi])^{\frac{p-1}{p}} \right) \Delta t \,.
\end{aligned}
$$

In particular, for fixed n, the family $((u^{m_k}(t), w_n))_{m_k > n}$ is equicontinuous. Together with the continuity of $t \mapsto (u(t), w_n)$, this ensures that the almost everywhere convergence of

$$
(u^{m_k}(t), w_n) \to (u(t), w_n) \,,
$$

can be improved to convergence for all $t \geq 0$.

For a fixed t, the family of functions $u^{m_k}(t)$ is bounded in the norm of V. Hence, on a subsequence we have weak convergence, and thanks to the convergence $(u^{m_k}(t), w_n) \to (u(t), w_n)$ for any $w_n \in W$, we know that the limit is $u(t)$. Due to the weak lower semi-continuity of the norm we have

$$
\|u(t)\|_p \leq \liminf_{k \to \infty} \|u^{m_k}(t)\|_p \,, \tag{1.58}
$$

and

$$
\|\nabla u(t)\|_2 \leq \liminf_{k \to \infty} \|\nabla u^{m_k}(t)\|_2 \,. \tag{1.59}
$$

As a result,

$$
E[u(t)] \leq \liminf_{k \to \infty} E[\phi^{m_k}] = E[\phi] \,. \tag{1.60}
$$

The absence of strong convergence does not allow us to establish a full conservation law, we replace it with the energy inequality.

Weak Continuity with Respect to V We still can improve the regularity of the solutions, in particular we want to show that the solution constructed in the previous paragraphs is weakly continuous from $[0, T]$ to V, i.e., for any $w \in V^*$, the map

$$t \mapsto \langle u(t), w \rangle$$

is continuous. In what follows we will denote by V_w the space V endowed with the weak topology.

In what follows we will use the space V_R defined as,

Definition 1.2.3. For $R > 0$, define

$$V_R = \{\, \xi \in V \mid E[\xi] \leq R \,\} \,.$$

As a subset of V, V_R sits inside a ball of radius $C_R = \max\{\, \sqrt{2R}, \sqrt[p]{pR}\,\}$. Moreover, since V_R is convex and closed in the strong topology of V, by Mazur's Lemma, it is closed in the weak topology of V.

The closed ball of radius C_R, denoted by $B_{C_R} \subset V$, is compact in the weak topology thanks to the Banach-Alaoglu theorem. Moreover, since V is separable, the weak topology in B_{C_R} is metrizable (see [25, Theorem 2.6.23].) We will construct a metric that induces the weak topology in V_R as follows:

- Fix a sequence $(h_k)_{k \geq 0}$, dense in B_{V^*}, the unit ball inside V^*, such that $(h_k) \subset V$ as well.

- Define, for any $v, w \in V_R$,

$$d_R(v, w) = \sum_{k=0}^{\infty} 2^{-k} |(v - w, h_k)| \,.$$

In this form, V_R turns into a compact metric space, and the topology induced by the metric is the weak topology innherited from V.

Proposition 1.2.4. *Given $\phi \in V$, the path constructed in Lemma 1.2.3 is weakly continuous with respect to V.*

Proof. Let $R > 0$ be such that $E[\phi] < R$. From the energy inequality we have that $u(t)$ remains in V_R for all $t \geq 0$. Moreover, u satisfies (1.56). In particular, for any $w \in V$ and Δt we have

$$|(u(t+\Delta t) - u(t), w)| \leq \int_t^{t+\Delta t} |(\nabla u(s), \nabla w)| + \left|\left\langle |u(s)|^{p-2} u(s), w \right\rangle\right| ds \tag{1.61}$$

let $A_R = ((2R)^{\frac{1}{2}} + (pR)^{\frac{p-1}{p}})$ then

$$|(u(t+\Delta t) - u(t), w)| \leq (\|w\|_V A_R)\ \Delta t\,. \tag{1.62}$$

Pick any $\varepsilon > 0$. We want to show that for sufficiently small Δt we have

$$d_R(u(t+\Delta t), u(t)) < \varepsilon\,,$$

which means continuity with respect to the topology induced by d_R which is the weak topology in V.

Pick N such that $C_R 2^{-(N-1)} < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} d_R(u(t+\Delta t), u(t)) &< \sum_{k=0}^{N} 2^{-k} \Delta t \|h_k\|_V A_R + C_R \sum_{k=N+1}^{\infty} 2^{-(k-1)} \\ &< \Delta t A_R \sum_{k=0}^{N} 2^{-k} \|h_k\|_V + 2^{-(N-1)} C_R\,. \end{aligned} \tag{1.63}$$

Since N is now fixed, we can pick Δt such that $\Delta t A_R \sum_{k=0}^{N} 2^{-k} \|h_k\|_V < \frac{\varepsilon}{2}$ to conclude that

$$d_R(u(t+\Delta t), u(t)) < \varepsilon \tag{1.64}$$

for sufficiently small Δt. □

We get to our first useful definition of weak solution of (1.10). For reasons that will be clear in the following chapter we will use two different types of solutions.

Definition 1.2.4 (Weak solution with the energy inequality)**.** For any $\phi \in V$, a function $u : [0, \infty) \to V$ will be called a weak solution of (1.10) satisfying the energy inequality with the initial condition ϕ, if it satisfies the following:

1. The function u belongs to $C([0, \infty) \to L^2) \cap L^\infty([0, \infty) \to V)$.

2. The function u belongs to $C([0, \infty) \to V_w)$, i.e., u is weakly continuous in V.

3. For any $\eta \in V$ and $t \geq 0$,

$$(u(t), \eta) = (\phi, \eta) - i \int_0^t (\nabla u(s), \nabla \eta) + \left\langle |u(s)|^{p-2} u(s), \eta \right\rangle ds \,. \tag{1.65}$$

4. The mass of u is conserved, i.e.,

$$\|u(t)\|_2 = \|\phi\|_2 \quad \forall t \geq 0 \,. \tag{1.66}$$

5. The energy of u satisfies the following inequality, i.e.,

$$E[u(t)] \leq E[\phi] \quad \forall t \geq 0 \,. \tag{1.67}$$

Definition 1.2.5 (Weak solution with bounded energy)**.** Given $R > 0$, for any $\phi \in V_R$, a function $u : [0, \infty) \to V_R$ will be called a weak solution of (1.10) with bounded energy and initial condition ϕ, if it satisfies the following:

1. The function u belongs to $C([0, \infty) \to L^2)$.

2. The function u belongs to $C([0, \infty) \to V_w)$.

3. For any $\eta \in V$ and $t \geq 0$,

$$(u(t), \eta) = (\phi, \eta) - i \int_0^t (\nabla u(s), \nabla \eta) + \left\langle |u(s)|^{p-2} u(s), \eta \right\rangle ds \,. \tag{1.68}$$

4. The mass of u is conserved, i.e.,

$$\|u(t)\|_2 = \|\phi\|_2 \quad \forall t \geq 0 \,. \tag{1.69}$$

Notice that a weak solution that satisfies the energy inequality is also a weak solution with bounded energy, but the opposite is not true.

Summarizing we obtain the following existence theorem:

Theorem 1.2.5. *Given $\phi \in V$, there exists at least one weak solution with the energy inequality with the initial condition ϕ. If $\phi \in V_R$, there exists at least one weak solution with the energy bounded by R with the initial condition ϕ.*

In the case when $p = 4$, i.e., the cubic NLS the existence, uniqueness and continuity of data has been extensively studied by J. Bourgain [5], J. Ginibre [14], and others, a more extensive bibliography can be found in [26].

1.3 Set-Valued Maps of Weak Solutions

From now on we assume that $d \geq 3$ and $p > 2$ is such that uniqueness of weak solutions is not known. In this section we discuss the properties of the sets of weak solutions. These results will be used in the study of statistical solutions and semiflow selections.

Let Ω be the set of weakly continuous paths from $[0, \infty)$ to V, endowed with the compact-open topology. Given $R > 0$, the set of paths in Ω taking values in V_R is denoted by Ω_R and inherits the topology from Ω.

Definition 1.3.1. For any $a \in V$, define $S(a) \subset \Omega$ as the set of all the weak solutions of NLS with the energy inequality with the initial condition a.

Definition 1.3.2. Given $R > 0$, and $a \in V_R \subset \Omega_R$, let $S_R(a)$ denote the set of all weak solutions of NLS with energy bounded by R with the initial condition a.

Lemma 1.3.1. *Let (u_n) be a sequence in Ω such that $u_n \to u \in \Omega$ as $n \to \infty$ in the topology of Ω. If (u_n) is uniformly bounded in $L^\infty([0,\infty) \to V)$, then $u_n \overset{\star}{\rightharpoonup} u$ in $L^\infty([0,\infty) \to V)$, $u^n \to u$ in $L^2_{loc}([0,\infty) \to L^2)$, and $u_n(t) \to u(t)$ strongly in L^2. Moreover, (u_n) admits a subsequence that converges almost everywhere in M_∞.*

Proof. Notice that the convergence in the compact-open topology of Ω means in particular that for any $t \geq 0$ the sequence $u_n(t)$ converges weakly to $u(t)$ as $n \to \infty$.

Since (u_n) is uniformly bounded in $L^\infty([0,\infty) \to V)$, there is a constant C such that $\|u_n(t)\|_V \leq C$.

Let φ be any element in $L^1([0,\infty) \to V^*)$. Consider the pairing $\langle u_n(t), \varphi(t)\rangle$, for which we have

$$|\langle u_n(t), \varphi(t)\rangle| \leq C\|\varphi(t)\|_{V^*}, \quad \text{a.e. } t \in [0,\infty). \tag{1.70}$$

In the other hand, since $u_n(t) \rightharpoonup u(t)$ for any t, we have

$$\langle u_n(t), \varphi(t)\rangle \to \langle u(t), \varphi(t)\rangle, \quad \text{a.e. } t \in [0,\infty). \tag{1.71}$$

Since the map $t \mapsto \|\varphi(t)\|_{V^*}$ is integrable on $[0,\infty)$, due to Lebesgue's dominated convergence theorem we have

$$\int_0^\infty \langle u_n(t), \varphi(t)\rangle dt \to \int_0^\infty \langle u(t), \varphi(t)\rangle dt \quad \text{as } n \to \infty. \tag{1.72}$$

Hence, $u_n \overset{\star}{\rightharpoonup} u$ in $L^\infty([0,\infty) \to V)$. Using Friedrichs's Lemma 1.1.3 we can obtain that $u_n \to u$ in $L^2_{\text{loc}}([0,\infty) \to L^2)$.

For each $t \geq 0$ we already have that $u_n(t) \rightharpoonup u(t)$ in V, using Rellich's lemma then implies $u_n(t) \to u(t)$ in L^2.

Finally, let $T = 1$, since $u_n \to u$ in $L^2(M_1)$ we can extract a subsequence converging almost everywhere to u in M_1. Extract further subsequences letting $T = 2, 3, \ldots$. The diagonal sequence then converges almost everywhere in M_∞.

□

We can now show that both sets $S(a)$ and $S_R(a)$ are compact in Ω.

Proposition 1.3.2. *For any $a \in V$, the sets $S(a)$ and $S_R(a)$ are compact subset of Ω. Moreover, given any weakly compact set $K \subset V_R$ the set*

$$S_R(K) = \bigcup_{a \in K} S_R(a)$$

is compact as well.

Proof. Consider first the case for $S(a)$. Since any $u \in S(a)$ satisfies the energy inequality (1.67), the set $S(a)$ is a subset of Ω_R, with $R = E[a]$. Also, u satisfies the integral equation (1.65), and the constants associated with the inequality (1.63) depend only on $E[a]$, hence, the set $S(a)$ is equicontinuous. Since V_R is a compact metric space, and the set $S(a)$ is equicontinuous, we can apply the Arzelà–Ascoli Theorem. Hence, $S(a)$ is relatively compact inside Ω_R, thus in Ω.

It remains to show that $S(a)$ is closed in the topology of Ω. Assume $(u_n)_{n>0}$ is a sequence of paths in $S(a)$ converging to u in the topology of Ω. By means of Lemma 1.3.1 we have that $u_n \to u$ weakly-$\star$ in $L^\infty([0, \infty) \to V)$ and strongly in $L^2_{\text{loc}}([0, \infty) \to L^2)$. For any $t \geq 0$, $u_n(t) \to u(t)$ weakly in V and strongly in L^2, and (u_n) admits a subsequence (u_{n_k}) that converges almost everywhere in M_∞. Thanks to the almost everywhere convergence of (u_{n_k}) we have the convergence described in (1.50) and (1.52). Hence, in the limit as $k \to \infty$, the path u satisfies (1.65). Since, for any $t \geq 0$, $u_n(t) \to u(t)$ strongly in L^2, and weakly in V, the convergence of mass and the energy inequality are preserved in the limit. Hence, u is a weak solution with the energy inequality starting at a, then $S(a)$ is closed.

For $S_R(a)$ the proof is similar. Any element $u \in S_R(a)$ satisfies the integral equation (1.68). Again, the constants associated with the inequality (1.63) depend only on R, hence, the set $S_R(a)$ is equicontinuous. As before, $S_R(a)$ is relatively compact inside Ω_R, thus in Ω. Assume $(u_n)_{n>0}$ is a sequence of paths in $S_R(a)$ converging to u in the topology of Ω. By means of Lemma 1.3.1 we have that $u_n \to u$ weakly-$\star$ in $L^\infty([0,\infty) \to V)$ and strongly in $L^2_{\text{loc}}([0,\infty) \to L^2)$. For any $t \geq 0$, $u_n(t) \to u(t)$ weakly in V_R and strongly in L^2, and (u_n) admits a subsequence (u_{n_k}) that converges almost everywhere in M_∞. Thanks to the almost everywhere convergence of (u_{n_k}) we have the convergence described in (1.50) and (1.52). Hence, in the limit as $k \to \infty$, the path u satisfies (1.68). Since, for any $t \geq 0$, $u_n(t) \to u(t)$ strongly in L^2, and weakly in V, the convergence of mass and the energy bound are preserved in the limit. Hence, u is a weak solution with the energy bounded by R starting at a, then $S_R(a)$ is closed.

To prove the second statement, notice that $S_R(K)$ is again equicontinuous. The constants of (1.63) depend only on R, thus $S_R(K)$ is relatively compact. If $u_n \to u$ in Ω, then $u_n(0) \rightharpoonup u(0)$ weakly in V, but this does not affect the limit to the integral equation (1.68).

$\square$

Chapter 2

Selection with Semigroup Property

Let X be a set, and let Ω be the space of all one-sided infinite paths in X, i.e., the maps $w : [0, +\infty) \to X$. Let $S = (S(a))_{a \in X}$ be a family of subsets of Ω such that each $S(a)$ is a subset of $\Omega_a = \{w \in \Omega : \ w(0) = a\}$. Think of $S(a)$ as of Kneser's integral funnel, the set of all possible solutions of some differential equation with the same initial condition a at time $t = 0$. A *selection* of the family S is a map $u : [0, +\infty) \times X \to X$ such that $u(\cdot, a) \in S(a)$ for every $a \in X$ (a selection picks a solution for every initial condition). We say that the selection u has the *semigroup property*, or that u is a *semiflow selection*, if

$$u(t_2, u(t_1, a)) = u(t_2 + t_1, a), \quad \forall t_1, t_2 \geq 0 \ \ \forall a \in X \, .$$

The main result in this chapter is Theorem 2.2.2. This theorem guarantees the existence of measurable selections with the semigroup property. This is a new selection theorem for set-valued maps. The theorem is used later to obtain measurable semiflows for the NLS.

2.1 Notation and Basic Concepts

2.1.1 Space of Continuous Paths

Let X be a *Polish* space, i.e., a complete separable metrizable space. Endow X with a metric ρ. Then, (X,ρ) is a complete separable metric space. The space of continuous paths in X, i.e., continuous maps from $[0,\infty)$ to X, is denoted by Ω. We endow Ω with a metric. Define the following family of pseudometrics:

$$d_T(u,v) = \sup_{t\in[0,T]} \rho(u(t),v(t))\,, \qquad T>0\,. \tag{2.1}$$

The metric on Ω is then

$$d(u,v) = \sum_{k=0}^{\infty} 2^{-k}\,\frac{d_k(u,v)}{1+d_k(u,v)}\,. \tag{2.2}$$

This metric induces the compact-open topology. With this metric the space Ω is again a complete separable metric space, see [19, Example A.10]. For any $a\in X$, the set of paths in Ω starting at a is denoted by Ω_a.

The Borel σ-algebra generated by the topology of X is denoted by $\mathcal{B}$ and the Borel σ-algebra of Ω is denoted by $\mathcal{F}$.

For elements in the space Ω we define the following operations of splicing and shifting of paths.

Definition 2.1.1 (Splicing of paths)**.** Given a path $w\in\Omega$, a time $t\geq 0$, and $v\in\Omega$ with $v(0)=w(t)$ define $w\lhd_t v$ as

$$(w\lhd_t v)(s) = \begin{cases} w(s) & s<t \\ v(s-t) & s\geq t \end{cases} \qquad \forall s\geq 0 \tag{2.3}$$

Notice that $w\lhd_t v\in\Omega$.

Definition 2.1.2. For $t \geq 0$, denote the usual shift on paths by $\theta_t : \Omega \to \Omega$, defined as

$$\theta_t(w) = w(t + \cdot) \tag{2.4}$$

2.1.2 Set-valued Maps and Hyperspaces

We give now some basic facts on set-valued maps and their properties. In what follows $(A, \mathcal{A})$ is a measurable space, i.e., A is a set, and $\mathcal{A}$ is a σ-algebra of subsets of A.

Definition 2.1.3 (Hyperspace)**.** For a Polish space X, the set of all of its non-empty compact subsets is called the *hyperspace of all of its compact subsets*, and is denoted by $\mathcal{H}_k(X)$. If X is also a vector space, or a subset of a vector space, the *hyperspace of all of its compact and convex subsets* is denoted by $\mathcal{H}_{ck}(X)$.

In cases where we don't need to specify which one we are using we just refer to a *hyperspace* as to a set of subsets of the space in question, in this case X, and we denote it by $\mathcal{H}(X)$.

Given a subset $K \subset X$ the following two sets of subsets are of great importance:

$$K^+ = \{ C \in \mathcal{H}(X) \,|\, C \subset K \} \ , \tag{2.5}$$

and

$$K^- = \{ C \in \mathcal{H}(X) \,|\, C \cap K \neq \varnothing \} \ . \tag{2.6}$$

Definition 2.1.4 (Vietoris topology)**.** The Vietoris topology on $\mathcal{H}(X)$ is generated by all the sets of the form A^+ and A^- with $A \subset X$ open.

One of the usual metrics defined on set of subsets is the Hausdorff metric. This is a standard construction, see [18, Definition 1.1.].

Definition 2.1.5 (Hausdorff metric)**.** If (X, ρ) is a metric space, the Hausdorff metric is defined as follows. For any $A, C \subset X$, let

$$h^*(A, C) = \sup_{a \in A} \rho(a, C)\,,$$

and

$$h^*(C, A) = \sup_{c \in C} \rho(c, A)\,.$$

Finally, the metric is

$$h(A, C) = \max\{h^*(A, C)\,,\ h^*(C, A)\}\,.$$

The following is an important result concerning $\mathcal{H}_k(X)$. See Chapter 1: Proposition 1.6, Theorem 1.30 and Corollary 1.26 in [18]

Proposition 2.1.1. *If (X, ρ) is a complete separable metric space, then, the Hausdorff metric topology and the Vietoris topology coincide, and $(\mathcal{H}_k(X), h)$ is a complete separable metric space.*

The Borel σ-algebra generated by these topologies allow us to study measurable set-valued maps. The Borel σ-algebra generated by the Vietoris (or Hausdorff) topology on $\mathcal{H}_k(X)$ is denoted by $\mathcal{N}$.

In what follows, we consider set-valued maps of the form $F : a \mapsto F(a) \subset X$, where $F(a)$ is compact (and convex) in X. It is convenient to view such maps as single-valued with values in the hyperspace $\mathcal{H}_k(X)$ (resp. $\mathcal{H}_{ck}(X)$).

In the hyperspace $\mathcal{H}_k(X)$, the definition of a measurable map reduces to the following, see Chapter 2: Proposition 1.10 in [18].

Definition 2.1.6 (Measurable set-valued map)**.** A map F from $(A, \mathcal{A})$ to $(\mathcal{H}_k(X), \mathcal{N})$, i.e., $F : A \to \mathcal{H}_k(X)$, is called measurable if

$$\{\, a \in A \,|\, F(a) \cap C \neq \varnothing \,\} \in \mathcal{A}\,, \qquad \forall \text{closed } C \subset X\,.$$

Definition 2.1.7 (Selections)**.** Given a set-valued map $F : a \mapsto F(a)$, a selection of F is a single-valued function $f : A \to X$ such that

$$f(a) \in F(a) \qquad \forall a \in A\,. \tag{2.7}$$

If such a function exists we say that F admits a selection.

One of the most important selection theorems is due to K. Kuratowski and C. Ryll-Nardzewski [22]. We restrict ourselves to the hyperspace $\mathcal{H}_k(X)$, but the theorem also applies in the case of the hyperspace of closed sets.

Theorem 2.1.2. *If $(A, \mathcal{A})$ is a measurable space, and if X a Polish space, any measurable set-valued map $F : A \to \mathcal{H}_k(X)$ admits a measurable selection.*

In the proof of our selection theorem, we need certain construction related to maximizers of continuous functions.

Definition 2.1.8 (Set of maximizers)**.** For a compact set K and a real-valued continuous function ξ on K, denote by $V_\xi[K]$ the set

$$V_\xi[K] = \left\{ \alpha \in K \,\middle|\, \xi(\alpha) = \max_{\beta \in K} \xi(\beta) \right\}. \tag{2.8}$$

We will refer to $V_\xi[K]$ as the set of maximizers. It is clear that $V_\xi[K]$ is a compact set.

Given a set-valued map $F : A \to \mathcal{H}_k(X)$ and a continuous function ξ on X, consider a new set-valued map

$$a \mapsto V_\xi[F(a)].$$

Abusing notation, we will denote this map by $V_\xi[F]$.

The following is an useful result, see Parthasarathy [27, Chapter 4, Lemma 4.3]

Lemma 2.1.3. *Assume X is compact. Given a continuous function ξ on X, if the set-valued map $F : A \to \mathcal{H}_k(X)$ is measurable then $V_\xi[F]$ is also measurable.*

Recall the definition of upper semi-continuous set-valued functions, see [18, Remark 2.4].

Definition 2.1.9. Let F be a set-valued map $X \to \mathcal{H}(Y)$. F is upper semi-continuous (u.s.c.) at $a \in X$ if for every neighborhood N_Y of $F(a) \subset Y$, there exists a neighborhood N_X of a such that

$$F(N_X) = \bigcup_{x \in N_X} F(x) \subset N_Y .$$

F is u.s.c. if it is u.s.c. at every point $a \in X$.

The following is an important result concerning u.s.c functions, see [18, Corollary 2.20]

Proposition 2.1.4. *Assume X and Y are complete separable metric spaces. If $F : X \to \mathcal{H}_k(Y)$ is u.s.c. and $K \subset X$ is compact, then*

$$F(K) = \bigcup_{a \in K} F(a)$$

is compact in Y.

As a consequence, we have.

Corollary. *Assume X and Y are complete separable metric spaces equipped with their Borel σ-algebras. Let $F : X \to \mathcal{H}_k(Y)$ be u.s.c. Then, F is measurable.*

Proof. For any subset $C \subset Y$, closed in the topology of Y, let

$$\hat{C} = \{ a \in X \mid F(a) \cap C \neq \varnothing \} \ .$$

To show that F is measurable it is enough to show that $\hat{C}$ is a measurable set. We will show that $\hat{C}$ is closed in the topology of X.

Assume (a_n) is a sequence in $\hat{C}$, and $a_n \to a$ as $n \to \infty$ in X. For each n we have $F(a_n) \cap C \neq \varnothing$, hence, we can pick $y_n \in F(a_n) \cap C$ for each n. Notice that the set $\{a\} \cup (a_n)$ is a compact set in X, hence $F(\{a\} \cup (a_n))$ is a compact subset of Y. Since (y_n) is a sequence in $F(\{a\} \cup (a_n))$, we can extract a convergent subsequence, hence, there is $y \in Y$ such that $y_{n_k} \to y$ in the topology of Y as $k \to \infty$. Now, for the subsequence y_{n_k} we have, $a_{n_k} \to a$ while $y_{n_k} \in F(a_{n_k})$ and $y_{n_k} \to y$. Since F is u.s.c. we conclude that $y \in F(a)$. Hence, the set $\hat{C}$ is closed.

□

2.2 Selection of Semiflows

Let X be a compact metric space. Consider a set-valued map $\mathcal{C} : X \to \mathcal{H}_k(\Omega)$. The following properties encode certain dynamical structure that will be exploited later.

Definition 2.2.1. For a set-valued map $\mathcal{C} : X \to \mathcal{H}_k(\Omega)$, consider the following properties:

S1 For any a the set $\mathcal{C}(a)$ is a compact subset of Ω_a.

S2 The map $\mathcal{C}$ is measurable, i.e., for any closed set $K \in \Omega$ the set $\{ a \in X \mid \mathcal{C}(a) \cap K \neq \varnothing \}$ is a Borel set in X.

S3 The map $\mathcal{C}$ is compatible with the shift operator, i.e.,

$$w \in \mathcal{C}(a) \implies \theta_t(w) \in \mathcal{C}(w(t)) \,, \quad \forall t \geq 0 \,, \forall a \in X \,. \tag{2.9}$$

S4 The map $\mathcal{C}$ is compatible with the splicing of paths i.e.,

$$w \in \mathcal{C}(a)\,, v \in \mathcal{C}(w(t)) \implies w \lhd_t v \in \mathcal{C}(a)\,, \quad \forall t \geq 0\,, \forall a \in X\,. \tag{2.10}$$

A selection of the map $\mathcal{C}$ is a map u from X to Ω, that maps a to a path $u(a)$ starting at x. This path is described by $t \mapsto u(a)(t) = u(t,a)$ (notice the change of order in the variables, we do this to keep some consistency with other sections of this thesis.)

Before stating the main theorem in this section, we present first an useful lemma.

Lemma 2.2.1. *Given a set-valued map $\mathcal{C} : X \to \mathcal{H}_k(\Omega)$, a number $\lambda > 0$ and a function $\varphi \in C_b(X)$, define*

$$\zeta(w) = \int_0^\infty e^{-\lambda s}\varphi(w(s))ds\,. \tag{2.11}$$

*If $\mathcal{C}$ satisfies properties **S1-S4**, then the set-valued map $V_\zeta[\mathcal{C}]$ satisfies the properties **S1-S4**.*

Proof. That properties **S1-S2** are satisfied is the result of Lemma 2.1.3 and the fact that the set of maximizers of a continuous function on a compact set is compact as well.

To establish **S3**, pick a path $w \in V_\zeta[\mathcal{C}(a)]$. We need to show that $\theta_t(w)$ is a maximizer in $\mathcal{C}(w(t))$ for any $t \geq 0$. Let v be any path in $\mathcal{C}(w(t))$. Then, we know that the spliced path $w \lhd_t v$ is in $\mathcal{C}(a)$. Since w is a maximizer of ζ in $\mathcal{C}(a)$, then $\zeta(w) \geq \zeta(w \lhd_t v)$, which implies

$$\int_t^\infty e^{-\lambda s}\varphi(w(s))ds \geq \int_t^\infty e^{-\lambda s}\varphi(v(s-t))ds = e^{-\lambda t}\int_0^\infty e^{-\lambda s}\varphi(v(s))ds = e^{-\lambda t}\zeta(v)\,.$$

Since

$$\zeta(\theta_t(w)) = \int_0^\infty e^{-\lambda s}\varphi(w(t+s))ds = e^{-\lambda t}\int_t^\infty e^{-\lambda s}\varphi(w(s))ds\,,$$

we have

$$\zeta(\theta_t(w)) \geq \zeta(v)\,.$$

This is true for all $v \in \mathcal{C}(w(t))$. Hence, $\theta_t(w)$ is a maximizer of ζ in $\mathcal{C}(w(t))$.

It remains to check property **S4** for $V_\zeta[\mathcal{C}]$. Let w be a maximizer in $\mathcal{C}(a)$ and let v be a maximizer in $\mathcal{C}(w(t))$. Consider the spliced path $w \lhd_t v$, then

$$\begin{aligned}
\zeta(w \lhd_t v) &= \int_0^t e^{-\lambda s}\varphi(w(s))ds + \int_t^\infty e^{-\lambda s}\varphi(v(s-t))ds \\
&= \int_0^t e^{\lambda s}\varphi(w(s))ds + e^{-\lambda t}\zeta(v) \\
&= \int_0^t e^{\lambda s}\varphi(w(s))ds + e^{-\lambda t}\zeta(\theta_t(w)) \\
&= \zeta(w)\,.
\end{aligned}$$

Since w is assumed to be a maximizer of ζ in $\mathcal{C}(a)$, the path $w \lhd_t v$ must be a maximizer as well. Note that $\zeta(v) = \zeta(\theta_t(w))$ because $\theta_t(w)$ is a maximizer as was shown above. □

The main result in this chapter is the existence of a selection with the semigroup property, i.e.,

$$u(t+s,a) = u(t,u(s,a))\,,\quad u(0,a) = a\,,\qquad \forall a \in X\,, \forall t,s \geq 0\,. \tag{2.12}$$

Theorem 2.2.2. *Given a set-valued map $\mathcal{C}$ taking values in subsets of Ω, if $\mathcal{C}$ satisfies properties **S1-4**, then, $\mathcal{C}$ admits a selection with the semigroup property (2.12)*

Proof. Fix a dense and countable subset Λ of $(0,\infty)$, and fix a countable family of separating functions $\Phi \subset C_b(X)$. Consider the Cartesian product $\Lambda \times \Phi$, which is a countable set, and fix an enumeration $(\lambda_n, \varphi_n)_{n>0}$ of $\Lambda \times \Phi$. Associate with each (λ_n, φ_n) the following continuous function on Ω

$$\zeta_n(w) = \int_0^\infty e^{-\lambda_n s}\varphi(w(s))ds\,. \tag{2.13}$$

Define recursively $\mathcal{C}^0 = \mathcal{C}$, and $\mathcal{C}^{n+1} = V_{\zeta_{n+1}}[\mathcal{C}^n]$. Thanks to Lemma 2.2.1 each set-valued map $\mathcal{C}^n$ satisfies properties **S1-S4**.

For each $a \in X$, the sequence of compact sets $(\mathcal{C}^n(a))_{n\geq 0}$ is monotone decreasing, hence the intersection is not empty, we denote it by $\mathcal{C}^\infty(a)$,

$$\mathcal{C}^\infty(a) = \bigcap_{n\geq 0} \mathcal{C}^n(a)\,.$$

By theorem 8.4 in [1], the set-valued map $\mathcal{C}^\infty$ is measurable.

Take any two elements $w_1, w_2 \in \mathcal{C}^\infty(a)$. Since both belong to each $\mathcal{C}^{n+1}(a)$, both maximize the functions ζ_n over the set $\mathcal{C}^n(a)$, hence

$$\zeta_n(w_1) = \zeta_n(w_2)\,, \quad \forall n \geq 0\,.$$

Since the sequence $(\lambda_n, \varphi_n)_{n>0}$ is an enumeration of $\Lambda \times \Phi$, then

$$\int_0^\infty e^{-\lambda s}\varphi(w_1(s))ds = \int_0^\infty e^{-\lambda s}\varphi(w_2(s))ds\,, \quad \forall \lambda \in \Lambda\,, \forall \varphi \in \Phi\,. \tag{2.14}$$

From the uniqueness of the Laplace transform and the continuity of w_1 and w_2 we have

$$\varphi(w_1(s)) = \varphi(w_2(s))\,, \quad \forall \varphi \in \Phi\,, \forall s \geq 0\,. \tag{2.15}$$

Finally, since the family of functions Φ is separating, (2.15) implies that

$$w_1(s) = w_2(s)\,, \quad \forall s \geq 0\,.$$

Then $w_1 = w_2$ and the set $\mathcal{C}^\infty(a)$ is a singleton. This is true for any $a \in X$.

The map $\mathcal{C}^\infty$ satisfies the properties **S1-S4**. Its only possible selection is measurable by the Kuratowski and Ryll-Nardzewski selection Theorem 2.1.2. Denote this selection by $u(\cdot, a)$. By property **S1**,

$$\theta_t[u(\cdot, a)] \in \mathcal{C}^\infty_{u(t,a)} , \tag{2.16}$$

and the only element in $\mathcal{C}^\infty_{u(t,a)}$ is the path $u(\cdot, u(t, a))$, hence

$$u(t + s, a) = u(s, u(t, a)) , \quad u(0, a) = a , \quad \forall t, s \in\geq 0 , \forall a \in X . \tag{2.17}$$

□

I refer to [7] for a variation of Theorem 2.2.2 on more general spaces. A similar result can be obtained for non-autonomous case, see [8].

2.3 Application of the Semigroup Selection to NLS

We are going to apply the Theorem 2.2.2 to the set-valued map of weak solutions of NLS defined in Chapter 1. Fix $R > 0$, the set $V_R = \{ \xi \in V \mid E[\xi] \leq R \}$ is a compact metric space space with the relative weak topology coming from V. V_R with weak topology will play the role of X. Ω_R is the space of continuous paths in V_R, with the compact-open topology. The space $\mathcal{H}_k(\Omega_R)$ is the hyperspace of all compact subsets of Ω_R.

In Section 1.3 we defined the set-valued map $S_R : V_R \to \mathcal{H}_k(\Omega_R)$, which maps initial conditions $a \in V_R$ to the set of all weak solutions of NLS with energy bounded by R starting at a. By Proposition 1.3.2 the map S_R takes values in $\mathcal{H}_k(\Omega_R)$.

Proposition 2.3.1. *The map S_R is u.s.c.*

Proof. Since S_R takes values in compact subsets of Ω_R, and, for any $U \subset V_R$ the set $S_R(U)$ is relatively compact (see the proof of Proposition 1.3.2), by Proposition 2.23 in [18], it is enough to show that, if $a_n \rightharpoonup a$, and $u_n \in S_R(a_n)$ with $u_n \to u$ in Ω_R then $u \in S_R(a)$.

By Lemma 1.3.1 we have that $u_n \to u$ weak-$\star$ in $L^\infty([0,\infty) \to V)$ and strongly in $L^2_{\text{loc}}([0,\infty) \to L^2)$, and, for any $t \geq 0$, $u_n(t) \to u(t)$ weakly in V and strongly in L^2. Then, there is a subsequence (u_{n_k}) that converges almost everywhere in M_∞. Thanks to the almost everywhere convergence of (u_{n_k}) we have the convergence described in (1.50) and (1.52), moreover we have that $a_n \rightharpoonup a$. Hence, in the limit as $k \to \infty$, the path u satisfies (1.49).

Since $u_n(t) \to u(t)$ strongly in L^2 and weakly in V, the convergence of mass is preserved in the limit, and, for any $t \geq 0$, we have that $E[u(t)] \leq \liminf_{n\to\infty} E[u_n(t)]$, hence $E[u(t)] \leq R$, hence $u_n(t) \in V_R$. Therefore, u is a weak solution with energy bounded by R starting at a. Thus, S_R is u.s.c.

□

Proposition 2.3.2. *The set-valued map S_R is measurable.*

Proof. Follows from Corollary 2.1.2. □

To apply our selection theorem we need to check all the properties **S1-S4**. So far we have properties **S1,S2**. Let us show first that the map S_R has the property **S3**.

Lemma 2.3.3. *If $u \in S_R(a)$, then $\theta_t(u) \in S_R(u(t))$.*

Proof. The path u satisfies

$$(u(t+r),\eta) = (\phi,\eta) - i\int_0^{t+r} (\nabla u(s),\nabla\eta) + \left\langle |u(s)|^{p-2}u(s),\eta\right\rangle ds\,. \tag{2.18}$$

Then,

$$\begin{aligned}(u(t+r),\eta) = (\phi,\eta) - i\int_0^{t} (\nabla u(s),\nabla\eta) + \left\langle |u(s)|^{p-2}u(s),\eta\right\rangle ds \\ - i\int_t^{t+r} (\nabla u(s),\nabla\eta) + \left\langle |u(s)|^{p-2}u(s),\eta\right\rangle ds\,.\end{aligned} \tag{2.19}$$

Since

$$(u(t),\eta) = (\phi,\eta) - i\int_0^{t} (\nabla u(s),\nabla\eta) + \left\langle |u(s)|^{p-2}u(s),\eta\right\rangle ds\,, \tag{2.20}$$

and $u(t+r) = (\theta_t u)(r)$, we have

$$((\theta_t u)(r), \eta) = (u(t), \eta) - i \int_0^r (\nabla(\theta_t u)(s), \nabla \eta) + \left\langle |(\theta_t u)(s)|^{p-2} (\theta_t u)(s), \eta \right\rangle ds \,. \quad (2.21)$$

Thus, $\theta_t u$ satisfies the integral equality (1.68) with the initial condition $u(t)$.

The path u preserves the mass of a, hence $\theta_t u$ preserves the mass as well. Finally, the energy of $\theta_t u$ is automatically bounded as well by R. Hence, $\theta_t u$ is a weak solution of (1.10) with energy bounded by R with the initial condition $u(t)$, thus $\theta_t u \in S_R(u(t))$. □

Let us show that S_R has the property **S4**.

Lemma 2.3.4. *For any $a \in V_R$, if $u \in S_R(a)$ and $v \in S_R(u(t))$, then, $u \lhd_t v \in S_R(a)$.*

Proof. Both u and v satisfies the integral equation (1.68). Set $w = u \lhd_t v$.

For $r < t$, the path w agrees with u, hence

$$(w(r), \eta) = (\phi, \eta) - i \int_0^r (\nabla w(s), \nabla \eta) + \left\langle |w(s)|^{p-2} w(s), \eta \right\rangle ds \,. \quad (2.22)$$

The path u satisfies

$$(u(t), \eta) = (\phi, \eta) - i \int_0^t (\nabla u(s), \nabla \eta) + \left\langle |u(s)|^{p-2} u(s), \eta \right\rangle ds \,. \quad (2.23)$$

For $r > t$ we have

$$(v(r-t), \eta) = (u(t), \eta) - i \int_0^{r-t} (\nabla v(s), \nabla \eta) + \left\langle |v(s)|^{p-2} v(s), \eta \right\rangle ds \,. \quad (2.24)$$

After a change of variable we have

$$(w(r), \eta) = (u(t), \eta) - i \int_t^r (\nabla w(s), \nabla \eta) + \left\langle |w(s)|^{p-2} w(s), \eta \right\rangle ds \,, \quad (2.25)$$

Replacing $(u(t),\eta)$ with (2.23) we get

$$\begin{aligned}(w(r),\eta) = (\phi,\eta) - i\int_0^t (\nabla w(s),\nabla\eta) + \left\langle |w(s)|^{p-2}w(s),\eta\right\rangle ds \\ - i\int_t^r (\nabla w(s),\nabla\eta) + \left\langle |w(s)|^{p-2}w(s),\eta\right\rangle ds\,.\end{aligned} \tag{2.26}$$

Finally,

$$(w(r),\eta) = (\phi,\eta) - i\int_0^r (\nabla w(s),\nabla\eta) + \left\langle |w(s)|^{p-2}w(s),\eta\right\rangle ds\,. \tag{2.27}$$

Hence, w satisfies (1.68)

Since u and v take values in V_R, then, the path w takes values in V_R. The path w is continuous, both in L^2 and weakly in V_R. Hence $w \in S_R(a)$. □

We can now apply the selection theorem.

Theorem 2.3.5. *For fixed $R > 0$, there is a Caratheodory map $u : V_R \times [0,\infty) \to V_R$, measurable in the first variable and continuous in the second, such that, for any $a \in V_R$, the path $t \mapsto u(t,a)$ is a weak solution of (1.10), of type II, with initial condition a, and u satisfies*

$$u(t+s,a) = u(s,u(t,a))\,, \quad \forall t,s \in [0,\infty)\,, \forall a \in V_R\,. \tag{2.28}$$

Proof. Apply Theorem 2.2.2 to the set-valued map $a \mapsto S_R(a)$. □

Chapter 3

Hopf Equation

In the seminal work **"Statistical hydromechanics and functional calculus"** [16], E. Hopf described a general framework to study the evolution of probability distributions of initial conditions for the Navier-Stokes equations.

When the existence, uniqueness and continuous dependence on the initial conditions for an evolution equation can be obtained, if a probability distribution on the space of initial conditions is given, one can study how this distribution evolves according to the equation. E. Hopf proposed to study the evolution of the characteristic functional of the probabilities and described a general equation in terms of variations satisfied by such functionals. This equation is known as the Hopf equation.

The first rigorous study of the Hopf equation for Navier-Stokes in two dimensions was given by O.A. Ladyzhenskaya and A.M. Vershik [24].

3.1 Abstract Derivation of Hopf Equation

Consider an abstract Cauchy problem

$$\dot{u}(t,a) = F(u(t,a))\,, \qquad u(0,a) = a\,, \tag{3.1}$$

where a belongs to a separable real Hilbert space X, and F maps X into itself. We think of (3.1) as a deterministic description of some physical process. Suppose it is known that (3.1) has a solution $t \mapsto u(t,a)$, for every $a \in X$, and that the map $S_t : a \mapsto u(t,a)$ is sufficiently nice. If the initial conditions are distributed randomly according to a probability distribution μ_0, then the push-forward

$$\mu_t(A) = \mu_0(S_t^{-1}(A))$$

will describe the distribution of the physical system at time t. Hopf proposed to look at the corresponding characteristic functionals, obtained as the Fourier transform,

$$\Phi(t,w) = \int_X e^{i(a,w)}\mu_t(da) = \int_X e^{i(u(t,a),w)}\mu_0(da)\,. \tag{3.2}$$

The characteristic functional Φ satisfies the equation

$$\partial_t \Phi(t,w) = \int_X e^{i(u(t,a),w)} i(\dot{u}(t,a),w)\mu_0(da) \tag{3.3}$$

which reduces to

$$\partial_t \Phi(t,w) = i\int_X e^{i(a,w)}(F(a),w)\mu_t(da)\,. \tag{3.4}$$

If X is a complex Hilbert space the characteristic functional is defined as

$$\Phi(t,w) = \int_X e^{i\,\mathrm{Re}((a,w))}\mu_t(da)\,,$$

and the equation is

$$\partial_t \Phi(t,w) = i\int_X e^{i\,\mathrm{Re}((a,w))}\,\mathrm{Re}\left((F(a),w)\right)\mu_t(da)\,. \tag{3.5}$$

Definition 3.1.1. A family of measures μ_t is said to be a Hopf statistical solution of (3.1) if its characteristic functional satisfies (3.4) (resp. (3.5)).

In some cases equation (3.4) can be rewritten in terms of Φ alone. If X is finite dimensional, we interpret (3.4) as a partial differential equation. A simple example of this case follows.

Example 3.1.1. Let $X = \mathbb{R}$ and consider the Cauchy problem

$$\dot{u}(t) = -u(t)\,, \quad u(0) = u_0\,.$$

The solution is $u(t) = u_0 e^{-t}$. Let S_t be the flow defined as $S_t(a) = ae^{-t}$. Let μ_0 be a probability measure on the measurable space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. For any $t > 0$, define μ_t as the push-forward of μ_0 through the map S_t i.e.,

$$\mu_t(A) = \mu_0(S_t^{-1}(A))\,.$$

The characteristic functional is

$$\Phi(t, w) = \int_{\mathbb{R}} e^{i\,aw} \mu_t(da) = \int_{\mathbb{R}} e^{i\,ae^{-t}w} \mu_0(da)\,. \tag{3.6}$$

Take a derivative with respect to t to obtain

$$\partial_t \Phi(t, w) = -i \int_{\mathbb{R}} e^{i\,aw} ae^{-t} w \mu_0(da) = -iw \int_{\mathbb{R}} e^{i\,aw} a\mu_t(da)\,, \tag{3.7}$$

and with respect to w,

$$\partial_w \Phi(t, w) = i \int_{\mathbb{R}} e^{i\,aw} ae^{-t} \mu_0(da) = i \int_{\mathbb{R}} e^{i\,aw} a\mu_t(da)\,. \tag{3.8}$$

Hence, the Hopf equation is

$$\partial_t \Phi(t,w) + w\partial_w \Phi(t,w) = 0\,, \qquad \Phi(0,\cdot) = \Phi_0\,. \tag{3.9}$$

Equation (3.9) is a linear first order differential equation with the solution

$$\Phi(t,w) = \Phi_0(we^{-t})\,. \tag{3.10}$$

If X is a infinite dimensional space, the Hopf equation can be formally interpreted as an equation in variations. Consider the following example.

Example 3.1.2. Let L^2 be the space of square integrable functions on the circle, i.e., $L^2 = L^2(\mathbb{T})$. Consider the Burgers' equation

$$\partial_t u(t,x) = \nu \Delta u(t,x) - \frac{1}{2}\partial_x(u(t,x)^2)\ \ (= F(u))\ , \qquad u(0,\cdot) = u_0 \in L^2\,. \tag{3.11}$$

A careful analysis of such an equation was given by Hopf in [17]. Let $S_t(a) = u(t,a)$ be the flow of Burgers' equation starting at the initial condition $a \in L^2$.

Let μ_0 be a probability measure on the space L^2. The characteristic functional is

$$\Phi(t,w) = \int_{L^2} e^{i\,(a,w)}\mu_t(da) = \int_{L^2} e^{i\,(u(t,a),w)}\mu_0(da)\,, \tag{3.12}$$

where $w \in L^2$, and $(\cdot,\cdot)$ is the inner product in L^2. Using the Fourier transform of w and $u(t,a)$, we can write

$$\Phi(t,w) = \int_{L^2} e^{i\sum_{n\in\mathbb{Z}} a_n w_n}\mu_t(da) = \int_{L^2} e^{i\sum_{n\in\mathbb{Z}} u_n(t,a) w_n}\mu_0(da)\,, \tag{3.13}$$

where

$$w_m(t) = \frac{1}{2\pi}\int_0^{2\pi} w(x)e^{-imx}dx\,. \tag{3.14}$$

The derivative of Φ with respect to t is

$$\begin{aligned}\partial_t\Phi(t,w) &= i\int_{L^2} e^{i\sum_{n\in\mathbb{Z}} u_n(t,a)w_n} \sum_{n\in\mathbb{Z}} \dot{u}_n(t,a)w_n\mu_0(da)\\ &= i\int_{L^2} e^{i\,(a,w)}(F(a),w)\mu_t(da)\,.\end{aligned} \tag{3.15}$$

Since $F(a) = \nu\Delta a - \frac{1}{2}\partial_x a^2$, we can compute $(F(a), w)$ as

$$(F(a),w) = \sum_{n\in\mathbb{Z}} -\nu n^2 u_n w_n - \frac{i}{2}\sum_{n\in\mathbb{Z}} n w_n \sum_{j+k=n} u_j u_k\,. \tag{3.16}$$

Which means

$$\begin{aligned}\partial_t\Phi(t,w) =& - i\nu\sum_{n\in\mathbb{Z}} n^2 w_n \int_{L^2} e^{i\,(u(t,a),w)} u_n(t,a)\mu_0(da)\\ &+ \frac{1}{2}\sum_{n\in\mathbb{Z}} n w_n \sum_{n=j+k}\int_{L^2} e^{i\,(u(t,a),w)} u_j(t,a)u_k(t,a)\mu_0(da)\,,\end{aligned} \tag{3.17}$$

hence,

$$\begin{aligned}\partial_t\Phi(t,w) =& - i\nu\sum_{n\in\mathbb{Z}} n^2 w_n \int_{L^2} e^{i\,(a,w)} a_n\mu_t(da)\\ &+ \frac{1}{2}\sum_{n\in\mathbb{Z}} n w_n \sum_{n=j+k}\int_{L^2} e^{i\,(a,w)} a_j a_k\mu_t(da)\,.\end{aligned} \tag{3.18}$$

The derivative of Φ with respect to w_n is

$$\partial_{w_n}\Phi(t,w) = i\int_{L^2} e^{i\,(a,w)} a_n\mu_t(da)\,, \tag{3.19}$$

and the second derivative of Φ with respect to w_j and w_k is

$$\partial_{w_j,w_k}\Phi(t,w) = -\int_{L^2} e^{i\,(a,w)} a_j a_k\mu_t(da)\,. \tag{3.20}$$

We can now write the Hopf equation as

$$\partial_t \Phi(t,w) = -\nu \sum_{n\in\mathbb{Z}} n^2 w_n \partial_{w_n}\Phi(t,w) - \frac{1}{2}\sum_{n\in\mathbb{Z}} n w_n \sum_{n=j+k} \partial_{w_j,w_k}\Phi(t,w)\,. \tag{3.21}$$

We interpret this equation as a variational equation written in “Fourier coordinates”.A more rigorous analysis is necessary to answer the following:

- Can we define the derivatives $\partial_{w_n}\Phi$ and $\partial_{w_j,w_k}\Phi$?
- Do the series in the equation (3.21) converge?

Hopf himself has written a similar equation for the Navier-Stokes equations in [16]. A rigorous analysis of that equation in two dimensions is given by Vershik and Ladyzhenskaya in [24].

The Hopf equations in Φ alone can be obtained for partial differential equations with algebraic nonlinearities. In those cases the equations are linear PDE's on infinite dimensional domains.

The Hopf equation by itself is worth being studied in the usual sense of existence and uniqueness, but this seems to be a difficult task.

It has been noticed by C. Foiaș and G. Prodi [12], Foiaș [11], Vishik and Komech [34] and Vishik and Fursikov [32] that the Hopf equation in the form (3.4) is much easier to analyze. In addition, one can prove the existence of solutions to (3.4) without uniqueness in the underlying equation (3.1). Moreover, the integral version of (3.4)

$$\Phi(t,w) - \Phi(0,w) = i\int_0^t \int_X e^{i\,(a,w)}(F(a),w)\,\mu_s(da)ds\,, \tag{3.22}$$

is easier to establish.

We are going to construct Hopf statistical solutions to NLS. The Banach space $V = \dot{H}^1 \cap L^p$ as defined in Chapter 1 is reflexive and separable. In what follows we work with regular probability measures in the measurable space $(V, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra generated by the weak topology of V. Note that, at the same time, $\mathcal{B}$ is the Borel σ-algebra generated by the strong topology. See Appendix A for some of the basic results.

Definition 3.1.2. Given a regular probability measure μ_0 on $(V, \mathcal{B})$, a family of regular probability measures $(\mu_t)_{t\geq 0}$ on $(V, \mathcal{B})$ is a Hopf statistical solution with the initial condition μ_0 for NLS, if it satisfies the following:

1. For any $v \in V$,

$$\Phi(t,v) - \Phi(0,v) = i \int_0^t \int_V e^{i\,\mathrm{Re}((a,v))}\, \mathrm{Im}\left((\nabla a, \nabla v) + \left\langle |a|^{p-2} a, v \right\rangle \right) \mu_s(da) ds\,, \tag{3.23}$$

2. For any $v_1, v_2 \in V$,

$$|\Phi(t,v_1) - \Phi(t,v_2)| \lesssim \|v_1 - v_2\|_{V^*}\,. \tag{3.24}$$

3. For any $t \geq 0$,

$$\int_V E[a]\, \mu_t(da) \leq \int_V E[a]\, \mu(da) \tag{3.25}$$

We proved the following existence result.

Theorem 3.1.1. *Let μ_0 be a regular probability measure on $(V, \mathcal{B})$. If μ_0 has finite mean energy, i.e.,*

$$\int_V \mathrm{E}[a]\, \mu_0(da) < \infty\,. \tag{3.26}$$

Then, there exists a Hopf statistical solution $(\mu_t)_{t\geq 0}$ for the NLS equation with initial condition μ_0.

3.2 Proof of Theorem 3.1.1

We first prove the existence of a Hopf statistical solution for probability measures supported on V_R for some $R > 0$. Recall that V_R is a compact metric space when considered as a subspace of V with the inherited weak topology. We assume the measure μ_0 has finite mean energy, i.e.,

$$M_E = \int_V E[a]\,\mu_0(da) < \infty\,.$$

For any $m > 0$, let μ_0^m be the push-forward of μ_0 through the map P^m, i.e.,

$$\mu_0^m(A) = \mu_0(\{\, a \in V \mid P^m(a) \in A \,\})\,, \quad \forall A \in \mathcal{B}\,.$$

Thanks to (1.42), P^m maps V_R into $V_{R'}$, with $R' = R\,C_p^p$, hence, for any $m > 0$, the measure μ_0^m is supported in $V_{R'}$. Moreover, $\mu_0^m \rightharpoonup \mu_0$ in the topology associated with strongly continuous functions, i.e.,

$$\int_V f(a)\mu_0^m(da) \xrightarrow{m\to\infty} \int_V f(a)\mu_0(da)\,, \quad \forall f \in C_b(V)\,. \tag{3.27}$$

For any $t \geq 0$ and $m > 0$, let μ_t^m be the push-forward of μ_0^m through the map S_t^m (see Remark 1.2.1), i.e.,

$$\mu_t^m(A) = \mu_0^m(\{\, a \in V_w \mid S_t^m(a) \in A \,\})\,, \quad \forall A \in \mathcal{B}\,.$$

The finite-dimensional flow S_t^m preserves the energy, hence

$$\int_V E[a]\,\mu_t^m(da) = \int_V E[a]\,\mu_0^m(da)\,.$$

For any $m > 0$, the characteristic functional of the family $(\mu_t^m)_{t\geq 0}$ is

$$\Phi^m(t,v) = \int_V e^{i\,\mathrm{Re}(\langle v,a\rangle)}\,\mu_t^m(da)\,, \quad \forall t\geq 0\,, \forall v\in V^*\,. \tag{3.28}$$

Since μ_t^m is carried by V^m, and $\langle v, a^m\rangle = \langle v^m, a^m\rangle$, we have

$$\Phi^m(t,v) = \int_V e^{i\,\mathrm{Re}(\langle v^m,a^m\rangle)}\,\mu_t^m(da)\,, \quad \forall t\geq 0\,, \forall v\in V^*\,. \tag{3.29}$$

The flow $(t,a)\mapsto S_t^m(a)$ satisfies (1.32), hence

$$\partial_t\Phi^m(t,v) = \int_V e^{i\,\mathrm{Re}(\langle v^m,a^m\rangle)} i\,\mathrm{Re}\left(\langle v^m, F^m(a^m)\rangle\right)\mu_t^m(da)\,, \tag{3.30}$$

where F^m is the map defined in Section 1.2.1. Recall that $F^m(a) = P^m(F(P^m(a)))$, where $F(a) = i\,\Delta a - i\,|a|^{p-2}a$, then

$$\partial_t\Phi^m(t,v) = \int_V e^{i\,\mathrm{Re}(\langle v^m,a^m\rangle)} i\,\mathrm{Re}\left(-i\left(\nabla a^m, \nabla v^m\right) - i\left\langle |a^m|^{p-2}a^m, v^m\right\rangle\right)\mu_t^m(da)\,. \tag{3.31}$$

Thus,

$$\partial_t\Phi^m(t,v) = i\int_V e^{i\,\mathrm{Re}(\langle v^m,a^m\rangle)}\,\mathrm{Im}\left(\left(\nabla a^m, \nabla v^m\right) + \left\langle |a^m|^{p-2}a^m, v^m\right\rangle\right)\mu_t^m(da)\,. \tag{3.32}$$

Finally, integrate over $[0,t]$ to obtain

$$\begin{aligned}\Phi^m(t,v) - \Phi^m(0,v) &= \\ &i\int_0^t\int_V e^{i\,\mathrm{Re}(\langle v^m,a^m\rangle)}\,\mathrm{Im}\left(\left(\nabla a^m, \nabla v^m\right) + \left\langle |a^m|^{p-2}a^m, v^m\right\rangle\right)\,\mu_s^m(da)\,ds\,.\end{aligned} \tag{3.33}$$

We have shown that for any $m > 0$ the family of measures $(\mu_t^m)_{t\geq 0}$ is a Hopf statistical solution for the Galerkin approximation.

The domain of the characteristic functionals Φ^m is $[0,\infty)\times V^*$. By analyzing first the restriction of them to $[0,\infty)\times V$ we can prove the following.

Lemma 3.2.1. *The family* (Φ^m) *is uniformly bounded and equicontinuous when restricted to* $[0,\infty)\times V$*. Hence,* (Φ^m) *admits a subsequence that converges to a function* Φ *on compact subsets of* $[0,\infty)\times V$*. Moreover, the functional* Φ *can be extended to* $[0,\infty)\times V^*$*.*

Proof. Notice that, for any $m>0$,

$$|\Phi^m(t,v)| \le \int_V \mu_t^m(da) = \mu_t^m(V) = 1\,, \tag{3.34}$$

hence, the family (Φ^m) is uniformly bounded.

For any $m>0$, Φ^m satisfies

$$\partial_t \Phi^m(t,v) = \int_V e^{i\,\mathrm{Re}(\langle v^m, a^m\rangle)} i\,\mathrm{Re}\left(\langle v^m, F^m(a^m)\rangle\right)\mu_t^m(da)\,. \tag{3.35}$$

Then,

$$\begin{aligned}
|\partial_t \Phi^m(t,v)| &\le \int_V |(\nabla a^m, \nabla v^m)| + |\langle |a^m|^{p-2}a^m, v^m\rangle| \mu_t^m(da)\\
&\le \|v^m\|_V \int_V \|\nabla a^m\|_2 + \|a^m\|_p^{p-1}\,\mu_t^m(da)\\
&\le \|v^m\|_V \int_V \left(\frac{1}{2(p-1)} + \frac{p-1}{2}\|\nabla a^m\|_2^2 + \frac{p-1}{p}\|a^m\|_p^p + \frac{1}{p}\right)\mu_t^m(da)\\
&\le \|v^m\|_V \int_V 1 + (p-1)E[a^m]\mu_t^m(da)
\end{aligned}$$

where we have used Young's inequality in the third step. Thanks to the conservation of energy for Galerkin approximations and the inequality (1.42), we have

$$|\partial_t \Phi^m(t,v)| \le C_p\|v\|_V \left(1 + C_p^p(p-1)M_E\right)\,. \tag{3.36}$$

Hence,

$$|\Phi^m(t+\Delta t, v) - \Phi^m(t,v)| \leq C_p\left(1 + C_p^p(p-1)M_E\right)\|v\|_V\,\Delta t\,. \tag{3.37}$$

At the same time, for any $t \geq 0$, we have

$$\begin{aligned}
|\Phi^m(t,v_1) - \Phi^m(t,v_2)| &\leq \int_V |e^{i\ \mathrm{Re}(\langle a, v_1 - v_2\rangle)} - 1|\,\mu_t^m(da)\\
&\leq \int_V |\mathrm{Re}\left(\langle a, v_1 - v_2\rangle\right)|\,\mu_t^m(da)\\
&\leq \|v_1 - v_2\|_{V^*}\int_V \|a\|_V\,\mu_t^m(da)\\
&\leq \|v_1 - v_2\|_{V^*}\int_V \frac{3}{2} + E[a]\,\mu_t^m(da)\,.
\end{aligned} \tag{3.38}$$

This,

$$|\Phi^m(t,v_1) - \Phi^m(t,v_2)| \leq \|v_1 - v_2\|_{V^*}\left(\frac{3}{2} + C_p^p M_E\right)\,. \tag{3.39}$$

Thanks to the bounds (3.37) and (3.39) we conclude that the family (Φ^m) is equicontinuous.

We apply Arzelà–Ascoli to extract from (Φ^m) a subsequence (Φ^{m_k}) that converges uniformly on compact subsets of $[0,\infty) \times V$ to a function Φ. In particular, for any $t \geq 0$ and $v \in V$

$$\Phi^{m_k}(t,v) \xrightarrow{k\to\infty} \Phi(t,v)\,.$$

Finally, the functional Φ continues to satisfy (3.39). Since V is dense in V^* we can extend Φ to $[0,\infty) \times V^*$ by linearity. □

Lemma 3.2.2. *The sequence of probability measures $\mu_t^{m_k}$ whose characteristic functionals Φ^{m_k} converge to Φ, converges weakly to the measure μ_t, whose characteristic functional is Φ.*

Proof. Recall that any family of probability measures supported on a compact metric space is relatively weak compact. The space $V_{R'}$ is a compact metric space when endowed with the weak topology inherited from V. The family (μ_t^n) is supported on

$V_{R'}$, hence is relatively weak compact. Since $\Phi^{m_k} \to \Phi$ by Lemma 3.2.1, the results follows from Theorem A.0.5. □

We still need to show that the limit obtained in Lemma 3.2.2 satisfies (3.23). We will show this in several steps.

Lemma 3.2.3. *For any $v \in V$, the map $a \mapsto (\nabla a, \nabla v) + \langle |a|^{p-2}a, v\rangle$ is weakly sequentially continuous.*

Proof. Take any sequence $a_n \rightharpoonup a$ in V. For the first term is enough to see that $\nabla a_n \rightharpoonup \nabla a$ in L^2, hence

$$(\nabla a_n, \nabla v) \to (\nabla a, \nabla v)\,.$$

By Rellichs's lemma we have that $a_n \to a$ strongly in L^2. Hence, (a_n) admits a subsequence (a_{n_k}) such that $a_{n_k} \to a$ almost everywhere.

Consider now the term $\langle |a_n|^{p-2}a_n, v\rangle$, i.e.,

$$\int_M |a_n(x)|^{p-2} a_n(x)\overline{v}(x)dx\,.$$

Thanks to Egorov's theorem, for any $\delta > 0$ there is a closed set $\Omega_\delta \subset M$, such that $a_n \to a$ uniformly on Ω_δ and $d(\Omega_\delta) < \delta$.

Notice that

$$\begin{aligned}\int_{\Omega_\delta^c} |a_n(x)|^{p-2} a_n(x)\overline{v}(x)dx &\leq \left(\int_{\Omega_\delta^c} |a_n(x)|^p dx\right)^{\frac{p-1}{p}} \left(\int_{\Omega_\delta^c} |v(x)|^p dx\right)^{\frac{1}{p}} \\ &\leq \|a_n\|_p^{p-1} \left(\int_{\Omega_\delta^c} |v(x)|^p dx\right)^{\frac{1}{p}}.\end{aligned} \tag{3.40}$$

Similarly,

$$\int_{\Omega_\delta^c} |a(x)|^{p-2} a(x)\overline{v}(x)dx \leq \|a\|_p^{p-1} \left(\int_{\Omega_\delta^c} |v(x)|^p dx\right)^{\frac{1}{p}}. \tag{3.41}$$

Since v is a fixed element in V, by the Lebesgue Theorem, we have

$$\int_{\Omega_\delta^c} |v(x)|^p dx \xrightarrow{\delta \to 0} 0\,.$$

Hence, for any $\varepsilon > 0$, we can find a $\delta > 0$ small enough such that

$$\int_{\Omega_\delta^c} \Big| |a_n(x)|^{p-2} a_n(x) - |a(x)|^{p-2} a(x) \Big| \overline{v}(x) dx < \frac{\varepsilon}{2}\,. \tag{3.42}$$

For any fixed $\delta > 0$, since $a_n \to a$ uniformly on Ω_δ, for sufficiently large n we have

$$\int_{\Omega_\delta} \Big| |a_n(x)|^{p-2} a_n(x) - |a(x)|^{p-2} a(x) \Big| \overline{v}(x) dx \ < \frac{\varepsilon}{2}\,. \tag{3.43}$$

Then, for any $\varepsilon > 0$ we can find n large enough such that

$$\int_V \Big| |a_n(x)|^{p-2} a_n(x) - |a(x)|^{p-2} a(x) \Big| \overline{v}(x) dx \ < \varepsilon\,. \tag{3.44}$$

□

The weak topology of an infinite dimensional Banach space is not first countable, hence, it is not immediately obvious that a sequentially continuous functions is also continuous. When restricted to any weak-compact subset of V we have the following result.

Lemma 3.2.4. *If $f : V \to \mathbb{C}$ is a weakly sequentially continuous map, then the restriction of f to V_R, for any $R > 0$, is continuous in the subspace topology of V_R induced from the weak topology.*

Proof. Recall that for any $R > 0$, the set V_R is a compact metric space, and the topology induced by its metric is equal to the subspace topology with respect to the weak topology on V. If f is restricted to V_R it remains to be sequentially continuous on V_R, hence continuous. □

Proposition 3.2.5. *The family of probability measures* $(\mu_t)_{t\geq 0}$ *obtained in Lemma 3.2.2 is a Hopf statistical solution for the NLS equation.*

Proof. Recall that the weak convergence of $\mu_0^m \rightharpoonup \mu_0$ is with respect to the strong topology of V. In particular, the map $a \mapsto E[a]$ is strongly continuous and, therefore,

$$\int_V E[a]\mu_0^m(da) \to \int_V E[a]\mu_0(da)\,.$$

Since the set V_R is weak-closed, the set $\{\, a \in V \mid E[a] > R\,\}$ is weak-open, hence the map $a \mapsto E[a]$ is weak lower semi-continuous.

Recall that we have obtained the convergence of probability measures $\mu_t^{m_k} \rightharpoonup \mu_t$ with respect to the weak topology of V. Due to the preservation of the energy of the finite-dimensional flow we have

$$M_E = \lim_{k\to\infty} \int_V E[a]\,\mu_0^{m_k}(da) = \lim_{k\to\infty} \int_V E[a]\,\mu_t^{m_k}(da)\,. \tag{3.45}$$

Since $a \mapsto E[a]$ is weak lower semi-continuous, and thanks to $\mu_t^{m_k} \rightharpoonup \mu_t$ with respect to the weak topology of V we have

$$\int_V E[a]\,\mu_t(da) \leq \liminf_{k\to\infty} \int_V E[a]\,\mu_t^{m_k}(da) = M_E\,, \tag{3.46}$$

hence,

$$\int_V E[a]\,\mu_t(da) \leq \int_V E[a]\,\mu_0(da)\,.$$

According to Definition 3.1.2 it only remains to show that the family of probability measures satisfies (3.23). Fix $v \in V$, and let

$$f(a) = (\nabla a, \nabla v) + \left\langle |a|^{p-2}a, v \right\rangle,$$

and

$$f_m(a) = (\nabla a, \nabla v^m) + \left\langle |a|^{p-2}a, v^m \right\rangle.$$

Then,

$$|f_m(a) - f(a)| < \left(\|\nabla a\|_2 + \|a\|_p^{p-1}\right) \|v^m - v\|_V. \tag{3.47}$$

Hence, $f_m \to f$ uniformly on bounded sets of V as $m \to \infty$.

For each $m > 0$, the family of probability measures $(\mu_t^m)_{t\geq 0}$ satisfies (3.33). Moreover, for any $m > 0$, and $t \geq 0$ the probability measure μ_t^m is supported in the weak-compact subset $V^m \cap V_{R'}$ of V. Hence, for each $s \geq 0$,

$$\int_V e^{i\ \mathrm{Re}(\langle v^m, a^m\rangle)}\,\mathrm{Im}\left(f_m(a^m)\right)\,\mu_s^m(da) = \int_V e^{i\ \mathrm{Re}(\langle v^m, a\rangle)}\,\mathrm{Im}\left(f_m(a)\right)\,\mu_s^m(da). \tag{3.48}$$

For all $m > 0$, and $t \geq 0$, the probability measure μ_t^m is supported in the weak-compact subset $V_{R'} \subset V$. Hence, the weak convergence of $\mu_t^{m_k} \rightharpoonup \mu_t$, as $k \to \infty$, can be considered to happen in the space of probability measures on $(V_{R'}, \mathcal{B}_{V_{R'}})$.

The uniform convergence of (f_m) together with the weak convergence of $\mu_t^m \rightharpoonup \mu_t$ for any $t \geq 0$, ensures that we can pass to the limit as $m \to \infty$, hence

$$\int_V e^{i\ \mathrm{Re}(\langle v^m, a\rangle)}\,\mathrm{Im}\left(f_m(a)\right)\,\mu_t^m(da) \xrightarrow{m\to\infty} \int_V e^{i\ \mathrm{Re}(\langle v, a\rangle)}\,\mathrm{Im}\left(f(a)\right)\,\mu_t(da). \tag{3.49}$$

Thus, $(\mu_t)_{t\geq 0}$ is a Hopf statistical solution for the NLS equation according to the Definition 3.1.2.

□

The passage to a regular probability measure μ_0 that is not supported in any V_R requires the following.

Lemma 3.2.6. *Let $(\alpha_n)_{n>0}$ be a sequence of non-negative numbers such that*

$$\sum_{n>0} \alpha_n = 1 \,.$$

Let $(\mu_0^n)_{n>0}$ be a sequence of regular probability measures on $(V, \mathcal{B})$. Assume that the mean energy of (μ_0^n) is uniformly bounded, i.e.,

$$\sup_{n>0} \int_V E[a]\, \mu_0^n(da) = M < \infty \,.$$

Moreover, let $(\mu_t^n)_{t\geq 0}$ be a Hopf statistical solution for the NLS equation with initial condition μ_0^n. Then, the family of probability measures

$$\left(\sum_{n>0} \alpha_n\, \mu_t^n \right)_{t\geq 0} ,$$

is a Hopf statistical solution for the NLS equation with initial condition

$$\sum_{n>0} \alpha_n\, \mu_0^n \,.$$

Proof. First, we need to clarify why the probability measures can be defined as infinite convex sums. The following argument works for any sequence of probability measures.

For any sequence (μ^n) of probability measures, the sequence (ν^N) defined by

$$\nu^N = \sum_{n=1}^{N} \alpha_n\, \mu^n$$

converges in total variation to a measure μ. Indeed, for any $B \in \mathcal{B}$, the sequence $(\nu^N(B))$ is Cauchy, thanks to $|\nu^K(B) - \nu^N(B)| \leq \sum_{n=N+1}^{K} \alpha_n$ for $N < K$. Hence, for any $B \in \mathcal{B}$ define $\mu(B)$ as the limit of the Cauchy sequence $(\nu^N(B))$. The convergence of the characteristic functionals follows.

Apply the previous argument to μ_0^n and $(\mu_t^n)_{t\geq 0}$ to obtain that both $\sum_{n>0} \alpha_n \, \mu_0^n$ and $\sum_{n>0} \alpha_n \, \mu_t^n$, for any $t \geq 0$ are well-defined probability measures.

Since the convergence $\mu_0^n \to \mu_0$ is in total variation we have

$$\lim_{n\to\infty} \int_V E[a] \, \mu_0^n(da) = \int_V E[a] \, \mu_0(da) = M_E \, .$$

Moreover, for any $t \geq 0$,

$$\int_V E[a] \, \mu_t(da) = \lim_{n\to\infty} \int_V E[a] \, \mu_t^n(da) \leq \lim_{n\to\infty} \int_V E[a] \, \mu_0^n(da) = M_E \, .$$

Hence,

$$\int_V E[a] \, \mu_t(da) \leq \int_V E[a] \, \mu_0(da) \, .$$

Finally, each of the Hopf statistical solutions $(\mu_t^n)_{t\geq 0}$ satisfies (3.23). The passage to the infinite convex sum is possible thanks to the uniform integrability of the term $e^{i \ \mathrm{Re}(\langle a,v\rangle)} \, \mathrm{Im} \, ((\nabla a, \nabla v) + \langle |a|^{p-2} a, v\rangle)$, i.e., for any $N > 0$ we have

$$\sum_{n=1}^{N} \alpha_n \Big| e^{i \ \mathrm{Re}(\langle a,v\rangle)} \, \mathrm{Im} \left((\nabla a, \nabla v) + \langle |a|^{p-2} a, v\rangle \right) \Big| \leq \|v\|_V \, (1 + (p-1)E[a]) \tag{3.50}$$

which is uniformly integrable because

$$\sup_{n>0} \int_V E[a] \, \mu_0^n(da) \leq M \, ,$$

and the energy inequality. Hence, an application of the dominated convergence theorem yields

$$\begin{aligned} \lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n \int_0^t \int_V e^{i \ \mathrm{Re}(\langle a,v\rangle)} \, \mathrm{Im} \left((\nabla a, \nabla v) + \langle |a|^{p-2} a, v\rangle \right) \, \mu_s^n(da) \, ds = \\ \int_0^t \int_V e^{i \ \mathrm{Re}(\langle a,v\rangle)} \, \mathrm{Im} \left((\nabla a, \nabla v) + \langle |a|^{p-2} a, v\rangle \right) \, \mu_s(da) \, ds \end{aligned} \tag{3.51}$$

□

Let μ_0 be a regular probability measure, not necessarily supported in some V_R. Find a increasing sequence of positive numbers (R_n), such that $\alpha_1 = \mu(V_{R_1}) > 0$, and

$$\alpha_{n+1} = \mu(V_{R_{n+1}} \setminus V_{R_n}) > 0\,, \quad \forall n > 0\,.$$

For any $B \in \mathcal{B}$, let

$$\mu^1(B) = \frac{1}{\alpha_1}\,\mu_0(B \cap V_{R_1})\,,$$

and

$$\mu^{n+1}(B) = \frac{1}{\alpha_{n+1}}\,\mu_0\left(B \cap (V_{R_{n+1}} \setminus V_{R_n})\right)\,, \quad \forall n > 0\,.$$

Notice that

$$\mu_0(B) = \sum_{n>0} \alpha_n\,\mu^n(B)\,, \quad \forall B \in \mathcal{B}\,.$$

Each μ^n is supported in V_{R_n}. Obtain the corresponding Hopf statistical solutions for each μ^n, and apply Lemma 3.2.6 to obtain a Hopf statistical solution with initial condition μ_0.

3.3 Foiaș Equation and Statistical Solutions

In [11] C. Foiaș proposed a slightly different approach to statistical solutions. Instead of $e^{i(a,w)}$ he is using a more general test functional $\phi(t,a)$. Define

$$L(t,\phi) = \int_X \phi(t,a)\mu_t(da) = \int_X \phi(t,u(t,a))\mu_0(da)\,. \tag{3.52}$$

Similarly to the Hopf equation the following must be satisfied for an appropriate set of test functionals

$$\frac{d}{dt}L(t,\phi) = \int_X \partial_t\phi(t,a) + \langle \partial_a\phi(t,a), F(a)\rangle \mu_t(da). \tag{3.53}$$

Definition 3.3.1. A family of measures μ_t is said to be a Foiaș statistical solution of (3.1) if (3.53) is satisfied for some appropriate set of test functionals.

In general if the solutions of (3.1) are unique, the statistical solutions are unique as well. However, if the solutions of (3.1) are not unique, there is no uniqueness for the statistical solutions (see [32]), and there is no natural way to select some solution out of many. Here is an instructive example:

Example 3.3.1. Consider the Cauchy problem for the equation

$$\dot{x} = H(x) \tag{3.54}$$

where the Heaviside function H is defined as $H(x) = 1$ if $x > 0$ and $H(x) = 0$ if $x \leq 0$. The solutions of this equation corresponding to the initial conditions $x(t_0) = 0$ are not unique.

Here is a family of Foiaș statistical solutions to (3.54):

$$\mu_t(dx) = \rho(t)\delta_0(dx) + \gamma(t-x)1_{[0,t]}dx\,, \tag{3.55}$$

where $\rho(t) = 1 - \int_0^t \gamma(s)\,ds$ and $\gamma(s)$ is an arbitrary continuous function.

3.3.1 The Homogeneous Markov Property

For autonomous equations such as (3.1) there is a formal semigroup property. The question is: Is there a similar property for statistical solutions? In the next chapter I introduce the notion of Markov statistical solutions. These solutions have the homogeneous Markov property which is the analog of the semigroup property.

In Example 3.3.1 the solutions with the homogeneous Markov property are

$$\mu_t(dx) = e^{-\alpha t}\delta_0(dx) + \alpha e^{-\alpha(t-x)} 1_{[0,t]} dx \,, \qquad \alpha \geq 0. \tag{3.56}$$

Intuitively this is obtained by noticing that the paths stay for a random time at the origin, and then they branch. That random time should be exponential in order to satisfy the Markov property.

In general, to state the homogeneous Markov property, we need to look at other forms of statistical solutions for (3.1) since neither Hopf nor Foiaș statistical solutions can capture the condition for the Markov property.

Chapter 4

Vishik-Fursikov Measures

Vishik and Fursikov introduced a new type of statistical solutions in [33] and [32]. These are, generally speaking, probability measures in some space of paths. In comparison, both Hopf and Foiaș statistical solutions are families of measures indexed by a "time" variable.

4.1 Abstract Construction

Let X be a Polish space, and $\Omega = C([0,\infty) \to X)$ endowed with the compact-open topology, i.e.Ω is the space of continuous paths from $[0,\infty)$ to X with the topology of uniform convergence on compact subsets of $[0,\infty)$. The space Ω is a Polish space, see Section 2.1.1. For any $t \geq 0$, let π_t be the map $\Omega \to X$ such that $\pi_t(w) = w(t)$. Let $\mathcal{F}$ be the Borel σ-algebra of Ω generated by the compact-open topology. For any $t \geq 0$, let $\mathcal{F}_t = \sigma(\{\, \pi_s \,|\, s \leq t \,\})$. We view $(\mathcal{F}_t)_{t\geq 0}$ as a filtration. The space Ω is one of the usual path spaces where the theory of random processes is well studied, see [29, Section 1.3] and [9].

Consider an abstract Cauchy problem

$$\dot{u}(t,a) = F(u(t,a)) , \qquad u(0,a) = a \in X , \tag{4.1}$$

where F maps X into itself. Assume some notion of solution of (4.2) is defined, and solutions are continuous paths, i.e., elements of Ω. We will study probability measures on $(\Omega, \mathcal{F})$.

Definition 4.1.1 (Abstract Vishik-Fursikov Measure)**.** A probability measure P on $(\Omega, \mathcal{F})$ is said to be a Vishik-Fursikov measure if it is carried by paths that are solutions of (4.2), i.e.

$$P(\{\text{solutions of } (4.2)\}) = 1\,.$$

Moreover, if a probability measure μ_0 is given and $P(\pi_0^{-1}(\cdot)) = \mu_0$ then P is said to be the Vishik-Fursikov measure of NLS with the initial condition μ_0.

We interpret any Vishik-Fursikov measure as a random process where the paths are almost surely (a.s.) solutions of (4.2), i.e., if P is a Vishik-Fursikov measure, there exists a set $N \subset \Omega$ such that $P(N) = 0$ and for any $u \notin N$, u is a solution of (4.2). In that sense it is useful to describe P as the family of probabilities $(P_x)_{x\in X}$, where P_x such that each is supported in the set $\{\, w \in \Omega \,|\, w(0) = x \,\}$, and P_x are related to the probability measure P via the equation

$$P[\,\cdot \mid \mathcal{F}_0\,](w) = P_{w(0)} \quad P\text{-a.s.}$$

Definition 4.1.2. A family of probability measures $(P_x)_{x\in X}$ on $(\Omega, \mathcal{F})$ is said to be a Vishik-Fursikov family if for any $x \in X$ the probability measure P_x is carried by paths that are solutions of (4.2) with initial condition x.

Let $(P_x)_{x\in X}$ be a Vishik-Fursikov family. Given a measure μ_0 on $(X, \mathcal{B})$, we can define a Vishik-Fursikov measure P on $(\Omega, \mathcal{F})$ by

$$P(A) = \int_X P_x(A)\mu_0(dx)\,, \quad \forall A \in \mathcal{F}\,.$$

4.2 Vishik-Fursikov Measures for NLS

For the NLS equation we were able to obtain solutions that are continuous with respect to L^2 with the strong topology and V with the weak topology, see Section 1.2.1. We are interested in the space V since it allows us to make sense of expressions of the form

$$(\nabla u(t), \nabla v) + \langle |u(t)|^{p-2} u(t), v \rangle \,, \quad \forall v \in V \,.$$

Sadly, V is not metrizable under the weak topology and we are forced to consider the spaces of the form V_R with $R > 0$ fixed.

Thus, fix $R > 0$. Consider the compact metric space V_R and the space of continuous paths $\Omega_R = C([0, \infty) \to V_R)$ endowed with the compact-open topology. The σ-algebras $\mathcal{F}$ and $\mathcal{F}_t$ for $t \geq 0$ are defined as before. We use the notation $\mathcal{B}_{\Omega_R}$ instead of $\mathcal{F}$ whenever we want to explicitly state the difference between paths taking values in V_R or V.

Adapting Definitions 4.1.1 and 4.1.2 to the case of the NLS equation, we call a probability measure P on $(\Omega_R, \mathcal{B}_{\Omega_R})$ a Vishik-Fursikov measure for the NLS equation if P is carried by a set of weak solutions of NLS with the energy bounded by R. If a probability measure μ_0 on $(V_R, \mathcal{B}_{V_R})$ is given, and if $P(\pi_0^{-1}(\cdot)) = \mu_0$ then P is a Vishik-Fursikov measure for the NLS with the initial condition μ_0.

A family of probability measures $(P_a)_{a \in V_R}$ on $(\Omega_R, \mathcal{B}_{\Omega_R})$ is a Vishik-Fursikov family of the NLS equation if, for any $a \in V_R$, P_a is carried by a set of weak solutions of NLS with the energy bounded by R and the initial condition a, i.e., for any $a \in X$, there is a set $\Gamma_a \subset \Omega_R$ such that $P_a(\Gamma_a) = 1$ and for any $u \in \Gamma_a$, u is a weak solution of the NLS equation with the energy bounded by R.

Recall the Definition 1.3.2 of the sets $S_R(a)$. For any $a \in V_R$, the set $S_R(a)$ is a compact subset of Ω_R, see Proposition 1.3.2.The space Ω_R is a complete metric space, and $S_R(a)$ itself is a complete metric space. The space of probability measures $\mathcal{P}(S_R(a))$ is a compact metric space.

For $a \in V_R$, denote by $\mathcal{C}_R(a)$ the set of probability measures supported in $S_R(a)$, i.e., $\mathcal{C}_R(a) = \mathcal{P}(S_R(a))$.

As a set-valued map $a \mapsto \mathcal{C}_R(a)$ takes values in compact subsets of $\mathcal{P}(\Omega_R)$. We have the following result.

Lemma 4.2.1. *The set-valued map $a \mapsto \mathcal{C}_R(a)$ is u.s.c.*

Proof. Suppose $a_n \rightharpoonup a$ in V_R, $P_n \in \mathcal{C}_R(a_n)$, and $P_n \overset{\star}{\rightharpoonup} P$ in $\mathcal{P}(\Omega_R)$. Then, it is known, see [2, Theorem 2.8]

$$\operatorname{supp}(P) \subset \bigcap_{n=1}^{\infty} \overline{\bigcup_{k>n} \operatorname{supp}(P_n)}\,,$$

and recall that the map $a \mapsto S_R(a)$ is u.s.c. Hence, $P \in \mathcal{C}_R(a)$. □

The existence of Vishik-Fursikov families is a result of the existence of a measurable selection for the map $\mathcal{C}_R$.

Lemma 4.2.2. *There is a Vishik-Fursikov family of probability measures for the NLS equation.*

Proof. The set-valued map $a \mapsto \mathcal{C}_R(a)$ takes values in the hyperspace of compact subsets of $\mathcal{P}(S_R(V_R)) \subset \mathcal{P}(\Omega_R)$, which is a compact metric space by itself. By Proposition 1.3.2 and Corollary 2.1.2 the map $\mathcal{C}_R$ is measurable. Thanks to Theorem 2.1.2 the map $\mathcal{C}_R$ admits a measurable selection. Any measurable selection of $\mathcal{C}_R$ is a family $P_a \in \mathcal{C}_R(a)$, hence P_a is carried by weak solutions of NLS with energy bounded by R starting at a. The family $(P_a)_{a\in X}$ is a Vishik-Fursikov family for the NLS equation. □

The beautiful aspect of having a Vishik-Fursikov family is that allow us to build Vishik-Fursikov measures from any possible initial condition μ_0.

Lemma 4.2.3. *Given a probability measure μ_0 on $(V_R, \mathcal{B}_{V_R})$ there is a Vishik-Fursikov measure for the NLS with initial condition μ_0.*

Proof. Let (P_a) be a Vishik-Fursikov family obtained as a measurable selection of $\mathcal{C}_R$. Define probability P on $(\Omega_R, \mathcal{B}_{\Omega_R})$ by the formula:

$$P(A) = \int_{V_R} P_a(A)\mu_0(da)\,, \quad \forall A \in \mathcal{B}_{\Omega_R}\,.$$

The probability measure P is carried by

$$\bigcup_{a\in V_R} \mathrm{supp}(P_a)\,,$$

which is a subset of weak solutions of the NLS equation with energy bounded by R. Moreover, thanks to the way P is defined we have $P(\pi_0^{-1}(\cdot)) = \mu_0$. Hence, P is a Vishik-Fursikov probability measure for the NLS with the initial condition μ_0. □

4.2.1 Another Construction of Vishik-Fursikov Probability Measures

Return back to the abstract Cauchy problem

$$\dot{u}(t,a) = F(u(t,a))\,, \qquad u(0,a) = a \in X\,, \tag{4.2}$$

where F maps X into itself.

There is a different way to construct an abstract Vishik-Fursikov probability measure for (4.2) if it is known that there is a measurable semiflow generated by solutions of (4.2).

Proposition 4.2.4. *Assume X is a Polish space with the Borel σ-algebra $\mathcal{B}$, and Ω is the space of continuous paths $C([0,\infty) \to X)$ equipped with the compact-open* *topology, with its Borel* σ*-algebra* $\mathcal{F}$*. Assume there is a Caratheodory map* $u(t,x)$ *with the semigroup property:*

$$u(t+s,x) = u(t,u(s,x))\,, \quad \forall t \geq 0 \;\, s \geq 0\,, x \in X\,,$$

such that $t \mapsto u(t,x)$ *solves (4.2) with the initial condition* $x \in X$*. Then, given any probability measure* μ_0 *there is a Vishik-Fursikov probability measure supported in the paths of the semiflow* u *with the initial probability measure* μ_0*.*

Proof. Since the map $x \mapsto u(\cdot,x)$ is measurable, define the push-forward of μ_0 from $(X,\mathcal{B})$ to $(\Omega,\mathcal{F})$ as

$$P(A) = \mu_0(\{\, x \in X \,|\, u(\cdot,x) \in A\,\}) \quad \forall A \in \mathcal{F}\,.$$

□

Here is a different, simpler proof of the existence of Vishik-Fursikov measures for the NLS.

Corollary. *The NLS has a Vishik-Fursikov probability measure.*

Proof. Use the measurable semiflow selection Theorem 2.2.2 on the set-valued map $S_R : V_R \to \mathcal{H}_k(\Omega_R)$ to obtain a measurable semiflow $u(t,x)$ and apply Proposition 4.2.4. □

Chapter 5

Markov Selection

Consider a Polish space X and the space of continuous paths $\Omega = C([0, \infty) \to X)$ with the compact-open topology. Given a set-valued map $a \mapsto \mathcal{C}(a) \subset \mathcal{P}(\Omega)$, a measurable selection $P_a \in \mathcal{C}(a)$ defines a random process. In some cases, such random process can be a Markov process. In [21] N. V. Krylov gave the first construction of a measurable Markov selection. D. W. Stroock and S. R. S. Varadhan presented a similar construction in the context of solutions to the martingale problem in [29]. In recent papers, F. Flandoli and M. Romito [10]; and B. Goldys et al [15], proved related versions of the Markov selection Theorem and applied it to the stochastic Navier-Stokes and porous media equation with additive Gaussian noise.

To prove the existence of Markov statistical solutions, in application of the NLS, I use the results of Flandoli and Romito. This is done in Section 5.2. In Section 5.3, I give a new abstract form of the Markov selection theorem, Theorem 5.3.6, which is closer in spirit to the work of Krylov.

5.1 Basic Notation

Throughout this chapter X is a Polish space. Let $\Omega = C([0, \infty) \to X)$ endowed with the compact-open topology.

We will need the following construction, see [29, Theorem 6.1.2]. Given $t \geq 0$, a probability measure $P \in \mathcal{P}(\Omega)$, and a $\mathcal{F}_t$-measurable map $Q_{(\cdot)} : \Omega \to \mathcal{P}(\Omega)$, such that Q_w is supported by $\{ u \in \Omega \,|\, u(0) = w(t) \}$. Define $P \otimes_t Q$ as the unique probability measure satisfying

$$P \otimes_t Q(A \cap \theta_t^{-1}(B)) = \int_A Q_w(B) P(dw) \,, \quad \forall A \in \mathcal{F}_t \,, B \in \mathcal{F} \,. \tag{5.1}$$

In particular, $P \otimes_t Q$ satisfies

$$P \otimes_t Q(A) = P(A) \,, \quad \forall A \in \mathcal{F}_t \,,$$

and

$$P \otimes_t Q \left[\theta_t^{-1}(\cdot) \,\middle|\, \mathcal{F}_t \right](w) = Q_w \quad P\text{-a.s.}$$

Recall the notation of the hyperspace of subsets of a set, $\mathcal{H}(\cdot)$. We are interested in the weak-$\star$ topology of $\mathcal{P}(\Omega)$, thus $\mathcal{H}_k(\mathcal{P}(\Omega))$ denotes the hyperspace of weak-$\star$ compact subsets of $\mathcal{P}(\Omega)$. The hyperspace of convex and weak-$\star$ compact subsets of a topological vector space X is denoted by $\mathcal{H}_{ck}(X)$.

5.2 Markov Statistical Solutions

We start with the definition of a Markov selection.

Definition 5.2.1 (Markov selection)**.** Consider a set-valued map $\mathcal{C} : X \to \mathcal{H}(\mathcal{P}(\Omega))$. A measurable selection $P_a \in \mathcal{C}(a)$ is said to be a Markov selection if, for any $a \in X$, it satisfies

$$P_a \left[\theta_t^{-1}(\cdot) \,\middle|\, \mathcal{F}_t \right](w) = P_{w(t)} \,, \quad P_a\text{-a.s.} \tag{5.2}$$

Consider the abstract Cauchy problem

$$\dot{u}(t,a) = F(u(t,a)) \,, \qquad u(0,a) = a \in X \,, \tag{5.3}$$

where F maps X into itself.

In the context of random processes, the property analogous to the semigroup property is the homogeneous Markov property. It is possible to have Vishik-Fursikov measure without the homogeneous Markov property.

I propose to study the Vishik-Fursikov measures with the homogeneous Markov property in order to recover the autonomy of the system in the sense of statistical solutions.

Definition 5.2.2 (Markov statistical solution)**.** A Vishik-Fursikov family $(P_a)_{a\in X}$ for the equation (5.3) will be called a Markov statistical solution of (5.3) if it has the Markov property (5.2).

The following theorem is a slight variation of the Markov selection as presented by F. Flandoli and M. Romito [10], and Goldys et al. [15]. A proof is presented as a corollary of Theorem 5.3.6 in Corollary 5.3.

Theorem 5.2.1. *Let $\mathcal{C}$ be a measurable set-valued map from X to $\mathcal{H}_{ck}(\mathcal{P}(\Omega))$ with the following three properties:*

MS1 *For any $a \in X$, any probability measure $P \in \mathcal{C}(a)$ is supported in $\{ \, w \in \Omega \,|\, w(0) = a \, \}$.*

MS2 *For any $a \in X$, if $P \in \mathcal{C}(a)$ and $t \geq 0$ there exists a P-null set $N \in \mathcal{F}_t$, such that*

$$P\left[\, \theta_t^{-1}(\cdot) \,\middle|\, \mathcal{F}_t \right](w) \in \mathcal{C}(w(t)) \,, \quad \forall w \notin N$$

MS3 *For any $a \in X$, if $P \in \mathcal{C}(a)$ and $Q_{(\cdot)}$ is a $\mathcal{F}_t$-measurable selection of $w \mapsto \mathcal{C}(w(t))$, then*

$$P \otimes_t Q \in \mathcal{C}(a) \,.$$

Then, $\mathcal{C}$ admits a Markov selection.

Here is an application of this theorem to statistical solutions of the NLS.

Theorem 5.2.2. *There exists a Markov statistical solution of the NLS.*

Proof. Fix $R > 0$. Define the set-valued map $a \mapsto \mathcal{C}_R(a)$ via

$$\mathcal{C}_R(a) = \mathcal{P}(S_R(a))\,.$$

This map is measurable, takes values in weak-$\star$ compact and convex subsets of $\mathcal{P}(\Omega_R)$, and satisfies **MS1**. See Section 4.2.

Let us prove that $\mathcal{C}_R$ satisfies **MS2**. For any $a \in V_R$, take any $P \in \mathcal{C}_R(a)$. The support of P is a closed set inside $S_R(a)$. Recall that Ω is a Polish space, hence the σ-algebra $\mathcal{F}_t$ is countably generated. From the properties of regular conditional probability (see [29, Theorem 1.1.8]), there is a set $N \in \mathcal{F}_t$ such that $N^c \subseteq S_R(a)$ and, for $w \notin N$, the probability measure $P\,[\,\cdot\,|\,\mathcal{F}_t\,]\,(w)$ has its support inside the closed set

$$\{\, u \in \Omega \,|\, u(s) = w(s)\,, 0 \le s \le t \,\} \bigcap S_R(a)\,.$$

Recall that if $u \in S_R(a)$ then $\theta_t(u) \in S_R(u(t))$. Thus

$$\{\, u \in \Omega \,|\, u(0) = w(t) \,\} \bigcap \theta_t(S_R(a)) \subseteq S_R(w(t))\,.$$

Hence,

$$P\left[\,\theta_t^{-1}(\cdot)\,\middle|\,\mathcal{F}_t\,\right](w) \in \mathcal{C}_R(w(t))\,, \quad \forall w \notin N\,.$$

Finally, let us prove that $\mathcal{C}_R$ satisfies **MS3**. Take $t \ge 0$. For any $a \in V_R$, let P be any element in $\mathcal{C}_R(a)$ and Q_w a $\mathcal{F}_t$-measurable selection of $w \mapsto \mathcal{C}_R(w(t))$. From the properties of $P \otimes_t Q$ there is a set $N \in \mathcal{F}_t$ such that $N^c \subseteq S_R(a)$ and

$$(P \otimes_t Q)\left[\theta_t^{-1}(\cdot)\,\middle|\,\mathcal{F}_t\right](w) = Q_w\,, \quad \forall w \notin N\,.$$

Notice that

$$\mathrm{supp}(Q_w) \subseteq S_R(w(t))\,, \quad \forall w \notin N\,,$$

then

$$\{\, w \lhd_t v \,|\, v \in \mathrm{supp}(Q_w)\,\} \subseteq \{\, w \lhd_t v \,|\, v \in S_R(w(t))\,\} \subseteq S_R(a)\,, \qquad \forall w \notin N\,.$$

Moreover,

$$\bigcup_{w \notin N} \{\, w \lhd_t v \,|\, v \in S_R(w(t))\,\} \subseteq S_R(a)\,.$$

Since

$$P \otimes_t Q \left(\bigcup_{w \notin N} \{\, w \lhd_t v \,|\, v \in S_R(w(t))\,\} \right) = 1\,,$$

$\mathrm{supp}(P \otimes_t Q) \subseteq S_R(a)$, which implies $P \otimes_t Q \in \mathcal{C}_R(a)$.

By Theorem 5.2.1 $\mathcal{C}_R$ admits a Markov selections. Any Markov selection is measurable, hence it is a Vishik-Fursikov family, and it has the Markov property, so, this is a Markov statistical solution.

□

5.3 Markov Selection Theorem

Our approach to prove the existence of a Markov selection is closer in spirit to the original construction due to N. V. Krylov [21]. We will use a theorem of V. Strassen that captures the import aspects of the required convexity of the set-valued map.

Given a set-valued map $\mathcal{C} : X \to \mathcal{H}_{ck}(\mathcal{P}(Y))$, let $h_{(\cdot)}$ be the support function of the convex set $\mathcal{C}(\cdot)$, i.e.

$$h_a[f] = \sup_{P \in \mathcal{C}(a)} \int_Y f(y)P(dy) , \quad \forall a \in X , f \in C_b(\Omega) .$$

We say that $h_{(\cdot)}$ is the support function of $\mathcal{C}$. If $\mathcal{X}$ is a σ-algebra on X and the map $\mathcal{C}$ is $\mathcal{X}$-measurable then, for any $f \in C_b(X)$, $x \mapsto h_x[f]$ is $\mathcal{X}$-measurable.

Definition 5.3.1 (Markov kernel)**.** Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be two measurable spaces. A Markov kernel from $(X, \mathcal{X})$ to $(Y, \mathcal{Y})$ is a function $Q : X \times \mathcal{Y} \to [0, 1]$ such that,

- for any $a \in X$, $Q(a, \cdot)$ is a probability measure on $\mathcal{Y}$, and,
- for any $B \in \mathcal{Y}$, $Q(\cdot, B)$ is a $\mathcal{X}$-measurable function.

A Markov kernel from $(X, \mathcal{X})$ to $(Y, \mathcal{Y})$ transforms probability measures on $(X, \mathcal{X})$ into probability measures on $(Y, \mathcal{Y})$. Indeed, if μ is a probability measure on $(X, \mathcal{X})$, a probability measure on $(Y, \mathcal{Y})$ can be defined as

$$\nu(B) = \int_X Q(a, B)\, \mu(da) , \quad \forall B \in \mathcal{Y} ,$$

and we denote this as $\nu = Q \cdot \mu$.

Assume now that Y is a Polish space and $\mathcal{Y}$ is its Borel σ-algebra. The following result of V. Strassen [28, Theorem 3] will be useful.

Theorem 5.3.1 (Strassen)**.** *Let $\mathcal{C}$ be a measurable set-valued map $X \to \mathcal{H}_{ck}(\mathcal{P}(Y))$ with the support function $h_{(\cdot)}$. Let μ be a probability measure on $(X, \mathcal{X})$ and let ν be a probability measure on $(Y, \mathcal{Y})$. In order that there exist a Markov kernel Q from $(X, \mathcal{X})$ to $(Y, \mathcal{Y})$ such that*

$$\nu = Q \cdot \mu$$

and

$$Q(x,\cdot) \in \mathcal{C}(x)\,, \qquad \textit{for } \mu\textit{-a.e. } x \in X\,,$$

it is necessary and sufficient that

$$\int_Y f(y)\,\nu(dy) \le \int_X h_x[f]\,\mu(dx)\,, \qquad \forall f \in C_b(Y)\,.$$

In what follows, let X be a Polish space, and let Ω be $C([0,\infty) \to X)$ with the compact-open topology. Recall that, for $t \ge 0$, the maps π_t and θ_t are defined as $\pi_t(w) = w(t)$ and $\theta_t(w) = w(t+\cdot)$. For any $t \ge 0$, let $\mathcal{F}_t = \sigma(\{\,\pi_s \,|\, 0 \le s \le t\,\})$, a filtration on Ω, and let $\mathcal{F}$ be the Borel σ-algebra generated by the compact-open topology of Ω. Notice that $\mathcal{F} = \sigma(\{\,\pi_t \,|\, t \ge 0\,\})$.

Definition 5.3.2. Let $\mathcal{C}$ be a measurable set-valued map $X \to \mathcal{H}_{ck}(\mathcal{P}(\Omega))$ with support function $h_{(\cdot)}$. For any $f \in C_b(\Omega)$ and $t \ge 0$ the map $w \mapsto h_{w(t)}[f]$ is $\sigma(\pi_t)$-measurable. Given any probability measure $P \in \mathcal{P}(\Omega)$, the map defined as

$$f \mapsto \int_\Omega h_{w(t)}[f]\,P(dw)$$

is also a support function, see [28]. It defines a convex subset of $\mathcal{P}(\Omega)$ that we will denote by $K_{\mathcal{C}}(P,t)$, i.e.

$$K_{\mathcal{C}}(P,t) = \left\{ Q \in \mathcal{P}(\Omega) \,\middle|\, Q[f] \le \int_\Omega h_{w(t)}[f]\,P(dw)\,,\ \forall f \in C_b(\Omega) \right\} \tag{5.4}$$

Lemma 5.3.2. *Let $\mathcal{C}$ be a measurable set-valued map $X \to \mathcal{H}_{ck}(\mathcal{P}(\Omega))$. For any $t \ge 0$ and any probability measure P on $(\Omega, \mathcal{F}_t)$, the following two are equivalent:*

1. *$Q \in K_{\mathcal{C}}(P,t)$.*

2. *There exists a Markov kernel* $Q_{(\cdot)}$ *from* $(\Omega, \mathcal{F}_t)$ *to* $(\Omega, \mathcal{F})$ *and a* P*-null set* $N \in \mathcal{F}_t$ *such that*

$$Q_w \in \mathcal{C}_{w(t)}\,, \qquad \forall w \notin N$$

and

$$Q = \int_\Omega Q_w(\cdot)\, P(dw)\,.$$

Proof. This is just a direct application of Strassen's theorem from $(\Omega, \mathcal{F}_t)$ to $(\Omega, \mathcal{F})$.

1) $\implies$ 2). Take $Q \in K_{\mathcal{C}}(P, t)$, then

$$\int_\Omega f(w)\, Q(dw) \le \int_\Omega h_{w(t)}[f]\, P(dw)\,, \qquad \forall f \in C_b(\Omega)\,.$$

By Strassen's theorem there is a P-null set $N \in \mathcal{F}_t$, and a Markov kernel $Q_{(\cdot)}$ from $(\Omega, \mathcal{F}_t)$ to $(\Omega, \mathcal{F})$, such that

$$Q_w \in \mathcal{C}_{w(t)}\,, \qquad \forall w \notin N\,,$$

and

$$Q = \int_\Omega Q_w(\cdot)\, P(dw)\,.$$

2) $\implies$ 1). If such a kernel $Q_{(\cdot)}$ exists, then, by Strassen's theorem

$$\int_\Omega f(w)\, Q(dw) \le \int_\Omega h_{w(t)}[f]\, P(dw)\,, \qquad \forall f \in C_b(\Omega)\,.$$

Hence, $Q \in K_{\mathcal{C}}(P, t)$. □

We will need certain family of linear bounded functionals on $\mathcal{P}(\Omega)$. For $\lambda > 0$ and $\varphi \in C_b(X)$, let $\zeta_{\lambda,\varphi}$ be a functional on $\mathcal{P}(\Omega)$, defined as

$$\zeta_{\lambda,\varphi}(P) = \int_\Omega \int_0^\infty e^{-\lambda s} \varphi(w(s))\, ds\, P(dw)\,.$$

For $t > 0$, let $\zeta^t_{\lambda,\varphi}$ be the functional on $\mathcal{P}(\Omega)$ defined by

$$\zeta^t_{\lambda,\varphi}(P) = \int_\Omega \int_0^t e^{-\lambda s} \varphi(w(s))\, ds\, P(dw)\,.$$

Definition 5.3.3. Assume (X,ρ) is a metric space. A collection of functions $M \subset C_b(X)$ is said to *strongly separates points* if for every $x \in X$ and $\delta > 0$ there exists a finite set $\{h_1, \dots, h_k\} \subset M$ such that

$$\inf_{y,:\, \rho(x,y) \geq \delta} \max_{1 \leq i \leq k} |h_i(y) - h_k(x)| > 0\,.$$

Let $\Phi \subset C_b(X)$ be a countable set of functions that strongly separates points of X and is closed under multiplications. Regarding the existence of such sets, see Lemma A.0.6. Let Λ be a countable and dense subset of $(0,\infty)$. Let (φ_n, λ_n) be an enumeration of $\Phi \times \Lambda$. Denote by ζ_n the functional defined as $\zeta_n = \zeta_{\lambda_n,\varphi_n}$; similarly let $\zeta^t_n = \zeta^t_{\lambda_n,\varphi_n}$ for any $t \geq 0$. An important property satisfied by these functionals is

$$\zeta(P) = \zeta^t(P) + e^{-\lambda t} \zeta(\theta_t P)\,, \qquad t \geq 0\,.$$

Lemma 5.3.3. *Let ζ be one of the functions $\zeta_{\lambda,\varphi}$. If $\mathcal{C}$ is a measurable set-valued map $X \to \mathcal{H}_{ck}(\mathcal{P}(\Omega))$ then*

$$V_\zeta[K_{\mathcal{C}}(P,t)] = K_{V_\zeta[\mathcal{C}]}(P,t)\,, \qquad \forall P \in \mathcal{P}(\Omega)\,, t \geq 0\,.$$

Proof. Let us proof first that

$$K_{V_\zeta[\mathcal{C}]}(P,t) \subseteq V_\zeta[K_{\mathcal{C}}(P,t)]\,, \qquad \forall P \in \mathcal{P}(\Omega)\,, t \geq 0\,.$$

Thanks to Lemma 5.3.2, if $Q_1 \in K_{V_\zeta[\mathcal{C}]}(P,t)$ and $Q_2 \in K_{\mathcal{C}}(P,t)$, there are maps

$$\begin{aligned} &w \mapsto Q_1(w,\cdot) \in V_\zeta[\mathcal{C}(w(t))] \\ &w \mapsto Q_2(w,\cdot) \in \mathcal{C}(w(t)) \end{aligned} \qquad P\text{-a.s.} \tag{5.5}$$

such that

$$\zeta(Q_1) = \int_\Omega \zeta(Q_1(w))P(dw)\,, \tag{5.6}$$

and

$$\zeta(Q_2) = \int_\Omega \eta(Q_2(w))P(dw)\,. \tag{5.7}$$

Since $\zeta(Q_1(w)) \geq \zeta(Q_2(w))$ P-a.s. , then $\zeta(Q_1) \geq \zeta(Q_2)$, and $Q_1 \in V_\zeta[K_{\mathcal{C}}(P,t)]$.

Suppose now that

$$V_\zeta[K_{\mathcal{C}}(P,t)] \not\subset K_{V_\zeta[\mathcal{C}]}(P,t)\,,$$

Let Q_1 be an element in $V_\zeta[K_{\mathcal{C}}(P,t)]$ but not in $K_{V_\zeta[\mathcal{C}]}(P,t)$. Due to Lemma 5.3.2 there is a map

$$w \mapsto Q_1(w) \in \mathcal{C}(w(t)) \qquad P\text{-a.s.}$$

such that

$$\zeta(Q_1) = \int_\Omega \zeta(Q_1(w))P(dw)\,. \tag{5.8}$$

The set

$$N = \left\{\, w \in \Omega \,\middle|\, Q_1(w) \in \mathcal{C}(w(t)) \bigcap V_\zeta[\mathcal{C}(w(t))]^c \,\right\}$$

is such that $P(N) > 0$, and $N \in \mathcal{F}_t$.

Take any other element $Q_2 \in K_{V_\zeta[\mathcal{C}]}(P,t)$. There is a map

$$w \mapsto Q_2(w) \in V_\zeta[\mathcal{C}(w(t))] \qquad P\text{-a.s.}$$

such that

$$\zeta(Q_2) = \int_\Omega \zeta(Q_2(w))P(dw)\,. \tag{5.9}$$

Define

$$w \mapsto Q_w = \begin{cases} Q_1(w) & w \notin N \\ Q_2(w) & w \in N \end{cases}, \tag{5.10}$$

Since $Q(w) \in \mathcal{C}(w(t))$ P-a.s. due to Lemma 5.3.2, the probability measure Q defined as $Q = Q_{(\cdot)} \cdot P$ is an element in $K_{\mathcal{C}}(P,t)$.

Since $P(N) > 0$,

$$\zeta(Q) - \zeta(Q_1) = \int_N (\zeta(Q_2(w)) - \zeta(Q_1(w))P(dw) > 0\,. \tag{5.11}$$

Then, $Q_1 \notin V_\zeta[K_{\mathcal{C}}(P,t)]$, which is a contradiction.

□

Let $\mathcal{C}$ be a measurable set-valued map $X \to \mathcal{H}_{ck}(\mathcal{P}(\Omega))$, let P be a probability measure in $\mathcal{P}(\Omega)$ and $t \geq 0$, define

$$\Gamma_{\mathcal{C}}(P,t,a) = \left\{ Q \in \mathcal{C}(a) \,\middle|\, \theta_t Q \in K_{\mathcal{C}}(P,t)\,;\, \zeta_n^t(Q) \geq \zeta_n^t(P)\,, \forall n > 0 \right\}\,.$$

Definition 5.3.4 (Krylov map)**.** A measurable set-valued map $\mathcal{C} : X \to \mathcal{H}_{ck}(\mathcal{P}(\Omega))$ will be called a *Krylov map* if it satisfies the following properties.

K1 For every $a \in X$, if $P \in \mathcal{C}(a)$ then

$$P(\{\, w \in \Omega \,|\, w(0) = a \,\}) = 1\,.$$

K2 For every $a \in X$, if $P \in \mathcal{C}(a)$ and $t \geq 0$ then

$$\theta_t P \in K_{\mathcal{C}}(P,t)\,.$$

K3 For every $a \in X$, if $P \in \mathcal{C}(a)$ and $t \geq 0$ then

$$\theta_t(\Gamma_{\mathcal{C}}(P, a, t)) = K_{\mathcal{C}}(P, t)\,.$$

Lemma 5.3.4. *Let ζ be one of the functions $\zeta_{\lambda,\varphi}$. If $\mathcal{C} : X \to \mathcal{H}_{ck}(\mathcal{P}(\Omega))$ is a Krylov map, the set-valued map $a \mapsto V_\zeta\left[\mathcal{C}(a)\right]$ is also a Krylov map.*

Proof. Let $\mathcal{C}$ be a Krylov map and consider the set-valued map $V_\zeta[\mathcal{C}] : x \mapsto V_\zeta[\mathcal{C}(x)]$. By [27, Lemma 4.3] the map is measurable, and clearly compact and convex valued. To verify that this map is Krylov, we check properties **K1 - K3** of Definition 5.3.4. Property **K1** is immediately satisfied.

To check **K2**, pick any $P \in V_\zeta[\mathcal{C}(x)]$ and an $t \geq 0$. We want to show that $\theta_t P \in K_{V_\zeta[\mathcal{C}]}(P, t)$. In view of Lemma 5.3.3, we need $\zeta(\theta_t P) \geq \zeta(Q)$ for all $Q \in K_{\mathcal{C}}(P, t)$. Condition **K3** for $\mathcal{C}$ implies that if $Q \in K_{\mathcal{C}}(P, t)$, then there exists $Q' \in \Gamma_{\mathcal{C}}(P, t, x)$ such that $\theta_t Q' = Q$. Since $Q' \in \mathcal{C}(x)$ and P maximizes ζ on $\mathcal{C}(x)$, we have

$$\zeta(P) \geq \zeta(Q')\,,$$

which we re-write as

$$\zeta^t(P) + e^{-\lambda t}\zeta(\theta_t P) \geq \zeta^t(Q') + e^{-\lambda t}\zeta(Q)\,.$$

On the other hand, since $Q' \in \Gamma_{\mathcal{C}}(P, t, x)$,

$$\zeta^t(P) \leq \zeta^t(Q')\,.$$

Combine this with the previous inequality to obtain

$$\zeta(\theta_t P) \geq \zeta(Q)\,,$$

which implies $\theta_t P \in V_\zeta[K_{\mathcal{C}}(P,t)]$, and property **K2** follows.

It remains to show that

$$\theta_t \left(\Gamma_{V_\zeta[\mathcal{C}]}(P,t,x)\right) = K_{V_\zeta[\mathcal{C}]}(P,t) \tag{5.12}$$

provided $P \in V_\zeta[\mathcal{C}(x)]$. Take any $Q \in K_{V_\zeta[\mathcal{C}]}(P,t)$. Then $Q \in K_{\mathcal{C}}(P,t)$, and, by condition **K3** for $\mathcal{C}$, there exists $Q' \in \Gamma_{\mathcal{C}}(P,t,x)$ such that $\theta_t Q' = Q$. Since both $\theta_t P$ and Q belong to $V_\zeta[K_{\mathcal{C}}(P,t)]$, we have $\zeta(\theta_t P) = \zeta(Q) = \zeta(\theta_t Q')$. Then

$$\zeta(Q') - \zeta(P) = \zeta^t(Q') + e^{-\lambda t}\zeta(\theta_t Q') - \zeta^t(P) - e^{-\lambda t}\zeta(\theta_t P) = \zeta^t(Q') - \zeta^t(P)\,.$$

Now recall that $Q' \in \Gamma_{\mathcal{C}}(P,t,x)$ which implies $\zeta^t(Q') \geq \zeta^t(P)$. Thus, $\zeta(Q') \geq \zeta(P)$. Since P maximizes ζ on $\mathcal{C}(x)$, so does Q'. Then $Q' \in \Gamma_{V_\zeta[\mathcal{C}]}(P,t,x)$. Inclusion $\subset$ in (5.12) is obvious. Indeed, let $P \in V_\zeta[\mathcal{C}(x)]$ and suppose $Q' \in \Gamma_{V_\zeta[\mathcal{C}]}(P,t,x)$. Then $\theta_t Q' \in K_{V_\zeta[\mathcal{C}]}(P,t)$ right from the definition of $\Gamma_{V_\zeta[\mathcal{C}]}(P,t,x)$.

□

Lemma 5.3.5 (Separating lemma)**.** *Let $\mathcal{C} : X \to \mathcal{H}_{ck}(\mathcal{P}(\Omega))$ be a Krylov map. Assume that, for any $x \in X$, and $P_1, P_2 \in \mathcal{C}(x)$ we have*

$$\int_\Omega f(u(t))P_1(du) = \int_\Omega f(u(t))P_2(du)\,, \quad \forall f \in C_b(X)\,, t \geq 0\,.$$

Then, for any $x \in X$, $\mathcal{C}(x)$ is a singleton.

Proof. The probability measures

$$p_t(x,\cdot) = P(\pi_t^{-1}(\cdot))\,, \quad \forall t \geq 0 \tag{5.13}$$

on X are independent of the choice of probability measure $P \in \mathcal{C}(x)$. Take one such P. An integral form of equation (5.13) is

$$\int_X \varphi(y)\, p_t(x, dy) = \int_\Omega \varphi(w(t))\, P(dw)\,, \quad \forall \varphi \in C_b(X)\,. \tag{5.14}$$

By property **K2**, $\theta_t P \in K_{\mathcal{C}}(P, t)$. Then Strassen's theorem guarantees that there exists a Markov kernel Q_w such that

$$\int_\Omega g(w)\, \theta_t P(dw) = \int_\Omega \int_\Omega g(v)\, Q_w(dv)\; P(dw)$$

for any $g \in C_b(\Omega)$, and $Q_w \in \mathcal{C}(w(t))$ $\;P$-a.s. , and the map $w \mapsto \int g(v)\, Q_w(dv)$ is $\mathcal{F}_t$-measurable. Take $g(w) = \varphi(\pi_s(w))$, where $\varphi \in C_b(X)$ and $s \geq 0$. Then the above equality reads

$$\int_\Omega \varphi(w(s+t))\, P(dw) = \int_\Omega \int_\Omega \varphi(\pi_s(v))\, Q_w(dv)\; P(dw)\,.$$

Using (5.13) and (5.14), we can re-write it as follows:

$$\int_X \varphi(y)\, p_{s+t}(x, dy) = \int_\Omega \int_X \varphi(y)\, p_s(w(t), dy)\; P(dw) = \int_X \int_X \varphi(y)\, p_s(z, dy)\, p_t(x, dz)\,.$$

In other words, the measures $p_t(x, \cdot)$ satisfy the Chapman-Kolmogorov equation,

$$p_{s+t}(x, dy) = \int_X p_s(z, dy)\, p_t(x, dz)\,. \tag{5.15}$$

The fact that $\theta_t P \in K_{\mathcal{C}}(P,t)$ for all $t \geq 0$ and equation (5.14) justify the following calculation, where ψ_1 and ψ_2 are arbitrary functions from $C_b(X)$:

$$\int \psi_1(w(t))\psi_2(w(t+s))\, P(dw) = \int \psi_1(w(0))\psi_2(w(s))\, (\theta_t P)(dw) =$$
$$\int \left(\int \psi_1(v(0))\psi_2(v(s))\, Q_w(dv) \right)\, P(dw) = \int \psi_1(w(t)) \left(\int \psi_2(v(s))\, Q_w(dv) \right)\, P(dw) =$$
$$\int \psi_1(w(t)) \left(\int_X \psi_2(x_2)\, p_s(w(t), dx_2) \right)\, P(dw) =$$
$$\int_X \psi_1(x_1) \left(\int_X \psi_2(x_2)\, p_s(x_1, dx_2) \right)\, p_t(x, dx_1)\,.$$

More generally, we obtain

$$\int_\Omega \psi_1(w(t_1))\psi_2(w(t_2)) \cdots \psi_n(w(t_n))\, P(dw) =$$
$$\int_{X^n} \psi_1(x_1)\psi_2(x_2) \cdots \psi_n(x_n)\, p_{t_1}(x, dx_1)\, p_{t_2-t_1}(x_1, dx_2) \cdots p_{t_n-t_{n-1}}(x_{n-1}, dx_n)\,.$$

This shows that the values of the measure P on sets of the form

$$\{\, w \in \Omega \,|\, w(t_1) \in A_1, \ldots, w(t_n) \in A_n, \text{ where } 0 \leq t_1 < \cdots \leq t_n,\ A_1, \ldots, A_n \in \mathcal{B} \,\}\ ,$$

are independent of the choice of P from $\mathcal{C}(x)$. Hence, $\mathcal{C}(x)$ is a singleton. □

Theorem 5.3.6. *If $\mathcal{C} : X \to \mathcal{H}_{ck}(\mathcal{P}(\Omega))$ is a Krylov map, then $\mathcal{C}$ admits a Markov selection.*

Proof. Define a sequence of Krylov maps recursively by setting $\mathcal{C}^0 = \mathcal{C}$ and $\mathcal{C}^{n+1} = V_{\zeta_{n+1}}[\mathcal{C}^n]$. For every $x \in X$, the sets $\mathcal{C}^n(x)$ form a decreasing sequences of nested convex compacta, hence their intersection, $\mathcal{C}^\infty(x)$, is a non-empty convex compact. Thus we obtain a reducted set-valued map $\mathcal{C}^\infty : x \mapsto \mathcal{C}^\infty(x)$. This map is again Krylov. We prove next that each set $\mathcal{C}^\infty(x)$ is a singleton.

Suppose $P_1, P_2 \in \mathcal{C}^\infty(x)$. Then $\zeta_n(P_1) = \zeta_n(P_2)$ for all n. The way the functionals ζ_n were defined, we first conclude (by taking into account uniqueness of the Laplace transform) that equality

$$\int_\Omega \varphi(w(t))\, P_1(dw) = \int_\Omega \varphi(w(t))\, P_2(dw)\,, \quad \forall t \geq 0, \tag{5.16}$$

is satisfied for all functions φ from the countable set Φ of bounded continuous functions on X that strongly separates points and are closed under multiplication. By Theorem 11 in [3], the set Φ separates probability measures, then $P_1(\pi_t^{-1}(\cdot)) = P_2(\pi_t^{-1}(\cdot))$. By Lemma 5.3.5, $\mathcal{C}^\infty(x)$ is a singleton.

The family of measures $P_x \in \mathcal{C}^\infty(x)$, $x \in X$, satisfies

$$\theta_t P_x(\cdot) = \int_\Omega P_{w(t)}(\cdot)\, P_x(dw)\,, \quad \forall t \geq 0\,,$$

which implies (5.2), hence it is Markov. □

Corollary. *Markov selection Theorem 5.2.1 follows from Theorem 5.3.6*

Proof. We check that conditions **MS1 - MS3** in the statement of theorem 5.2.1 guarantee that conditions **K1 - K3** for the map $\mathcal{C}$ to be Krylov are satisfied. Property **K1** is just **MS1**. To verify **K2**, we use assumption **MS2**. Pick $x \in X$ and a $P \in \mathcal{C}(x)$. From the definition of conditional probability,

$$P(A \cap \theta_r^{-1}(B)) = \int_A P[\theta_r^{-1}(B)|\mathcal{F}_r](w)P(dw)$$

for $A \in \mathcal{F}_t$ and $B \in \mathcal{F}$. If $A = \Omega$,

$$\theta_t P(B) = \int P[\theta_t^{-1}(B)|\mathcal{F}_t](w)P(dw)\,.$$

Assumption **MS2** and Strassen's theorem then shows that $\theta_t P \in K_{\mathcal{C}}(P,t)$, hence property **K2** is satisfied. Notice here that we need the construction of $K_{\mathcal{C}}(P,t)$ to use P restricted to $\mathcal{F}_t$ and not just to $\sigma(\pi_t)$, because $w \mapsto P[\theta_t^{-1}(B)|\mathcal{F}_t](w)$ is $\mathcal{F}_t$-measurable but not $\sigma(\pi_t)$-measurable.

Finally, with $P \in \mathcal{C}(x)$, any element in $K_{\mathcal{C}}(P,t)$ comes from some measurable selection $Q.$ of the set-valued map $w \mapsto \mathcal{C}(w(s))$. According to **MS3**, $P \otimes_t Q \in \mathcal{C}(x)$, and

$$\theta_t(P \otimes_t Q)(\cdot) = \int Q_w(\cdot) P(dw)\,.$$

Therefore, $\theta_t(P \otimes_s Q) \in K_{\mathcal{C}}(P,t)$. Since $P(A) = (P \otimes_t Q)(A)$ for any $A \in \mathcal{F}_t$, we have $\zeta_n^t(P) = \zeta_n^t(P \otimes_s Q)$. Hence, $P \otimes_s Q \in \Gamma_{\mathcal{C}}(P,t,x)$. Thus, $K_{\mathcal{C}}(P,t) = \theta_t(\Gamma_{\mathcal{C}}(P,t,x))$ and property **K3** is satisfied. $\square$

Appendices

Appendix A

Probability Measures on Topological Spaces

We are mostly interested on probability measures on Banach spaces with the weak and weak-$\star$ topology. Recall that this space is not metrizable. The usual literature in probability theory is presented for metric spaces, extra care is needed to use some of the fundamental results in this cases. We follow the definitions and results of [4], [20], [25] and [31].

Definition A.0.1. Let X be a topological space, and let $\mathcal{B}$ be its Borel σ-algebra.

A Borel measure on $(X, \mathcal{B})$ is said to be **regular** if for any set $B \in \mathcal{B}$ the measure satisfies

$$\mu(B) = \sup \{ \mu(F) \,|\, B \supseteq F \text{ is closed} \} . \tag{A.1}$$

Definition A.0.2. Let X be a topological space, and let $\mathcal{B}$ be its Borel σ-algebra. A Borel measure on $(X, \mathcal{B})$ is said to be **tight** if for any $\varepsilon > 0$ there is a compact subset $K_\varepsilon \subseteq X$ such that $\mu(K_\varepsilon) > 1 - \varepsilon$.

Definition A.0.3. Let X be a topological space, and let $\mathcal{B}$ be its Borel σ-algebra. A Borel measure on $(X, \mathcal{B})$ is said to be a **Radon measure**, if for any $B \in \mathcal{B}$ and $\varepsilon > 0$ there is a compact subset $K_\varepsilon \subseteq B$ such that $\mu(B \setminus K_\varepsilon) < \varepsilon$.

Lemma A.0.1. *If X is a separable Banach space, then the Borel σ-algebras associated with the weak and the strong topologies agree.*

We will denote the Borel σ-algebra by $\mathcal{B}$ whenever the contexts allows.

Lemma A.0.2. *Let X be a reflexive and separable Banach space. If μ is a finite Borel measure on X, then μ is tight. Moreover, if μ is regular with respect to the weak topology, then μ is also a Radon measure.*

Proof. Let B_r be the closed ball of radius $r > 0$ in X, i.e.$B_r = \{ x \in X \mid \|x\|_X \leq r \}$. Since X is reflexive, the set B_r is weak-compact thanks to the Banach-Alaoglu theorem.

Let $A_1 = B_1$ and for any $n \geq 1$ let $A_{n+1} = B_{n+1} \setminus B_n$. Notice that the space X can be exhausted with the disjoint family $(A_n)_{n\geq 1}$. Since the measure is countably additive we have

$$\sum_{n=1}^{\infty} \mu(A_n) = \mu(X) < \infty \, .$$

For any $\varepsilon > 0$, there exists some N such that

$$\sum_{n=N+1}^{\infty} \mu(A_n) < \varepsilon \, ,$$

hence $\mu(B_N) > 1 - \varepsilon$ and B_N is compact.

Assume now that μ is regular. For any $B \in \mathcal{B}$ let F_ε be a weak-closed set such that $\mu(B \setminus F_\varepsilon) < \frac{\varepsilon}{2}$ and let K_ε be a weak-compact set such that $\mu(K_\varepsilon) > 1 - \frac{\varepsilon}{2}$.

Consider the set $C = F_\varepsilon \cap K_\varepsilon \subset B$ a weak-compact set satisfying $\mu(B \setminus C) < \varepsilon$, hence μ is Radon.

□

Theorem A.0.3. *Let X be a reflexive and separable Banach space. If (μ_n) is a tight sequence of regular probability measures, then it admits a weak convergent subsequence.*

Proof. If X is separable and reflexive, for any $r > 0$ the closed ball of radius r, B_r, is weak-compact. Moreover, when endowed with the subspace topology it is a metrizable topological space. See [25]. Assume all B_r have been endowed with a metric.

From the Banach-Steinhaus theorem, any weak-compact subset $K \subset X$ is bounded, hence it is a subset of B_r for some $r > 0$. Moreover, since B_r is a metric space, then K is a compact metric space as well.

From Lemma A.0.2 the family of regular probability measures is a family of Radon measures. Since X is completely regular, see [25], and due to Theorem 8.6.7 in [4] we conclude that the sequence (μ_n) admits a weakly convergent subsequence. $\square$

Recall that, given a separable Banach space X, its dual X^* when endowed with either the weak or weak-$\star$ topology is a completely regular locally convex topological space, see [25]. We use this fact to rewrite must of the results needed from [31] that usually hold for completely regular topological spaces.

Theorem A.0.4. *Let X be a separable Banach space, and let μ_1 and μ_2 be probability measures on $(X^*, \mathcal{B}_{X^*})$. If $\hat{\mu}_1(x) = \hat{\mu}_1(x)$ for any $x \in X$, then $\mu_1 = \mu_2$.*

The following results is essential in analyzing the weak-convergence of measures on infinite-dimensional topological vector spaces. For a more general result, see [31, Chapter IV, Theorem 3.1].

Theorem A.0.5. *Let X be a separable and reflexive Banach space. Let (μ_n) be a sequence of probability measures on $(X, \mathcal{B})$, regular with respect to the weak topology of X. Let $(\hat{\mu}_n)$ be the corresponding sequence of characteristic functionals of (μ_n). Finally, let Φ be a functional $X^* \to \mathbb{C}$.*

If the sequence (μ_n) is weakly relatively compact, and $(\hat{\mu}_n)$ converges pointwise to Φ, then the whole sequence (μ_n) is weakly convergent to a measure μ, moreover $\hat{\mu} = \Phi$.

Proof. Let μ be any limit of the sequence (μ_n), i.e.there is a subsequence μ_{n_k} weakly convergent to μ.

For any $y \in X$, the map $x \mapsto e^{i\ \mathrm{Re}(\langle x,y\rangle)}$ is continuous with respect to the weak-$\star$ topology of X, on the subsequence μ_{n_k} we have

$$\hat{\mu}_{n_k}(y) = \int_{X^*} e^{i\ \mathrm{Re}(\langle x,y\rangle)}\mu_{n_k}(dx) \xrightarrow{k\to\infty} \int_{X^*} e^{i\ \mathrm{Re}(\langle x,y\rangle)}\mu(dx) = \hat{\mu}(y)\,.$$

Since the sequence $\hat{\mu}_n$ converges pointwise to Φ, we have $\hat{\mu}(y) = \Phi(y)$. This is true for any limit point of (μ_n), due to Theorem A.0.4 any two limits are the same, hence (μ_n) has only one limit point.

□

Lemma A.0.6. *Let (X,ρ) be a complete separable metric space. There exists a countable family of bounded uniformly continuous functions, closed under multiplication which strongly separates points of X.*

Proof. Consider the following functions $\{g_{y_n,k}(\cdot) = (1 - k\rho(\cdot, y_n) \vee 0\}$, where $k \in \mathbb{N}$ and y_n runs through the dense sequence in X. This family itself strongly separates points, see example after Lemma 4 in [3]. Define Φ as the family of functions $g_{y_n,k}$ and all of their finite products. □

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